Quantum theory is exclusive: a distributed computing setup

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The framework of distributed computing, whose constituents are the several spatially separated input-output servers, has immense importance in distant data manipulation. The most challenging part for such a setting is to optimize the use of information transmission lines among these distant servers. In this work, we have modeled such a physically motivated distributed computing setup for which quantum communication outperforms its classical counterpart, in terms of a limited usage of perfect transmission lines. Moreover, a broader class of communication entities, which allow state-effect description more exotic than quantum and described within the framework of general probability theory, also fails to meet the strength of quantum theory. The computational strength of quantum communication further justified in terms of a stronger version of this task, namely the delayed-choice distributed computation. The proposed task thus provides a new approach to operationally single out quantum theory in the theory-space and hence promises a novel perspective towards the axiomatic derivation of Hilbert space quantum mechanics.

INTRODUCTION

Computation is one of the most profound achievements of human scientific endeavour that shapes the modern era. A comprehensive understanding of its naive foundations demands interdisciplinary study on mathematics and logic [1], computer science [2], cognitive sciences [3], and physics [4–8]. Mathematically, a computation can be represented as a function from some input string to output string. In a physical model of computation, the input strings are encoded in a physical system on which, depending upon the computable function, some physical processes are performed to obtain the desired output value. By its name, the distributed computation suggests a highly complex, however practically relevant, computing scenario, where the input strings are distributed among multiple numbers of non-communicating receiving servers and the outputs are, in general, computed at the output servers – some distant computers. Every individual input servers hence are allowed to transmit their received data to the distant computers via information transmission lines. This scenario mimics the framework of

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distant data manipulation, which challenges modern technology to decrease the resource requirement for transferring the data from individual input servers. For instance, when a massive celestial object is observed by multiple telescopes at different geographic locations on earth and their observed data is finally computed at a distant computing lab, one needs to access a huge numbers of perfect transmission lines between those input servers and the computing lab. Although the inputs and outputs in this distributed setup are considered to be the strings of classical bits, but while the question of transferring those information among different servers arises, they can be encoded in the states of different systems described by different operational theories, viz., classical, quantum or even more exotic systems than quantum [5, 6, 10, 11]. The transmission lines between the input and output servers should be then chosen accordingly to be compatible with the concerning theories and finally the computation is accomplished by performing a suitably chosen measurement on the encoded systems.

Depending upon the configuration of these servers in spacetime, the encoding and decoding step can be of two types - global and local. Global implementation of encoding and decoding requires all the servers to be at same spacetime point so that any joint physical processes can be performed for the required computation, whereas in local case the servers are spatially separated and accordingly their actions are limited. This results in four broad class of computational scenarios – (i) local-local, (ii) global-local, (iii) global-global and (iv) local-global; the first type characterizes the encoding procedure and the second stands for decoding procedure. For instance, local discrimination of multipartite product states can be considered as local-local computational scenario where classical information encoded in product states needs to be decoded locally. The phenomenon of 'nonlocality without entanglement' studied in quantum theory [13–16] as well as in generalized probability theory (GPT) [17] confirms instances where perfect success is not possible if the spatially separated parties are constrained to communicate classically only (restriction on the type of communication). Similarly, the local distinguishability of orthogonal entangled states [18–20] and recently proposed local marking of such states [21] constitute a scenario for global-local computation, while the recently proposed 'hyper-signaling game' [11] and 'pairwise distinguishability' game [22] can be considered as a prototype of global-global computational scenario. In the present work we propose a computational scenario that stands as an appropriate example of the fourth type, *i.e.* the local-global computational scenario. The marginal constituents of the encoded system possessed by each receiving server will be constrained by their type and by their information carrying capacity which motivates us to call this computing scenario *distributed computing with limited communication* (DCLC). Quite interestingly, we find DCLC tasks that can be computed exactly in quantum theory, but the classical theory as well as several other GPTs allowing more exotic state or/and effect space structure than quantum theory fail to do the computations. We also provide a characterization of such tasks that can be done perfectly in quantum theory. We then study a variant of the proposed the DCLC task where part of the computing function will be known after the communication from the servers to the computer is completed. We called this variant *delayed choice*-DCLC, *i.e.* DCDCLC and in short denote it as DC^{2}LC. Interestingly, perfect accomplishment of certain DC^{2}LC depends on the structure of the operational theory considered to model the communicating systems. The present work therefore initiates

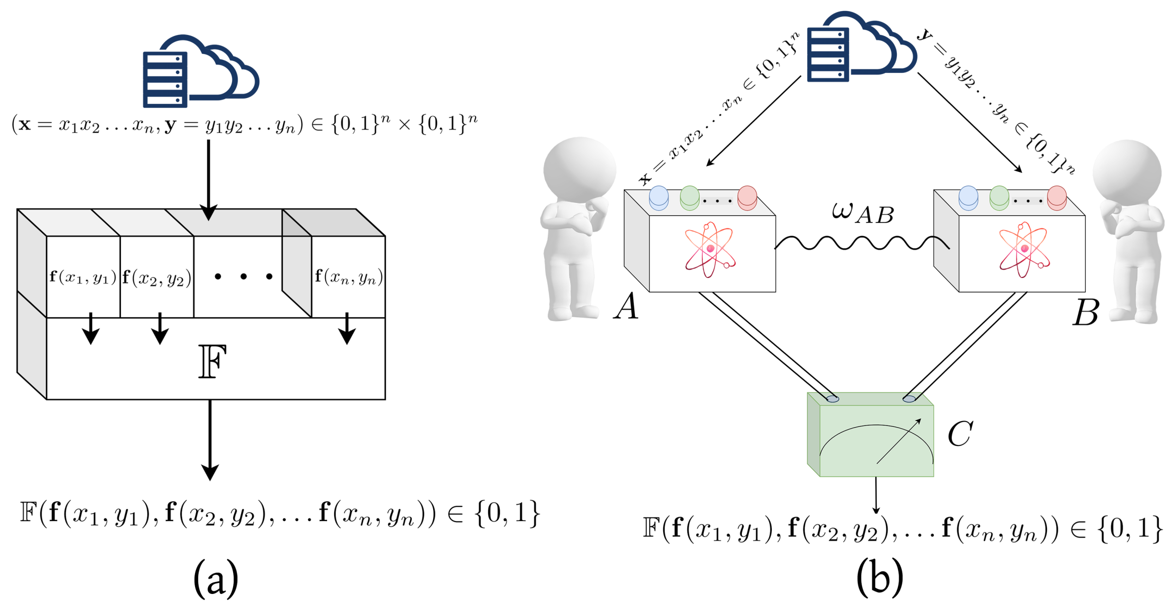

Figure 1. (Color on-line) (a) *Dual layer computing device* receives two independent and uniformly random n-bit strings x and y from a server and outputs a bit. First, it computes the function $\mathbf{f}$ on the bits taken pairwise from $\mathbf{x}$ and $\mathbf{y}$ and finally computes $\mathbb{F}$ on the outputs of the first layer, *i.e.*, the dual layer computation can be represented as $(\mathbb{F}, \mathbf{f}) : \{0,1\}^n \times \{0,1\}^n \mapsto \{0,1\}$. (b) Corresponding *distributed computing scenario*: Non-communicating labs A and B receive two n-bit strings $\mathbf{x}$ and $\mathbf{y}$ respectively from a server. A and B are allowed to encode their strings' information in the state of some systems S_A and S_B that can initially be prepared in some correlated state ω_{AB}. The computer C needs to perform a measurement on systems received from A and B to simulate the dual layer computing device. In delayed choice version of this task, the function $\mathbb{F}$ is declared at a later stage after the communications from A to C and from B to C are done.

a novel approach in identifying quantum theory as an island in the theory-space. At this point it should be noted that imposing restriction on the type or the capacity of the individual transmission lines is crucial to obtain distinction among different theories. Without any such such restriction any distributed computation can always be performed perfectly since input and output strings are of classical bits.

RESULTS

We start by introducing the *distributed computation with limited communication* scenario for two n-bit input strings.

Distributed computing scenario:

The scenario consists of dual layer functions $\mathcal{F} \equiv (\mathbb{F}, \mathbf{f}) : \{0,1\}^{2n} \to \{0,1\}$ and input strings distributed among two non communicating servers A and B personified as Alice and Bob, respectively. The strings $\mathbf{x} \equiv x_1 \cdots x_n \in \{0,1\}^n$ and $\mathbf{y} \equiv y_1 \cdots y_n \in \{0,1\}^n$

are sampled independently and randomly from a cloud server and distributed to Alice and Bob respectively. Alice and Bob encodes their inputs in the state of their respective systems (denoted as S_A and S_B), and send the system to a distant computer C, personified as Charlie, where the final computation $\mathcal{F}(\mathbf{x},\mathbf{y}) \equiv \mathbb{F}(z_1,\cdots,z_n)$ with $z_i = \mathbf{f}(x_i, y_i)$ takes place and produces a single bit output (see Fig.1). A more general structure of the aforementioned dual-layer computation can be proposed where $z_i = \mathbf{f}_i(x_i, y_i)$ and $\mathbf{f}_i$'s are different functions for different $i \in \{1, 2, \cdots, n\}$. However, in this work we will be restricted to the case where all f_i's are identical. Restriction on the information carrying capacities of S_A and S_B make this computational scenario nontrivial. For instance, not all DCLC(n) tasks can be done perfectly if only $(n-1)$-cbits are allowed from each of Alice and Bob to Charlie. Here and henceforward, we will use the notation DCLC(n) to indicate that the inputs $\mathbf{x}$ and $\mathbf{y}$ are n-bit strings. For an arbitrary theory, the limitation on communication can be put through some operational means. One such way is to restrict the *operational dimension* (OD) of the encoding systems which corresponds to the maximum number of states that can be perfectly discriminated in a single-shot measurement (see Appendix B for mathematically rigorous definition). Restriction on communication might also be imposed from other information theoretic motivations [11, 23, 24]. Our DCLC task can be seen as a close cousin of the well known *simultaneous message passing model* [25]. The difference is that we consider the computation to be two-layered. Furthermore, while the studies of simultaneous message passing model are so far limited with classical and quantum resources [26–29] and to show quantum advantage over the classical counterpart, here we study the DCLC task in a more versatile framework popularly known as generalized probabilistic theory (GPT).

