Finite-dimensional quantum observables are the special symmetric †-Frobenius algebras of CP maps

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We use *purity*, a principle borrowed from the foundations of quantum information, to show that all special symmetric †-Frobenius algebras in CPM (fHilb)—and, in particular, all classical structures—are *canonical*, i.e. that they arise by doubling of special symmetric †-Frobenius algebras in fHilb.

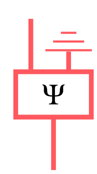
1 Introduction

The exact correspondence [9] between finite-dimensional C*-algebras and special symmetric †-Frobenius algebras (†-SSFAs) in fHilb—the dagger compact category of finite-dimensional complex Hilbert spaces and linear maps—is a cornerstone result in the categorical treatment of quantum theory [1, 4, 7]. A corresponding characterisation in CPM (fHilb)—the dagger compact category of finite-dimensional Hilbert spaces and completely positive maps [8]—has been an open question for around 10 years [5, 6]—of interest, for example, in the investigation of the robustness of sequentialisable quantum protocols [2, 6].

In this work we use *purity*, a principle borrowed from the foundations of quantum information [3], to answer this question once and for all: the †-SSFAs in CPM (fHilb) are exactly the *canonical* ones, the ones arising by doubling of †-SSFAs in fHilb. This is a notable result in that it allows fundamental notions from quantum theory—notably, those of quantum observable and measurement—to be defined directly in the diagrammatic language of CP maps, without reference to fHilb or the CPM construction, and without relying on the biproduct or convex-linear structure of CPM (fHilb).

2 Purity

The very definition of morphisms in the category CPM (fHilb) means that a CP map $\Phi : \mathscr{H} \to \mathscr{K}$ can always be *purified*, i.e. that it can be written in terms of a *pure* map $\Psi : \mathscr{H} \to \mathscr{K} \otimes \mathscr{E}$ and discarding of an "environment" system $\mathscr{E}$:

$\Phi \quad :=$ 

In this work, by a *pure* CP map $\Psi : \mathscr{H} \to \mathscr{K}$ we mean a CP map in the form $\Psi = \mathrm{CPM}(\psi)$, arising by doubling of a morphism $\psi : \mathscr{H} \to \mathscr{K}$ in fHilb. This "diagrammatic" notion of purity is connected to the notion of purity used in the foundations of quantum information by the following *purity principle*.

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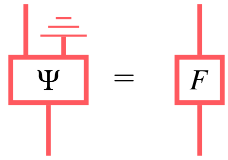
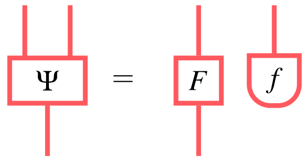
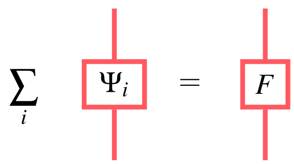
Proposition 1 (Purity Principle)**.**
If the following holds for some pure CP maps $\Psi : \mathscr{H} \to \mathscr{K} \otimes \mathscr{E}$ *and* $F : \mathscr{H} \to \mathscr{K}$*:*



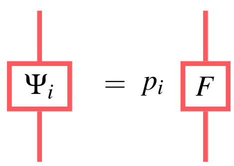
then there is a normalised pure state f *on* $\mathscr{E}$ *such that:*



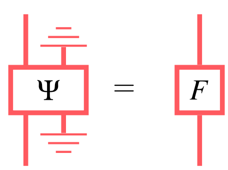
By expanding the discarding map in terms of some orthonormal basis of pure states, one straightforwardly gets an equivalent formulation of the principle in terms of sums.
Proposition 2 (Purity Principle, sums version)**.**
If the following holds for pure CP maps $(\Psi_i : \mathscr{H} \to \mathscr{K})_i$ *and* $F : \mathscr{H} \to \mathscr{K}$*:*



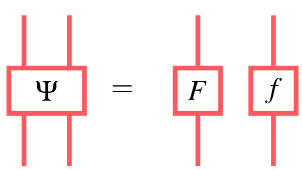
then there are coefficients $p_i \in \mathbb{R}^+$ *summing to* 1 *such that the following holds for all i:*