In the computational scenario, introduced above, all the parties (Alice, Bob, and Charlie) know both the functions $\mathbb{F}$ & $\mathbf{f}$ and choose their encoding and decoding strategies accordingly. An interesting variation of the task can be introduced where part of the computing function remain oblivious to Alice and Bob prior to their communication(s) to charlie. More particularly, the function $\mathbf{f}$ is known to all apriori, but Alice and Bob learn about the function $\mathbb{F}$ only after they communicate to Charlie – DC^{2}LC variant. Interestingly, we establish that perfect accomplishment of some DC^{2}LC task demands specific structures in the state and effect spaces of the operational theories. Depending upon whether the systems S_A and S_B are taken to be classical or quantum or the elements of post-quantum GPT, the strategies executing a DCLC/ DC^{2}LC are respectively called classical, quantum, and post-quantum strategies.

Trivial computation:

The formal definition of a trivial computation in the aforesaid distributed scenario is given below.

Definition 1. *A dual layer computation $(\mathbb{F},\mathbf{f}) \in$DCLC(n) is said to be trivial whenever there exists a classical strategy executing the computation exactly, otherwise it is said to be nontrivial.*

For a given n, there are total $2^{2^n} \times 2^{2^2}$ number of different DCLC(n) tasks - some of them are trivial and others nontrivial. For instance, a computation $(\mathbb{F},\mathbf{f})$ is trivial whenever at least one of the functions is a constant function. Importantly, there exist trivial computations where neither $\mathbb{F}$ nor $\mathbf{f}$ is a constant. One such example is

($\mathbb{F} \equiv \oplus, \mathbf{f} \equiv \oplus$), where '$\oplus$' denotes the logical exclusive disjunction (XOR) operation. Triviality follows from the fact that $\oplus_{i=1}^{n} z_i = \oplus_{i=1}^{n}(x_i \oplus y_i) = (\oplus_{i=1}^{n} x_i) \oplus (\oplus_{i=1}^{n} y_i)$, *i.e.* Charlie can do the required computation if Alice and Bob inform parity of their respective strings which requires only 1-cbit communication from each of the transmitters to Charlie. The general class of DCLC(2) trivial computations has been characterized in the *Methods* section.

Nontrivial computations and their computability:

We will now explore the possibility of accomplishing a nontrivial DCLC task in a broader class of GPTs. In this framework, a system is specified by its state space, effect space, and by the allowed transformations acting on the states and effects. For instance, the state space of a classical system with d distinct state is a $(d-1)$ simplex, while for a d-label quantum system it is the set $\mathcal{D}(\mathcal{H}^d)$ of density operators acting on d-dimensional complex Hilbert space $\mathcal{H}^d$. Due to the convex structure of the allowed states and effects, this framework is also known as *convex operational theories* (see Appendix B for detailed description). A GPT also specifies the description of composite systems which is given by some *tensor product* of the component subsystems. Composite systems can be prepared in entangled states that can not be decomposed as convex mixtures of product states of the component subsystems. Importantly, such entanglement is considered to be one of the most crucial non classical signatures. Mathematically, choice of the tensor product structure in not unique, and at this point, role of physical/information principles become crucial to single out the desired structure [6, 7, 22].

To accomplish a distributed computation, Alice and Bob start their protocol with a shared bipartite state $\omega^{AB} \in \Omega^{AB}$, where Ω^{AB} is the state space for composite system with the subsystems S_A & S_B satisfying the constraints imposed on their operational dimension. Depending upon the inputs $\mathbf{x}$ and $\mathbf{y}$, Alice and Bob will apply some local encoding operations which consists of some local reversible transformations $\mathcal{T}_{\mathbf{x}}^{A}$ and $\mathcal{T}_{\mathbf{y}}^{B}$, respectively. At this encoding stage. one might consider more general local operations that are not reversible. However such operations turn out be less efficient as the non-classical correlation, eg. entanglement, in the bipartite state generally decreases under such operations. On the other hand application of such operations is thermodynamically costlier than reversible operations. Once Charlie the encoded systems from Alice and Bob, he performs some decoding measurement $\mathcal{M}^{AB} \equiv \{e_k^{AB} \mid e_k^{AB} \in \mathcal{E}^{AB}\ \&\ \sum_k e_k^{AB} = u^{AB}\}$ on the received bipartite state, where $\mathcal{E}^{AB}$ is the set of all bipartite effects with u^{AB} being the unit effect (see Appendix B). Post processing of the measurement outcomes completes the final computation $\mathbb{F}(f(x_1, y_1), \cdots, f(x_n, y_n))$.

Among the broad class of GPTs quantum theory is an example that allows entangled states as well as entangled measurements. However, quantum theory is not the only model with this features. An interesting class of toy models was introduced in [31] where state spaces of elementary systems are described by symmetric polygons. Bipartite compositions of these models can allow entangled states and entangled effects and both [11]. In the *Methods* section we establish an interesting result that perfect accomplishment of a non-trivial DCLC(n) computation in a theory necessitates the presence of entanglement. Naturally, the question arises whether all the nontrivial DCLC tasks can be perfectly accomplished in a GPT that allows entanglement in its state and/or effect space. In the next, we will see that this is not the case in general. To this aim, we characterize the DCLC(n) tasks that can be perfectly done in quantum theory. Recall that due to the restriction on OD, each of Alice and Bob can communicate some quantum state $\rho \in \mathcal{D}(\mathbb{C}^d)$ to Charlie, where $d \leq 2^{(n-1)}$. Of course, they can start the protocol with some bipartite entangled state $\rho^{AB} \in \mathcal{D}(\mathbb{C}^d \otimes \mathbb{C}^d)$.

Theorem 1. *A nontrivial dual layer computation* $(\mathbb{F}, \mathbf{f}) \in DCLC(2)$ *is perfectly computable in quantum theory if and only if* $\mathbf{f}$ *is a balanced function.*

Part of the proof of the theorem is discussed in the *Methods* section and the other part is detailed in Appendix C. Once Theorem 1 identifies the nontrivial computations that can be perfectly accomplished with quantum resource, we can now proceed to see whether these computations can be accomplished in other GPTs that allow entanglement is their bipartite description. This question is important to establish exclusiveness of QT over a broader class of GPTs.

In accordance with the limitation imposed on OD, we consider a class of GPTs for the DCLC(2) tasks where the state space Ω_k for a single system is described by symmetric polygons with k vertices ($k \geq 4$) [31]. Their bipartite composition can be constructed in several ways that are more enriched than quantum theory in some sense. For instance, the Popescu & Rohrlich (PR)-model is one extreme composition for $k = 4$ that allows all possible product and entangled states [10]. This model exhibits stronger nonlocal behaviour than quantum theory which gets depicted in the Clauser-Horne-Shimony-Holt inequality violation. On the other extreme, the hyper signaling (HS)-model allows only the product states, but incorporates all possible product and entangled effects. While this model is local by construction, it exhibits some striking feature by allowing stronger *time-like* correlations than that are allowed in than quantum theory [11]. Notably, every possible bipartite compositions for any GPT model can lie in between two of extreme configurations – (i) Minimal tensor product (SEP), which is comprised of only product states but with all possible effects (product as well as, entangled) and (ii) Maximal tensor product ($\overline{SEP}$), containing only product effects with all possible descriptions of states (product and entangled). In Appendix D, we construct extreme bipartite compositions for all the polygonal models and establish the following *no-go* result.

Theorem 2. *None of the nontrivial computations in DCLC(2) can be perfectly done in the extreme bipartite models with marginal subsystems described by symmetric polygon model.*

Apart from the above two extremes, bipartite polygon models can also be composed in several ways that lie in between. The sets of possible reversible transformations for these intermediate models are extremely restricted in comparison to the extremal compositions. This restriction appears due to the *consistency requirement* that demands positivity of outcome probabilities. In fact, the transformations are so limited that even not all the product states (effects) can be converted among themselves under the allowed reversible transformations, and thus makes these intermediate compositions less interesting. We conjecture that the *no-go* statement of Theorem 2 holds true for any such intermediate composite models. Support to this conjecture comes from the study, made in [32], where it has been shown that the intermediate compositions for the square-bit model (*i.e.,* Ω_4) cannot perfectly compute the equality problem, *i.e.*, the computation

Figure 2. (Color on-line) The class T denotes the trivial computations (see Definition 1). QD (in between red and blue curves) are the nontrivial computations that can be perfectly done in quantum theory (Theorem 1 & Corollary 2) and they establish quantum advantage over classical as well as exotic GPTs (Theorem 2). H (in between blue and black curves) represents the class of hard computations that cannot be done perfectly even in quantum theory [e.g. $(\mathbb{F} \equiv \oplus, \mathbf{f} \equiv \vee)$]. There might exist a finer class QA in between QD and H (dashed green curve) where quantum theory provides probabilistic advantage. Example of such a computation is yet to be identified.

$(\mathbb{F} \equiv \vee, \mathbf{f} \equiv \oplus) \in \mathrm{DCLC}(2)$.

DISCUSSION

Apart from its immense practical importance in present day distant data manipulation, our work is a potential operational task to single out the bipartite composite structure of quantum theory. In spite of its remarkable efficiencies in the domain of computation and information, it still remains illusive from a foundational point of view: *why the nature is quantum*? There is no general consensus why our physical world should be modeled by Hilbert space quantum mechanics, which, from a mathematical standpoint, is just an example of model among several other possibilities [1, 2]. During last two decades some novel approaches has been developed to find the seemingly impossible consequences of post-quantum models and thus reject them to be the possible theory of the physical world [38–48]. In this direction, our study yields non-trivial example of distributive computations, where quantum theory performs better than every possible two-dimensional polygon theories, while all these theories are very close to the $\mathbb{C}^2$ Hilbert space quantum description both from geometric and informational perspectives [31]. In this respect the works in Refs. [49–51] are worth mentioning. There it has been established that some 3-party quantum correlations cannot be produced by Popescu-Rohrlich (PR) correlation [10] while quantum theory can produce the correlation and hence outperforms no-signaling theories allowing stronger than quantum nonlocal correlation. The computational limitation of the GPTs established in those work stem from the impossibility of entanglement swapping for generalized nonlocal correlations [52]. The present work has stronger implications as it shows that some 2-party correlation outside the quantum realm is not as good as 2-party quantum correlation without invoking any 3-party correlation. Furthermore, those works only address correlations in space-like scenario whereas the present work considers correlation in space-like as well as in time-like scenario.

It is worth mentioning that the task of *Quantum fingerprinting* [27] is a special case $(\mathbb{F} \equiv \vee, \mathbf{f} \equiv \oplus)$ of our DCLC paradigm, where C is asked to calculate the function $e(\mathbf{x}, \mathbf{y}) := 1$ (if $\mathbf{x} = \mathbf{y}$) and $e(\mathbf{x}, \mathbf{y}) := 0$ (if $\mathbf{x} \neq \mathbf{y}$) using the minimum communication from Alice and Bob who are given two random n bit strings $\mathbf{x}$ and $\mathbf{y}$, respectively. Although an exponetial gap between classical and quantum resources have been established there on an average, our result concerns with the single-shot version (Corollary 2 in the *Methods* section).

Our result is complete in the framework of DCLC(2) upto the trivial ones and those with perfect quantum accomplishment. However, the class of computations, for which there are probabilistic quantum advantages over the polygonal GPTs, is still unknown (see Fig 2). It is also instructive to completely characterize these classes for DCLC(n) settings. Lastly, the distributed computing scenario with higher numbers (≥ 3) of input ports can be a potential candidate for further research.

METHODS

We will start this technical section by a complete characterization of the general trivial DCLC(2) tasks.

Proposition 1. *A dual layer computation* $(\mathbb{F}, \mathbf{f}) \in DCLC(2)$ *is trivial if and only if any one of the following criteria is satisfied:*
(i) *at-least one of the two functions is a constant function;*
(ii) *at-least one of them is a single bit function;*
(iii) $\mathbb{F}$ *is symmetric on inputs and* $\mathbf{f}$ *can be realized through* $\mathbb{F}$ *[and with single-bit NOT operation], i.e.* $\mathbf{f}(a_1, a_2) = \mathbb{F}(a_1, a_2)$ *[*$\mathbf{f}(a_1, a_2) = \mathbb{F}(\bar{a}_1, \bar{a}_2)$*].*

A function $\mathbb{G} : \{0,1\}^n \to \{0,1\}$ will be called a single bit function if $\forall \mathbf{a} \in \{0,1\}^n$ the functional value $\mathbb{G}(\mathbf{a})$ only depends on a single bit a_i for some fixed $i \in \{1, \cdots, n\}$. Such a function will be called symmetric if it is of the form either $\mathbb{G}(\mathbf{a}) = a_1 \star a_2 \star \cdots \star a_n$ or $\mathbb{G}(\mathbf{a}) = \bar{a}_1 \star \bar{a}_2 \star \cdots \star \bar{a}_n$ for some binary operation $\star$. Otherwise, it is called non-symmetric. The detailed proof of the proposition is provided in Appendix **??**. In this case, out of 256 computations 176 turns out to be trivial and the rests are nontrivial. Furthermore, among the trivial computations 60 can be accomplished even without any communication from Alice & Bob to Charlie, 56 require communication from only one of Alice and Bob to Charlie, and the rests require communications from both. The proof technique used for Proposition 1 leads us to the following generalized result.

Corollary 1. *A dual layer computation* $(\mathbb{F}, \mathbf{f}) \in DCLC(n)$*, for arbitrary* n (≥ 2)*, is trivial if any one of the criteria in Proposition 1 is satisfied.*

Unlike Proposition 1, Corollary 1 characterizes only a class of trivial computations for $n > 2$. Our next statement specifies the basic resource requirement for a nontrivial DCLC task.

Proposition 2. *Any nontrivial computation* $(\mathbb{F}, \mathbf{f}) \in DCLC(n)$ *in a GPT necessitates presence of entanglement in bipartite state and/or effect spaces of that theory.*

Proof. Recall that a DCLC is trivial (nontrivial) if it can (cannot) be perfectly accomplished by some (any) classical strategy. In the language of GPT, the state and effect spaces of a d-level classical system is specified by a $(d-1)$-simplex. A bipartite system, composed of two such classical systems, is described uniquely by the minimal tensor product. In other words, the composite system has unique state space, as in this case we have, $\Omega^A \otimes_{\min} \Omega^B = \Omega^A \otimes_{\max} \Omega^B$ [8]; hence, the composite system allows no entanglement neither in states nor in effects. Barker's conjecture [54] concerns with the converse question, *i.e.*, for what kind of convex sets the tensor product is unique. Recently, Aubrun *et al.* provide an affirmative proof to the Barker's conjecture that the minimal and maximal tensor products of two finite-dimensional proper cones coincide *if and only if* one of the two cones is generated by a linearly independent set, *i.e.*, one of the state spaces is classical [55]. The only if part of this result assures the present Proposition. ∎

Notably, the above argument is true upto the assumption of *no restriction hypothesis* [6, 56], which states that with a particular choice of state space all possible effects which gives positive probability measure on this set should be physically realizable.

Now we will give a proof for *if* part of our main theorem (Theorem 1), while the *only if* part is presented in Appendix C.

Theorem 1. (*if* part)

Note that there are ${}^4C_2$ balanced Boolean functions $\{0,1\}^2 \mapsto \{0,1\}$; out of which 4 are single bit function and hence trivial (Proposition 1). The remaining two functions are XOR and X-NOR. We first discuss the protocol for the case $(\mathbb{F} \equiv \vee, \mathbf{f} \equiv \oplus)$. Alice and Bob start the protocol with the 2-qubit maximally entangled state $|\phi^+\rangle_{AB} := \frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB})$. Depending on the inputs $\mathbf{x}$ and $\mathbf{y}$, they apply local unitary operation σ_i^A and σ_j^B on their respective part of the entangled state, where $i := 2x_1 + x_2$ & $j := 2y_1 + y_2$, and $\sigma_0 = \mathbb{I}$, $\sigma_1 = \sigma_X$, $\sigma_2 = \sigma_X\sigma_Z$ and $\sigma_3 = \sigma_Z$. Whenever $\mathbf{x} = \mathbf{y}$, Charlie receives the state $|\phi^+\rangle$ else a state $\perp$ to $|\phi^+\rangle$. The required computation can be exactly done by performing the 2-outcome measurement, $\mathrm{M}_{[2]} \equiv \{\mathrm{P}_{\phi^+}, \mathbb{I}_4 - \mathrm{P}_{\phi^+}\}$ and declaring the outcome as $\mathrm{P}_{\phi^+} \to 0$ and $\neg\, \mathrm{P}_{\phi^+} \to 1$. Other nontrivial computations follow a similar protocol with suitable relabeling of encoding and decoding (see Table II).

$\mathbf{f} \equiv$ **XNOR**	$\mathbf{f} \equiv$ **XOR**	σ_j^B	**Outcome**
$\mathbb{F}(z_1,z_2) \equiv (\overline{z_1 \wedge z_2})$	$\mathbb{F}(z_1,z_2) \equiv z_1 \vee z_2$	$j = 2y_1 + y_2$	$\mathrm{P}_{\phi^+} \to 0$
$\mathbb{F}(z_1,z_2) \equiv z_1 \wedge z_2$	$\mathbb{F}(z_1,z_2) \equiv (\overline{z_1 \vee z_2})$	$j = 2y_1 + y_2$	$\mathrm{P}_{\phi^+} \to 1$
$\mathbb{F}(z_1,z_2) \equiv (\overline{z_1 \vee z_2})$	$\mathbb{F}(z_1,z_2) \equiv z_1 \wedge z_2$	$j = 2\bar{y}_1 + \bar{y}_2$	$\mathrm{P}_{\phi^+} \to 1$
$\mathbb{F}(z_1,z_2) \equiv z_1 \vee z_2$	$\mathbb{F}(z_1,z_2) \equiv (\overline{z_1 \wedge z_2})$	$j = 2\bar{y}_1 + \bar{y}_2$	$\mathrm{P}_{\phi^+} \to 0$
$\mathbb{F}(z_1,z_2) \equiv \bar{z}_1 \wedge z_2$	$\mathbb{F}(z_1,z_2) \equiv z_1 \wedge \bar{z}_2$	$j = 2\bar{y}_1 + y_2$	$\mathrm{P}_{\phi^+} \to 1$
$\mathbb{F}(z_1,z_2) \equiv z_1 \vee \bar{z}_2$	$\mathbb{F}(z_1,z_2) \equiv \bar{z}_1 \vee z_2$	$j = 2\bar{y}_1 + y_2$	$\mathrm{P}_{\phi^+} \to 0$
$\mathbb{F}(z_1,z_2) \equiv z_1 \wedge \bar{z}_2$	$\mathbb{F}(z_1,z_2) \equiv \bar{z}_1 \wedge z_2$	$j = 2y_1 + \bar{y}_2$	$\mathrm{P}_{\phi^+} \to 1$
$\mathbb{F}(z_1,z_2) \equiv \bar{z}_1 \vee z_2$	$\mathbb{F}(z_1,z_2) \equiv z_1 \vee \bar{z}_2$	$j = 2y_1 + \bar{y}_2$	$\mathrm{P}_{\phi^+} \to 0$

Table I. Quantum strategies for nontrivial DCLC(2) tasks where $\mathbf{f}$ is balanced. Alice's encoding operations $\{\sigma_j^A\}$ are same as used in $(\mathbb{F} \equiv \vee, \mathbf{f} \equiv \oplus)$.

Corollary 2. *All the computations of Theorem 1 are also computable in quantum theory if we consider their delayed-choice version, i.e., DC^{2}LC(2).*

The protocol is discussed in detail in Appendix C. In this case, Alice and Bob follow the same encoding procedure(s) as of Theorem 1. However, this time, Charlie performs a 4-outcome measurement $\mathrm{M}_{[4]} \equiv \{\mathrm{P}_{\phi^+}, \mathrm{P}_{\phi^-}, \mathrm{P}_{\psi^+}, \mathrm{P}_{\psi^-}\}$, where $|\phi^\pm\rangle := (|00\rangle \pm |11\rangle)/\sqrt{2}$ and $|\psi^\pm\rangle := (|01\rangle \pm |10\rangle)/\sqrt{2}$.

Corollary 3. *Any nontrivial dual layer computation* $(\mathbb{F}, \mathbf{f}) \in DCLC(n)$ *is perfectly computable, along with their delayed-choice version, i.e., $DC^2LC(n)$ in quantum theory whenever* $\mathbf{f}$ *is a balanced function.*

The proof is presented in Appendix C.

Unlike Theorem 1, Corollary 3 characterizes only a class of nontrivial DCLC(n) tasks for $n > 2$. Identifying all the quantum computable nontrivial DCLC(n) we leave here as an open question. Importantly, only $\lceil \frac{n}{2} \rceil$-qubit communication, from each of Alice and Bob to Charlie, suffices to execute all the computations in Corollary 3. This amounts to (nearly) half of the maximum allowed communication, i.e., $(n-1)$-qubit communication from each. It is of further interest to explore whether a lesser amount of quantum communication suffices the purpose or not. Our next result deals with nontrivial DC^{2}LC in quantum theory.

Theorem 2. (Outline of the proof)

The complete proof of Theorem 2 can be found in Appendix E and here we will discuss only an outline of it.

To start with, it is important to observe the following fact regarding the nontrivial DCLC(2) tasks.

Observation 1. *Consider a set* $\mathcal{G} := \{\mathbf{x}, \mathbf{x}', \mathbf{y}, \mathbf{y}'\}$, *where* $\mathbf{x} \neq \mathbf{x}'$ *are the inputs at* A *while* $\mathbf{y} \neq \mathbf{y}'$ *at* B *for a* $DCLC(2)$ *task. Altogether,* $\binom{4}{2} \times \binom{4}{2} = 36$ *different such sets are possible. Evidently, the strings in* $\mathcal{G}$ *will be mapped into the bit values 0 and 1 respectively in the ratio* $4:0$, $2:2$, $0:4$, $1:3$ *and* $3:1$. *It turns out that at-least one* $\mathcal{G}$, *among the* 36 *possibilities, must have the aforesaid ratio either* $1:3$ *or* $3:1$ *for every nontrivial* $DCLC(2)$ *task.*

It is then shown explicitly in Appendix E that none of the extreme bipartite compositions for *even*, as well as *odd*-gon theories can satisfy this requirement, which prohibits them to execute any nontrivial DCLC(2).

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AUTHOR CONTRIBUTIONS

All the authors conceived the idea, derived the technical results, discussed all stages of the project, and prepared the manuscript and figures.

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The authors declare no competing interests.

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Appendix A: Proof of Proposition 1

(i) If $\mathbb{F}$ is a constant function, Charlie can yield the required output which is independent of the bit strings received from the transmitters. On the other hand, when **f**

is constant, a single input pair (z_1, z_2), with $z_1 = z_2$, will be fed into the computer C, effectively. In both of these cases, the dual layer computation $(\mathbb{F}, \mathbf{f})$ can be done perfectly even without any communication from the transmitters to the computer.

(ii) If $\mathbb{F}$ is a single bit function, Charlie only requires information about one of z_1 and z_2 to execute the dual layer computation. The transmitters will accordingly send the corresponding bit of their strings. If $\mathbf{f}$ is a single bit function, the DCLC $(\mathbb{F}, \mathbf{f})$ will effectively depend on one of the input strings - $\mathbf{x}$ or $\mathbf{y}$. Now, Alice or Bob will perform the required computation accordingly and communicate the 1-bit output to Charlie.

(iii) Let $\mathbf{f}$ (denoted by $\star$) be realized by $\mathbb{F}$ (denoted by $\circ$) and single bit NOT operation. Since, we consider $\mathbb{F}$ as symmetric, therefore $\mathbf{f}(\alpha, \beta) = \alpha \star \beta = \bar{\alpha} \circ \bar{\beta} = \mathbb{F}(\bar{\alpha}, \bar{\beta})$, where $\alpha, \beta \in \{0, 1\}$. In the dual-layer computation, we have,

$$\begin{aligned}
\mathbb{F}(z_1, z_2) &= z_1 \circ z_2 = \mathbf{f}(x_1, y_1) \circ \mathbf{f}(x_2, y_2) \\
&= (x_1 \star y_1) \circ (x_2 \star y_2) = (\bar{x}_1 \circ \bar{y}_1) \circ (\bar{x}_2 \circ \bar{y}_2) \\
&= (\bar{x}_1 \circ \bar{x}_2) \circ (\bar{y}_1 \circ \bar{y}_2).
\end{aligned}$$

Alice and Bob thus compute a single bit data from their respective inputs and send it to Charlie. Same holds true if $\mathbf{f}(\alpha, \beta) = \mathbb{F}(\alpha, \beta)$.

Note that **(i)** are the trivial computations where no communication is required. In **(ii)**, if $\mathbf{f}$ is a single bit function, we need to use the communication channel partially (from a single transmitter to the computer C). For the computations in **(iii)** and the remaining possibilities in **(ii)** (*i.e.*, where $\mathbb{F}$ is a single bit function), we need 1-bit communication from both the transmitters to the computer. All other remaining dual layer computations are nontrivial.

Now, we will prove the *only if* part using the contradiction, i.e., by assuming that there exist a pair of dual-layer computation $(\mathbb{F}^*, f^*)$ trivially computable with classical strategy, however, does not satisfy any of the conditions (i)-(iii). Note that, for local encoding on a pair of correlated classical bit can not do any better over the uncorrelated one. Therefore, without loss of generality, we can assume that both Alice and Bob can encode their individual bit values in two independent classical bits. Also Alice and Bob being restricted to communicate with each other, they can send $g_1(x_1, x_2)$ and $g_2(y_1, y_2)$ to Charlie, where g_1 and g_2 can be any arbitrary Boolean functions on their respective input bits. On the other hand, after getting these two bits Charlie can compute another Boolean operation $\mathbb{H}$ upon these and hence obtain,

$$\mathbb{H}(g_1(x_1, x_2), g_2(y_1, y_2)) = \mathbb{F}^*(f^*(x_1, y_1), f^*(x_2, y_2)), \ \forall x_1, x_2, y_1, y_2 \tag{A1}$$

Note that, being neither a constant nor a single-bit function, only possibilities left for f^* and $\mathbb{F}^*$ are $\{\vee, \wedge, \oplus\}$ (upto local NOT operations), which we will consider here case by case.

Let us first consider $f^*(x_i, y_i) = x_i \vee y_i$, also due to the assumption that condition (iii) does not hold, $\mathbb{F}^*(z_1, z_2)$ can be anything except $z_1 \vee z_2$ and $\bar{z}_1 \wedge \bar{z}_2$. If $\mathbb{F}^*(z_1, z_2) = \bar{z}_1 \vee z_2$, then

$\mathbb{F}^*(f^*(x_1, y_1), f^*(x_2, y_2)) = (\overline{x_1 \vee y_1}) \vee (x_2 \vee y_2) = (\bar{x}_1 \wedge \bar{y}_1) \vee (x_2 \vee y_2)$,

which fails to attain the form of Eq.(A1), following the distributive properties of binary operations. Also the similar arguments runs for other representations of f^* with corresponding $\mathbb{F}^*$, which don't satisfy the condition (iii) in the Proposition.

Remark 1. *Any dual layer function of DCLC(2) with balanced final output will be a trivial computation.*

Proof. Suppose that the function f is neither constant, nor balanced. Then for $f(x,y)$ the 'Output bit value 0: Output bit value 1'= 1 : 3 (or the reverse). Consequently, for such a f the values of z_1z_2 will be $\{00,\ 01,\ 10,\ 11\}$ in a ratio $1:3:3:9$, identifying $z_i = f(x_i, y_i)$. Now, observe that no $\mathbb{F}$ can be defined on the arguments $\{z_1, z_2\}$, which will produce the final output 0 and 1 in 1 : 1 ratio. Therefore, for final output to be 1 : 1, either f is constant or, balanced. If f is constant, then from Proposition. 1 the computation $(\mathbb{F}, f)$ is trivial. On the other hand, if f is balanced then the pair $\{z_1, z_2\}$ takes the values 00 : 01 : 10 : 11 uniformly. Then the function $\mathbb{F}$ should also be either a single-bit (which is again trivial from Proposition. 1), or balanced function to generate the final output in 1 : 1 ratio. Hence, if both of the function $\mathbb{F}$ and f are balanced but not single bit, then they can be XOR or, XNOR and in both cases they are trivial according to condition (iii) of Proposition. 1. This finally helps us to conclude that the final output of the pair $(\mathbb{F}, f)$ is 1 : 1 only if the computation is trivial. ■

Appendix B: Generalized probabilistic theory

Origin of this framework dates back to 1960's [1–3], and it has gained renewed interest in the recent past [4–6]. A GPT is specified by a list of system types and the composition rules specifying combination of several systems. A system state ω specifies outcome probabilities for all measurements that can be performed on it. For a given system, the set of possible normalized states forms a compact and convex set Ω embedded in a positive convex cone V_+ of some real vector space V. Convexity of Ω assures that any statistical mixture of states is a valid state. The extremal points of Ω, that do not allow any convex decomposition in terms of other states, are called pure states or states of maximal knowledge. An effect e is a linear functional on Ω that maps each state onto a probability, *i.e.*, $e : \Omega \mapsto [0,1]$ by a pre-defined rule $p(e|\omega) = Tr(e^T.\omega)$. The set of effects $\mathcal{E}$ is embedded in the positive dual cone $(V^\star)_+$. The normalization of Ω is determined by u which is defined as the unit effect and a specified element of $(V^\star)_+$, such that, $p(u|\omega_i) = Tr(u^T.\omega) = 1, \forall \omega \in \Omega$. A d-outcome measurement is specified by a collection of d effects, $M \equiv \{e_j \mid \sum_{j=1}^{d} e_j = u\}$, such that, $\sum_{j=1}^{d} p(e_j|\omega) = 1$, for all valid states ω. Another much needed component to complete the mathematical structure for GPT is the reversible transformation $\mathcal{T}$ which maps states to states, *i.e.*, $\mathcal{T}(\Omega) = \Omega$. They are linear in order to preserve the statistical mixtures, and they cannot increase the total probability. In a GPT one can introduce the idea of distinguishable states from an operational perspective which consequently leads to the concept of *Operational dimension*.

Definition 2. *Operational dimension of a system (S) is the largest cardinality of the subset of states, $\{\omega_i\}_{i=1}^{n} \subset \Omega$, that can be perfectly distinguished by a single measurement, i.e., there exists a measurement, $M \equiv \{e_j \mid \sum_{j=1}^{n} e_j = u\}$, such that, $p(e_j|\omega_i) = \delta_{ij}$.*

Importantly, operational dimension is different from the dimension of the vector space V in which the state space Ω is embedded. For instance, for qubit the state space, the set of density operators $\mathcal{D}(\mathbb{C}^2)$ acting on $\mathbb{C}^2$ is embedded in $\mathbb{R}^3$. However, the operational dimension of this system is 2, as at most two qubit state can be perfectly distinguished by a single measurement – *e.g.* $\{|0\rangle, |1\rangle\}$ states can be perfectly distinguished by Pauli measurement along z-direction.

Composite systems of a GPT must be constructed in accordance with no signaling (NS) principle that prohibits instantaneous communication between two distant locations. This, along with the assumption of *tomographic locality* [7], assures that the composite state space lies in between two extremes - the maximal and the minimal tensor product [8].

Definition 3. *The maximal tensor product, $\Omega^A \otimes_{\max} \Omega^B$, is the set of all bi-linear functionals, $\phi : (V^A)^\star \otimes (V^B)^\star \mapsto \mathbb{R}$, such that, (i) $\phi(e^A, e^B) \geq 0$, for all $e^A \in \mathcal{E}^A$ and $e^B \in \mathcal{E}^B$, and (ii) $\phi(u^A, u^B) = 1$, where u^A and u^B are unit effects for system A and B respectively.*

Definition 4. *The minimal tensor product, $\Omega^A \otimes_{\min} \Omega^B$, is the convex hull of the product states $\omega^{A\otimes B}(= \omega^A \otimes \omega^B)$.*

States belonging in $\Omega^A \otimes_{\min} \Omega^B$ are called separable; otherwise, they are entangled. One can define effect spaces for the composite systems in a similar manner. The quantum mechanical tensor product is neither the minimal one nor the maximal; it lies strictly in between.

Appendix C: Proof of Theorem 1

–:only if part:–

First we prove the following lemma.

Lemma 1. *Any nontrivial dual layer computation $(\mathbb{F}, \mathbf{f}) \in DCLC(2)$ maps the set of input stings $\{\mathbf{x}, \mathbf{y}\}$ into the binary bit values in $1:3$ ratio if and only if $\mathbf{f}$ is balanced.*

Proof. A $\{0,1\}^2 \to \{0,1\}$ Boolean function produces a binary output in either of the three possible ratios – $4:0$ (constant), $1:1$ (balanced) and $1:3$ (unbalanced). If one of the functions between $\mathbb{F}$ and $\mathbf{f}$ is constant or single-bit or both of them are balanced (XOR or, XNOR), then the dual layer computation ($\mathbb{F}$, $\mathbf{f}$) is trivial (Proposition 1). However, if $\mathbf{f}$ is balanced but not a single-bit function along with an unbalanced $\mathbb{F}$, the outputs of ($\mathbb{F}$, $\mathbf{f}$) are in $1:3$ and the dual layer computation is nontrivial establishing the *if* part.

If $\mathbf{f}$ is unbalanced and $\mathbb{F}$ is balanced, the output will be either $1:3$ or $3:5$. Alternatively, it will be among $1:15$, $3:13$ and $7:9$ for both of $\mathbf{f}$ and $\mathbb{F}$ being unbalanced. Now, consider an unbalanced $\mathbf{f}$ where the 'output bit value 0 : output bit value $1 = 1:3$'. In this case, bit values of $z_1 z_2$ follow the ratio $00:01:10:11 = 1:3:3:9$. If $\mathbb{F}$ is a balanced function, such that, $\{00, 01\} \to 0/1$ or $\{00, 10\} \to 0/1$, output of ($\mathbb{F}$, $\mathbf{f}$) is in $1:3$ ratio. But in this case, $\mathbb{F}$ being a single-bit function makes ($\mathbb{F}$, $\mathbf{f}$) trivial (Proposition 1). Therefore, no other nontrivial $(\mathbb{F}, \mathbf{f})$ can be in $1:3$ ratio except $\mathbf{f}$ being a balanced function. ∎

Lemma 1 assures that to prove the *only if* part of Theorem 1, it is sufficient to prove that no quantum strategy (entangled states along with local unitaries and two outcome measurements) can produce two disjoint subspaces containing states other than 1 :

3 or 1 : 1. Consider that Alice and Bob start their protocol with a pure entangled state $|\psi\rangle = a|00\rangle + b|11\rangle$, where, $\{a,b\} \in \mathbb{R}$, s.t., $a^2+b^2=1$ without loss of generality . They have some unitary encoding strategies, $\{U_i^A\}_{i=0}^3$ and $\{U_j^B\}_{j=0}^3$, respectively. Now, according to the Observation 1 (see the *Methods* section), for every nontrivial DCLC(2) there exists at least a group of four input stings $\mathcal{G} := \{\mathbf{x},\mathbf{x}',\mathbf{y},\mathbf{y}'\}$ ($\mathbf{x} \neq \mathbf{x}'$ & $\mathbf{y} \neq \mathbf{y}'$) that follows the ratio 1 : 3. Considering Alice's and Bob's encoding as U_0^A, U_1^A and U_0^B, U_1^B, the resulting encoded states read

$$\begin{aligned}
\mathbf{x},\mathbf{y} &\longmapsto |\xi_1\rangle = a|\psi_0\phi_0\rangle + b|\psi_0^\perp\phi_0^\perp\rangle\,,\\
\mathbf{x}',\mathbf{y} &\longmapsto |\xi_2\rangle = a|\psi_1\phi_0\rangle + b|\psi_1^\perp\phi_0^\perp\rangle\,,\\
\mathbf{x},\mathbf{y}' &\longmapsto |\xi_3\rangle = a|\psi_0\phi_1\rangle + b|\psi_0^\perp\phi_1^\perp\rangle\,,\\
\mathbf{x}',\mathbf{y}' &\longmapsto |\xi_4\rangle = a|\psi_1\phi_1\rangle + b|\psi_1^\perp\phi_1^\perp\rangle\,,
\end{aligned}$$

where, $|\psi_i\rangle = U_i^A|0\rangle$ & $|\phi_i\rangle = U_i^B|0\rangle$, for $i \in \{0,1\}$. The orthogonality conditions $\langle\xi_1|\xi_j\rangle = 0$, $\forall\, j \in \{2,3,4\}$, imply $|\psi_1\rangle = |\psi_0^\perp\rangle, |\psi_1^\perp\rangle = \pm|\psi_0\rangle$ and $|\phi_1\rangle = |\phi_0^\perp\rangle, |\phi_1^\perp\rangle = \mp|\phi_0\rangle$. In other worlds, both U_0^A & U_1^A (U_0^B & U_1^B) map the states $\{|0\rangle, |1\rangle\}$ into same orthogonal pairs $\{|\psi_0\rangle, |\psi_0^\perp\rangle\}$ ($\{|\phi_0\rangle, |\phi_0^\perp\rangle\}$). For decoding, Charlie performs the measurement, $\mathrm{M}_{[2]}^{\xi_1} \equiv \{\mathrm{P}_{\xi_1}, \mathbb{I}_4 - \mathrm{P}_{\xi_1}\}$ and assigns the outcome as $\mathrm{P}_{\xi_1} \to 0$ and $\neg\, \mathrm{P}_{\xi_1} \to 1$. For every input string $\mathbf{x}$, either ($\mathbf{x}$, $\mathbf{y}$) yields outcome 0, whereas ($\mathbf{x}$, $\mathbf{y}'$) yields 1 (where $\mathbf{y} \neq \mathbf{y}'$ belong to Bob), or $\mathbf{x}$ forms a group like $\mathcal{G}$. In both these cases, all the Unitaries $\{U_i^A\}_{i=0}^3$ ($\{U_j^B\}_{j=0}^3$) map the states $\{|0\rangle, |1\rangle\}$ into same orthogonal pairs $\{|\psi_0\rangle, |\psi_0^\perp\rangle\}$ ($\{|\phi_0\rangle, |\phi_0^\perp\rangle\}$). Therefore, the encoding by Alice and Bob, without loss of any generality, can be chosen to be the Pauli matrices $\{\sigma_i\}_{i=0}^3$. Let us denote $(\sigma_i^A \otimes \sigma_j^B)|\psi\rangle = |\xi\rangle_{ij}$, where, $\{i,j\} \in \{0,...,3\}$. Note that $|\xi\rangle_{ij} \sim |\xi\rangle_{kl}$ (up-to global phase) if $i+k = j+l = 3$. To compute DCLC (2) perfectly, Charlie performs a measurement that divides the communicated bipartite states in two orthogonal subspace. Evidently, there exists only one such measurement which divides the above states in $\{|00\rangle, |11\rangle\}$ and $\{|01\rangle, |10\rangle\}$ subspace in 1 : 1 ratio, hence achieves a trivial computation (see Remark 1). Therefore, the dual-layer computations for which the final outcome is balanced, *i.e.*, only the trivial ones can be performed with non-maximally pure entangled states. Whenever $a = b = \frac{1}{\sqrt{2}}$, the subspace can also be divided in 1 : 3 ratio and *no other* choice is possible at all.

Remark 2. *Proof of Corollary 2:*

Proof. Let us consider $\mathbf{f}$ as a balanced function in $(\mathbb{F}, \mathbf{f})$, while $\mathbb{F}$ is delayed-choice, i.e., declared later, once after Alice and Bob communicate their respective bit information to Charlie. The encoding protocol, here, is similar to that of Theorem 1. Depending upon the input strings $\mathbf{x}$ & $\mathbf{y}$, Charlie receives the bipartite state as follows:

inputs $\mathbf{x}$, $\mathbf{y}$	$\sigma^A_{2x_1+x_2} \otimes \sigma^B_{2y_1+y_2}\, \lvert\phi^+\rangle_{AB}$
$x_1 = y_1$ & $x_2 = y_2$	$\lvert\phi^+\rangle_{AB}$
$x_1 \neq y_1$ & $x_2 = y_2$	$\lvert\phi^-\rangle_{AB} := \frac{1}{\sqrt{2}}(\lvert 00\rangle_{AB} - \lvert 11\rangle_{AB})$
$x_1 = y_1$ & $x_2 \neq y_2$	$\lvert\psi^+\rangle_{AB} := \frac{1}{\sqrt{2}}(\lvert 01\rangle_{AB} + \lvert 10\rangle_{AB})$
$x_1 \neq y_1$ & $x_2 \neq y_2$	$\lvert\psi^-\rangle_{AB} := \frac{1}{\sqrt{2}}(\lvert 01\rangle_{AB} - \lvert 10\rangle_{AB})$

For decoding, Charlie performs the 4-outcome Bell measurement, $\mathrm{M}_{[4]} \equiv \{\mathrm{P}_{\phi^+}, \mathrm{P}_{\phi^-}, \mathrm{P}_{\psi^+}, \mathrm{P}_{\psi^-}\}$. He, then, calculates $z_i = \mathbf{f}(x_i, y_i)$ for $\mathbf{f} \in \{\mathrm{XOR}, \mathrm{XNOR}\}$ and computes the final outcome $\mathbb{F}(z_1, z_2)$. This suffices to compute all the nontrivial computations as in Theorem. 1 in a delayed-choice manner. ■

Remark 3. *Proof of Corollary 3:*

Proof. Note that, it is clear from Remark-2 that by performing the complete Bell measurement, Charlie is able to obtain the individual $z_i = f(x_i, y_i)$, whenever the function f is XOR or, XNOR. Therefore, following the same DCLC(2) encoding protocol, for each two successive bits of their n-bit string, Alice and Bob will use a maximally entangled state and thus they require $n/2$-ebits for even n. Alternatively, For odd n, each of them requires $(n-1)/2$-ebits for first $(n-1)$-bits and 1 product qubit for their last bit of information. After getting the z_i values Charlie can evidently compute the given $\mathbb{F}$ and this completes the proof for both DCLC(n) and DC^{2}LC(n) whenever f is balanced. ■

Appendix D: Polygon theory

Single system: For an elementary system, the state space Ω_n is a regular polygon with n vertices. For a fixed n, Ω_n is the convex hull of n pure states $\{\omega_i\}_{i=0}^{n-1}$, where,

$$\omega_i = \begin{pmatrix} r_n \cos\frac{2\pi i}{n} \\ r_n \sin\frac{2\pi i}{n} \\ 1 \end{pmatrix}, \text{ with } r_n = \sqrt{\sec(\pi/n)}. \tag{D1}$$

On the other hand, corresponding effect space $\mathcal{E}_n$, collection of all the possible measurement effects, is the convex hull of the null effect O, unit effect u, the extremal effects $\{e_j\}_{j=0}^{n-1}$, and their complementary effects $\{\bar{e}_j\}_{j=0}^{n-1}$, where, $\bar{e}_j := u - e_j$. The null and unit effects are respectively given by $O = (0,0,0)^{\mathrm{T}}$ and $u = (0,0,1)^{\mathrm{T}}$, where, T denotes the matrix transposition. The effects $\{e_j\}_{j=0}^{n-1}$ are given by,

Even-gon	Odd-gon
$e_j = \frac{1}{2}\begin{pmatrix} r_n \cos\frac{(2j-1)\pi}{n} \\ r_n \sin\frac{(2j-1)\pi}{n} \\ 1 \end{pmatrix}$	$e_j = \frac{1}{1+r_n^2}\begin{pmatrix} r_n \cos\frac{2j\pi}{n} \\ r_n \sin\frac{2j\pi}{n} \\ 1 \end{pmatrix}$

For even-gon, it turns out that, $\bar{e}_j := u - e_j = e_{(j \oplus_n \frac{n}{2})}$, where, $\oplus_n$ denotes *addition modulo n*. Therefore, the effects $\{e_j\}_{j=0}^{n-1}$ as well as their complementary effects are not the pure effects only, but they are the ray extremals of the effect cone $(V^\star)_+$ also. In contrast, for odd-gon, only $\{e_j := u - e_j\}_{j=0}^{n-1}$ are the ray extremals, whereas their complementary effects $\{\bar{e}_j := u - e_j\}_{j=0}^{n-1}$ are not despite being pure.

For any n-gon theory, the set of the reversible transformations (RT), $\mathbb{T}_n$, is the dihedral group of order $2n$ containing n reflections and n rotations, *i.e.*,

$$\mathbb{T}_n \equiv \left\{\mathcal{T}_k^p \mid k = 0, \cdots, n-1;\ \&\ p \in \{+,-\}\right\},$$

$$\mathcal{T}_k^p := \begin{pmatrix} \cos\frac{2\pi k}{n} & -p\sin\frac{2\pi k}{n} & 0 \\ \sin\frac{2\pi k}{n} & p\cos\frac{2\pi k}{n} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{D2}$$

with $p = +$ corresponds to the rotation and $p = -$ to the reflection.

Bipartite system: Any bipartite composition of n-gon systems must include n^2 factorized states,

$$\Omega^{product} := \left\{\omega_{ni+j}^{A\otimes B} := \omega_i^A \otimes \omega_j^B \mid i,j \in \{0,\cdots,n-1\}\right\} \subset \Omega_{n^{\otimes 2}} := \Omega_n^{AB}. \tag{D3}$$

We will use the superscript $A \otimes B$ to denote factorizability. For the bipartite system, the product effects are of the form,

$$\mathcal{E}^{product} := \left\{g^A \otimes g^B\right\} \subset \mathcal{E}_n^{AB}, \tag{D4}$$

$$\text{where,} \qquad g^X \in \left\{O^X, u^X\right\} \bigcup \left\{e_i^X,\ \bar{e}_i^X\right\}_{i=0}^{n-1}; \quad X \in \{A,B\}.$$

Since, $p(e^{A\otimes B}|\omega^{A\otimes B}) = p(e^A|\omega^A)p(e^B|\omega^B)$, therefore,

$$0 \le p(e^{A\otimes B}|\omega^{A\otimes B}) \le 1;\ \ \forall\ e^{A\otimes B} \in \mathcal{E}^{product}\ \&\ \forall\ \omega^{A\otimes B} \in \Omega^{product}. \tag{D5}$$

Apart from these factorized states and effects, a bipartite system may also allow non-factorized (entangled) states and effects that we will denote as ω^{AB} and e^{AB} respectively. Of course, they must satisfy the consistency requirements:

$$0 \le p(e^{A\otimes B}|\omega^{AB}) \le 1,\ \forall\ e^{A\otimes B} \in \mathcal{E}^{product}, \tag{D6}$$

$$0 \le p(e^{AB}|\omega^{A\otimes B}) \le 1,\ \forall\ \omega^{A\otimes B} \in \Omega^{product}. \tag{D7}$$

In Ref.[31], the authors have introduced an maximally entangled state for bipartite n-gon theories both for *odd* and *even* n. Applying all possible local RTs $\left\{\mathcal{T}_k^p\right\}$ on Alice's part and $\left\{\mathcal{T}_l^q\right\}$ on Bob's part, we can get all the other entangled states as follows :

<u>Odd n:</u>

$$\begin{aligned} \omega_{kl}^{AB}(p,q) &:= \begin{pmatrix} \cos\left(\frac{2\pi}{n}(k-l)\right) & -\sin\left(\frac{2\pi}{n}(k-l)\right) & 0 \\ \sin\left(\frac{2\pi}{n}(k-l)\right) & \cos\left(\frac{2\pi}{n}(k-l)\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}; \ \text{when} \ \ p = q, \\ \omega_{kl}^{AB}(p,q) &:= \begin{pmatrix} \cos\left(\frac{2\pi}{n}(k+l)\right) & \sin\left(\frac{2\pi}{n}(k+l)\right) & 0 \\ \sin\left(\frac{2\pi}{n}(k+l)\right) & -\cos\left(\frac{2\pi}{n}(k+l)\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}; \ \text{when} \ \ p \neq q. \end{aligned} \tag{D8}$$

<u>Even n:</u>

$$\begin{aligned} \omega_{kl}^{AB}(p,q) &:= \begin{pmatrix} \cos\left(\frac{2\pi}{n}(k-l) - p\frac{\pi}{n}\right) & -\sin\left(\frac{2\pi}{n}(k-l) - p\frac{\pi}{n}\right) & 0 \\ \sin\left(\frac{2\pi}{n}(k-l) - p\frac{\pi}{n}\right) & \cos\left(\frac{2\pi}{n}(k-l) - p\frac{\pi}{n}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}; \ \text{when} \ \ p = q, \\ \omega_{kl}^{AB}(p,q) &:= \begin{pmatrix} \cos\left(\frac{2\pi}{n}(k+l) - p\frac{\pi}{n}\right) & \sin\left(\frac{2\pi}{n}(k+l) - p\frac{\pi}{n}\right) & 0 \\ \sin\left(\frac{2\pi}{n}(k+l) - p\frac{\pi}{n}\right) & -\cos\left(\frac{2\pi}{n}(k+l) - p\frac{\pi}{n}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}; \ \text{when} \ \ p \neq q. \end{aligned} \tag{D9}$$

Similarly, all the possible maximally entangled effects are given by,
Odd n:

$$
\begin{aligned}
e^{AB}_{kl}(p,q) &:= \frac{1}{1+r_n^2}\begin{pmatrix}\cos\left(\frac{2\pi}{n}(k-l)\right) & -\sin\left(\frac{2\pi}{n}(k-l)\right) & 0\\ \sin\left(\frac{2\pi}{n}(k-l)\right) & \cos\left(\frac{2\pi}{n}(k-l)\right) & 0\\ 0 & 0 & 1\end{pmatrix};\ \text{when}\ \ p=q,\\
e^{AB}_{kl}(p,q) &:= \frac{1}{1+r_n^2}\begin{pmatrix}\cos\left(\frac{2\pi}{n}(k+l)\right) & \sin\left(\frac{2\pi}{n}(k+l)\right) & 0\\ \sin\left(\frac{2\pi}{n}(k+l)\right) & -\cos\left(\frac{2\pi}{n}(k+l)\right) & 0\\ 0 & 0 & 1\end{pmatrix};\ \text{when}\ \ p\neq q.
\end{aligned}
\tag{D10}
$$

Even n:

$$
\begin{aligned}
e^{AB}_{kl}(p,q) &:= \frac{1}{2}\begin{pmatrix}\cos\left(\frac{2\pi}{n}(k-l)-p\frac{\pi}{n}\right) & -\sin\left(\frac{2\pi}{n}(k-l)-p\frac{\pi}{n}\right) & 0\\ \sin\left(\frac{2\pi}{n}(k-l)-p\frac{\pi}{n}\right) & \cos\left(\frac{2\pi}{n}(k-l)-p\frac{\pi}{n}\right) & 0\\ 0 & 0 & 1\end{pmatrix};\ \text{when}\ \ p=q,\\
e^{AB}_{kl}(p,q) &:= \frac{1}{2}\begin{pmatrix}\cos\left(\frac{2\pi}{n}(k+l)-p\frac{\pi}{n}\right) & \sin\left(\frac{2\pi}{n}(k+l)-p\frac{\pi}{n}\right) & 0\\ \sin\left(\frac{2\pi}{n}(k+l)-p\frac{\pi}{n}\right) & -\cos\left(\frac{2\pi}{n}(k+l)-p\frac{\pi}{n}\right) & 0\\ 0 & 0 & 1\end{pmatrix};\ \text{when}\ \ p\neq q.
\end{aligned}
\tag{D11}
$$

For an arbitrary n, the set of all possible RTs for bipartite system is given by

$$
\begin{aligned}
\mathbb{T}_{n^{\otimes 2}} := \mathbb{T}^{AB} &\equiv \left\{\mathcal{S}, \mathcal{T}^p_k\otimes\mathcal{T}^q_l\right\},\\
k,l &\in \{0,\cdots,n-1\};\quad p,q\in\{+,-\}.
\end{aligned}
\tag{D12}
$$

$\mathcal{S}$ is the SWAP map whose action is defined as,

$$
\mathcal{S}(\omega^A\otimes\omega^B)=\omega^B\otimes\omega^A;\ \ \forall\ \omega^A\in\Omega^A\ \&\ \omega^B\in\Omega^B. \tag{D13}
$$

Remark 4. *The SWAP map is a global transformation, i.e., it cannot be implemented locally. Alternatively, any local transformation $\mathcal{T}\in\mathbb{T}^{AB}$ never maps a product state (effect) to an entangled one and vice versa. In other words, the set of product states (effects) and the set of entangled states (effects) are two disconnected islands under the reversible transformation $\mathbb{T}^{AB}$.*

Positivity of the predicted probabilities imposes restrictions on the states, effects and transformations that can be allowed together in a composite system. Satisfying this consistency requirement, several composite models are possible. These models can be classified into three main types as discussed below.

Type-I: *Entangled states product effects model* : In this case, all possible product as well as entangled states (listed above) are allowed, *i.e.*, Ω^{AB}_n is the convex hull of the set $\left\{\omega^{A\otimes B}_{ni+j}, \omega^{AB}_{kl}(p,q)\mid i,j,k,l\in\{0,\cdots,n-1\};\ p,q\in\{+,-\}\right\}$, whereas the effects are only product in nature. Due to the presence of entangled states, such model can exhibit Bell *nonlocality* [9]. In fact, such a model can be stronger in *space-like* correlation by revealing more nonlocal behaviour than quantum theory [10].

Type-II: *Product states entangled effects model* : It allows only the product states, *i.e.*, Ω^{AB} is the convex hull of the set $\left\{\omega^{A\otimes B}_{ni+j}\mid i,j\in\{0,\cdots,n-1\}\right\}$. However, it allows all possible product as well as entangled effects. Such a model is *local* by construction. Due to the presence of all possible entangled effects, this model can also exhibit peculiar feature. For instance, the authors have shown in Ref. [11] that such a model can allow *time-like* correlations that are stronger than quantum theory.

Type-III: *Dynamically restricted models* : There can be some models which allow some entangled states along with some (suitably chosen) entangled effects unlike the **Type-I** and **Type-II** models. all the transformations in Eq.(D12) are allowed. Further, due to the consistency requirement (*i.e.*, positivity of the outcome probability), not all the reversible transformations can be allowed when both of teh entangled states and entangled effects are incorporated; hence, it is named as 'dynamically restricted models'. Such restriction prevents even all the pure states (effects) to be mapped to each other under reversible transformation which makes **Type-III** models quite uninteresting.

Appendix E: Proof of Theorem 2

Evidently, operational dimension of any polygonal model $(\Omega_n, \mathcal{E}_n, \mathbb{T}_n)$ is 2. Therefore, both Alice and Bob are allowed to communicate one such system while performing a DCLC(2) task. However, they can consider some composite models allowing entanglement. Now, for encoding, they will apply some reversible transformations on their respective part. Therefore, Remark 4 leads us as follows :

Remark 5. *Encoded states, received by Charlie, are all either entangled or product.*

We will now prove the Theorem 2 for **Type-I** and **Type-II** composite model. For odd-gon and even-gon, the proof will be discussed separately.

Odd-gon theory

Type-I Models : According to Remark 5, the encoded states, received by Charlie, are product states if Alice and Bob start their protocol with a product one. If a non-trivial DCLC(2) task can be accomplished perfectly by such a strategy, it is perfectly computable in classical theory too which is not possible at all. On the other hand, Alice and Bob may start the protocol with an entangled state. Then, Charlie receives all the encoded states as entangled, followed by Remark 5. A straightforward calculation shows – $p(\bar{e}_i^A \otimes \bar{e}_j^B|\omega_{kl}^{AB}(p,q)) \neq 0$, $\forall i,j,k,l \in \{0,...,(n-1)\}$ & $\forall\ \omega_{kl}^{AB}(p,q)$ in (D8) which leads to an unavoidable ambiguity while decoding by Charlie.

Type-II Models : A decoding strategy with product effects ensures the equivalence with a classical strategy for a nontrivial DCLC(2), and hence perfect accomplishment is impossible. Let us then move to entangled decoding strategies, and we consider, without loss of generality, Charlie's decoding measurement, $\mathcal{M}^{AB} \equiv \{e_{00}^{AB}(++), \bar{e}_{00}^{AB}(++)\}$. Suppose, Alice encodes her strings $\mathbf{x}$ and $\mathbf{x}'$ $(\neq \mathbf{x})$ into the states ω_k^A and ω_l^A respectively, whereas the strings $\mathbf{y}$ and $\mathbf{y}'$ $(\neq \mathbf{y})$, in Bob's side, are encoded by ω_s^B and ω_t^B respectively. For a particular encoded state, any of these two effects should get clicked perfectly in case of unambiguous decoding which leads to the following restrictions (see Table II). It can be easily shown that $e_{00}^{AB}(++)$ and $\bar{e}_{00}^{AB}(++)$ will get clicked in $1:1$ ratio for any

Input strings	$e_{00}^{AB}(++)$ clicks	$\bar{e}_{00}^{AB}(++)$ clicks
$\mathbf{x},\mathbf{y}$	$k=s$	$k=s\oplus_n \frac{n\pm1}{2}$
$\mathbf{x}',\mathbf{y}'$	$l=t$	$l=t\oplus_n \frac{n\pm1}{2}$
$\mathbf{x},\mathbf{y}'$	$k=t$	$k=t\oplus_n \frac{n\pm1}{2}$
$\mathbf{x}',\mathbf{y}$	$l=s$	$l=s\oplus_n \frac{n\pm1}{2}$

Table II. Conditions for which either of the entangled effects click sharply.

combination of these conditions. However, for a nontrivial computation, there should be at least a group $\mathcal{G}=\{\mathbf{x},\mathbf{x}',\mathbf{y},\mathbf{y}'\}$ with $\mathbf{x}\neq\mathbf{x}'$ & $\mathbf{y}\neq\mathbf{y}'$ such that the string pairs are mapped into $1:3$ ratio (Observation 1). Hence, no nontrivial computation can be performed perfectly in any Type-II *odd-gon* theory.

Even-gon theory

Type-I Models: The effects, $e_i^A\otimes e_j^B$ and $\bar{e}_i^A\otimes\bar{e}_j^B$, can be clubbed together to get a single effect, $E_{i\otimes j}:=e_i^A\otimes e_j^B+\bar{e}_i^A\otimes\bar{e}_j^B$ since, $p(e_i^A\otimes e_j^B|\omega_{kl}^{AB}(p,q))=p(\bar{e}_i^A\otimes\bar{e}_j^B|\omega_{kl}^{AB}(p,q)),\ \forall\, i,j,k,l,p,q.$ Similarly, we have, $\bar{E}_{i\otimes j}:=e_i^A\otimes\bar{e}_j^B+\bar{e}_i^A\otimes e_j^B$. Clubbing the effects in a different manner will do nothing but increase the ambiguity. Consider that Alice and Bob start the protocol with $\omega_{00}^{AB}(++)$ and Charlie performs the decoding measurement, $\mathcal{M}^{A\otimes B}\equiv\{E_{0\otimes0},\bar{E}_{0\otimes0}\}$ without loss of generality. Encoding of different bit-strings can be accomplished by applying the proper reversible transformations on $\omega_{00}^{AB}(++)$. The probabilities to obtain the effect $E_{0\otimes0}$ on different entangled states are given by,

i) $p\left(E_{0\otimes0}|\omega_{kl}^{AB}(++)\right)=\frac{1}{2}\left[1+r_n^2\cos\left(\frac{\pi}{n}-\frac{2\pi}{n}(k-l)\right)\right]$,

ii) $p\left(E_{0\otimes0}|\omega_{kl}^{AB}(--)\right)=\frac{1}{2}\left[[1+r_n^2\cos\left(\frac{\pi}{n}+\frac{2\pi}{n}(k-l)\right)\right]$,

iii) $p\left(E_{0\otimes0}|\omega_{kl}^{AB}(+-)\right)=\frac{1}{2}\left[[1+r_n^2\cos\left(\frac{\pi}{n}+\frac{2\pi}{n}(k+l)\right)\right]$,

iv) $p\left(E_{0\otimes0}|\omega_{kl}^{AB}(-+)\right)=\frac{1}{2}\left[[1+r_n^2\cos\left(\frac{3\pi}{n}+\frac{2\pi}{n}(k+l)\right)\right]$,

where, $\omega_{kl}^{AB}(p,q)=\left(\mathcal{T}_k^p\otimes\mathcal{T}_l^q\right)\omega_{00}^{AB}(++)$. To avoid the ambiguity, we have to choose the proper reversible transformations maintaining the restrictions listed in Table III. Suppose,

	Restrictions on k and l when the states are			
	$\omega_{kl}^{AB}(++)$	$\omega_{kl}^{AB}(--)$	$\omega_{kl}^{AB}(+-)$	$\omega_{kl}^{AB}(-+)$
$p[E_{0\otimes0}\vert\omega_{kl}^{s_As_B}]=1$	$k=l,\,k=l\oplus_n 1$	$k=l,\,k=l\oplus_n(n-1)$	$k=-l\oplus_n n,\,k=-l\oplus_n(n-1)$	$k=-l\oplus_n(n-1),\,k=-l\oplus_n 2(n-1)$
$p[\bar{E}_{0\otimes0}\vert\omega_{kl}^{s_As_B}]=1$	$k=l\oplus_n\frac{n}{2},k=l\oplus_n(\frac{n}{2}+1)$	$k=l\oplus_n\frac{n}{2},k=l\oplus_n(\frac{n}{2}-1)$	$k=-l\oplus_n\frac{n}{2},k=-l\oplus_n(\frac{n}{2}-1)$	$k=-l\oplus_n(\frac{n}{2}-1),\,k=-l\oplus_n(\frac{n}{2}-2)$

Table III. The allowed integer values of k and l have been depicted for which either of the entangled effects clicks sharply.

Alice applies $\mathcal{T}_k^p$ & $\mathcal{T}_{k'}^{p'}$ when she receives the strings $\mathbf{x}$ & $\mathbf{x}'$ $(\neq\mathbf{x})$ respectively, and Bob applies $\mathcal{T}_l^q$ & $\mathcal{T}_{l'}^{q'}$ for the strings $\mathbf{y}$ & $\mathbf{y}'$ ($\neq \mathbf{y}$) similarly on their shared state $\omega_{00}^{AB}(++)$. Compared to the *odd-gon* theories, there are more possibilities for encoding in *even-gon* cases as the number of entangled states are more. We consider a particular case with $p = +$, $p' = -$, $q = +$, and $q' = -$. With the help of the Table III, we arrive at the following conditions (see Table IV) for unambiguous decoding. These conditions lead

Input strings	Encoded State	Conditions
$\mathbf{x}, \mathbf{y}$	$\omega_{kl}^{AB}(++)$	$k = l$ & $k = l \oplus_n 1$ or $k = l \oplus_n \frac{n}{2}$ & $k = l \oplus_n (\frac{n}{2} + 1)$
$\mathbf{x}', \mathbf{y}'$	$\omega_{k'l'}^{AB}(--)$	$k' = l'$ & $k' = l' \oplus_n (n-1)$ or $k' = l' \oplus_n \frac{n}{2}$ & $k' = l' \oplus_n (\frac{n}{2} - 1)$
$\mathbf{x}, \mathbf{y}'$	$\omega_{kl'}^{AB}(+-)$	$k = -l' \oplus_n n$ & $k = -l' \oplus_n (n-1)$ or $k = -l' \oplus_n \frac{n}{2}$ & $k = -l' \oplus_n (\frac{n}{2} - 1)$
$\mathbf{x}', \mathbf{y}$	$\omega_{k'l}^{AB}(-+)$	$k' = -l \oplus_n (n-1)$ & $k' = -l \oplus_n 2(n-1)$ or $k' = -l \oplus_n (\frac{n}{2} - 1)$ & $k' = -l \oplus_n (\frac{n}{2} - 2)$

Table IV. Conditions for unambiguous decoding.

to the fact that the effects $E_{0\otimes 0}$ and $\bar{E}_{0\otimes 0}$ will get clicked either in $1:1$ or in $1:0$ ratio resulting a trivial computation. In a similar fashion, one can argue the same for any choice of $p, p', q, q' \in \{+, -\}$. Hence, no nontrivial computations can be executed by this type of theories.

Type-II Models: In this case, the arguments go as in the case of **Type-II** *odd-gon* models and it turns out that the decoding effects get clicked in $1:1$ ratio. Hence, all the computations, which can be accomplished, are trivial by this kind of theories. This completes proof of the Theorem 2.

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