Because CPM(fHilb) is dagger compact, finally, one can also straightforwardly obtain a more general formulation of the principle which replaces the purifying state f with a purifying operator f.
Proposition 3 (Purity Principle, operator version)**.**
If the following holds for some pure CP maps $\Psi : \mathscr{H} \otimes \mathscr{F} \to \mathscr{K} \otimes \mathscr{E}$ *and* $F : \mathscr{H} \to \mathscr{K}$*:*



then there is a pure CP map $f : \mathscr{F} \to \mathscr{E}$ *such that:*

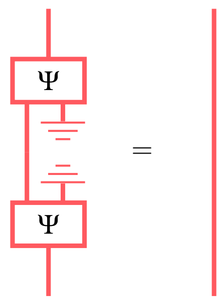


3 Isometries of CP maps

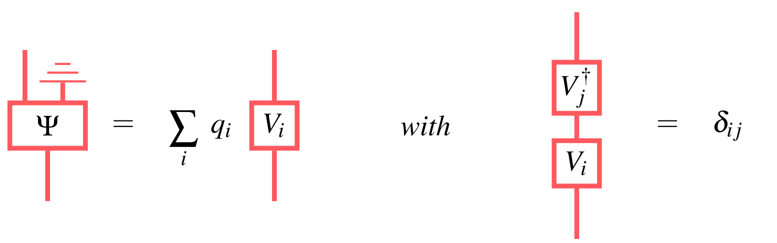
Theorem 4 (Purification of CP isometries)**.**
Every isometry Φ *in* CPM(fHilb) *can be written as a* $\mathbb{R}^+$*-linear combination of pure isometries* V_i *with pairwise orthogonal images.*

$$\Phi^\dagger \circ \Phi = \mathbb{1} \qquad \Rightarrow \qquad \Phi = \sum_i q_i V_i \quad \textit{with} \quad V_i^\dagger \circ V_j = \delta_{ij} \mathbb{1}$$

In pictures, this means that if the following equation holds, where Ψ *is any purification of* Φ*:*

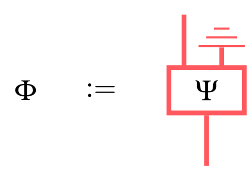


then we must in fact have:

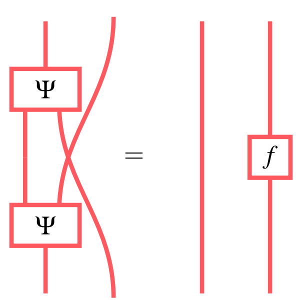


Furthermore, the coefficients satisfy $\sum_i (q_i)^2 = 1$*, and they can be chosen to be all non-zero.*

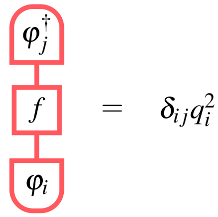
Proof. By the existence of purifications, we can obtain $\Phi : \mathscr{H} \to \mathscr{K}$ by discarding an "environment" system $\mathscr{E}$ from some pure $\Psi : \mathscr{H} \to \mathscr{K} \otimes \mathscr{E}$:



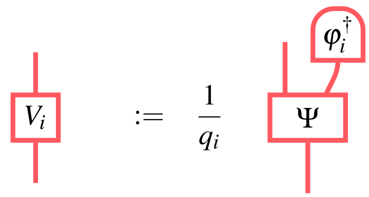
By the operator version of the purity principle, the isometry equation $\Phi^\dagger \circ \Phi = \mathbb{1}$ for Φ implies the following equation for Ψ, where f is some pure CP map $\mathscr{E} \to \mathscr{E}$:



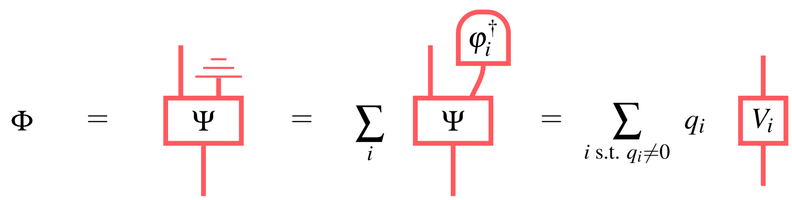
The pure CP map f is self-adjoint, which means that we can find an orthonormal basis of pure states $(\varphi_i)_{i=1}^{\dim \mathscr{E}}$ and associated non-negative real numbers $(q_i)_i$ such that:



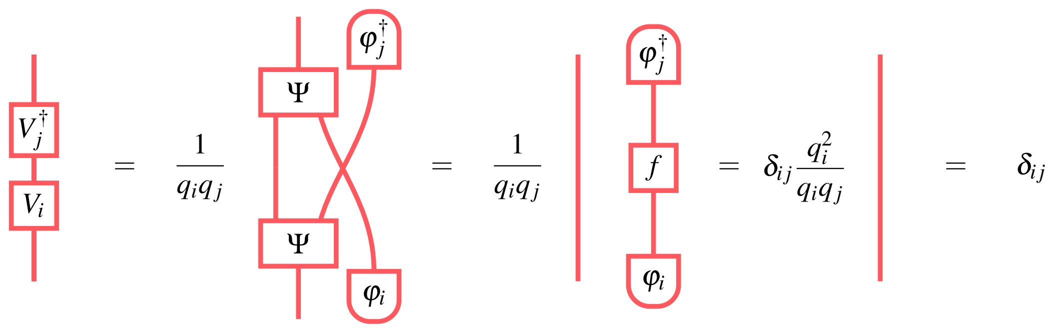
for all $i, j = 1, \ldots, \dim \mathscr{E}$. For each i with $q_i \neq 0$, define:



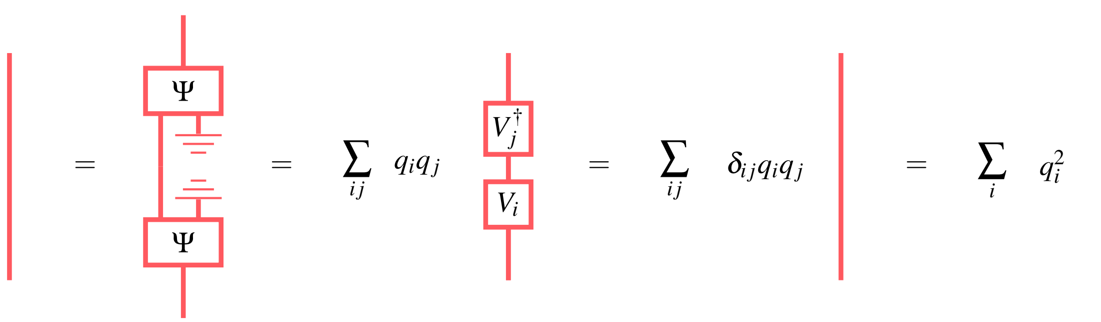
This way we get:



Furthermore, the V_i pure maps we just defined are isometries with orthogonal images:



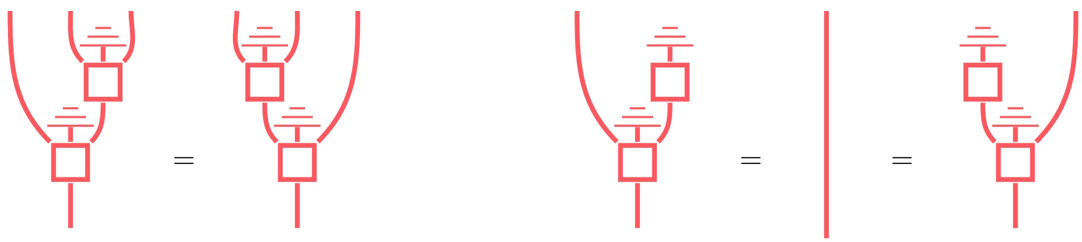
Finally, we have that:



so we conclude that $\sum_i (q_i)^2 = 1$. □

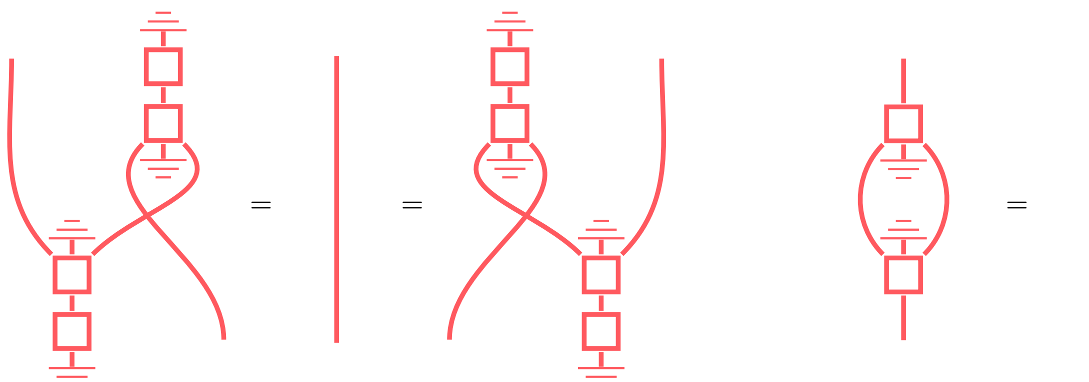
4 Main result

A comonoid (Δ, E) on an object $\mathscr{H}$ of the dagger compact category $\mathrm{CPM}(\mathrm{fHilb})$ is a pair of completely positive maps satisfying the associativity and unit laws:



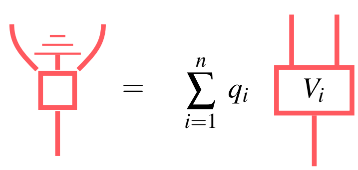
associativity law unit laws

Theorem 5 (Ophidian isometric CP comonoids are pure)**.**
If (Δ, E) *is a comonoid in* $\mathrm{CPM}(\mathrm{fHilb})$ *which is* isometric *and satisfies the* snake equations*, then the CP maps* Δ *and* E *are both pure.*[1]



snake equations *isometry*

Proof. By Theorem 4 and the isometry condition, we know that:

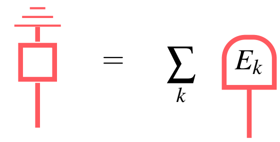


$$= \sum_{i=1}^{n} q_i \; V_i$$

for some pure isometries V_i of fHilb with $V_i^\dagger \circ V_j = \delta_{ij} \mathbb{1}$. The coefficients q_i satisfy $\underline{q} \cdot \underline{q} = \sum_i (q_i)^2 = 1$, and we choose them to be all non-zero. By the existence of purifications, we decompose E as a sum of

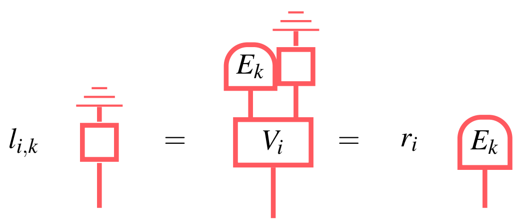
[1] Associativity is not actually necessary for this result to hold.

non-zero pure effects:

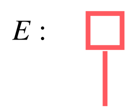


By the purity principle and the unit laws for the comonoid, we deduce that there are coefficients $r_{i,k}, l_{i,k} \in \mathbb{R}^+$ such that:

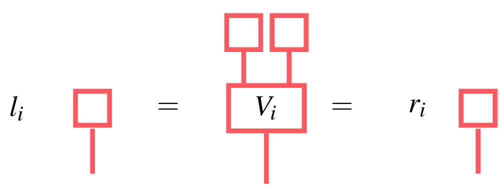
Writing $l_i := \sum_k l_{i,k}$ and $r_i := \sum_k r_{i,k}$, the *unit laws* imply that $\underline{q} \cdot \underline{r} = \sum_i q_i r_i = 1 = \sum_i q_i l_i = \underline{q} \cdot \underline{l}$: it cannot therefore be that for all i,k we have $l_{i,k} = 0$ or that for all i,k we have $r_{i,k} = 0$. Picking some i,k such that $l_{i,k} \neq 0$, we deduce that E is, in fact, a non-zero pure effect:



Because E is a pure effect, we will henceforth drop the discarding map in our notation:

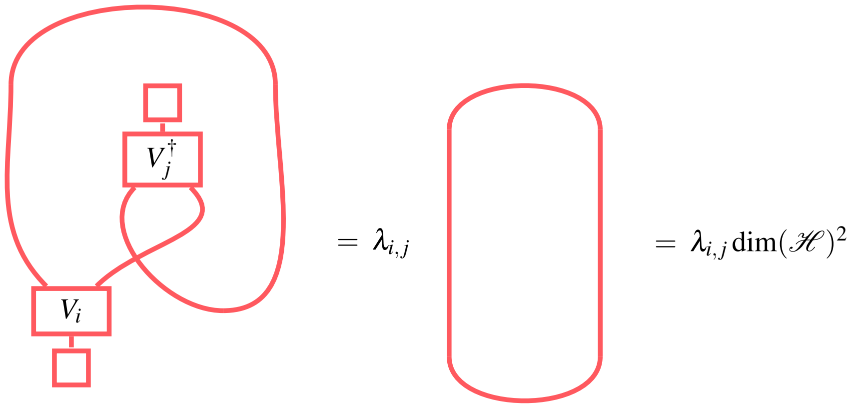


The same trick can then be used to show that $r_i = l_i$ for all i:

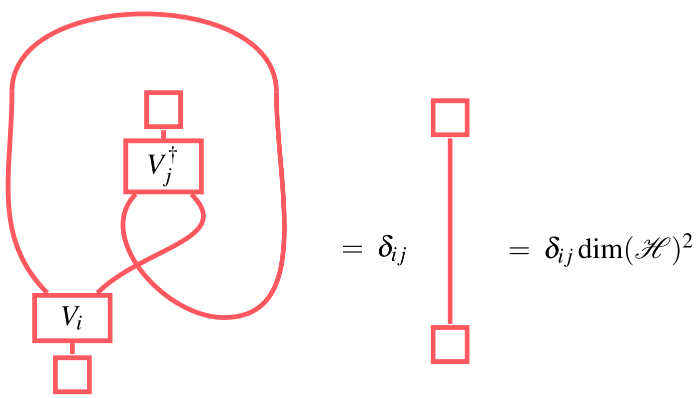


Having established that the counit is pure, we now move on to establish that the comultiplication is pure as well. From the snake equations, the purity principle implies the existence of coefficients $\lambda_{i,j}, \rho_{i,j} \in \mathbb{R}^+$ such that:

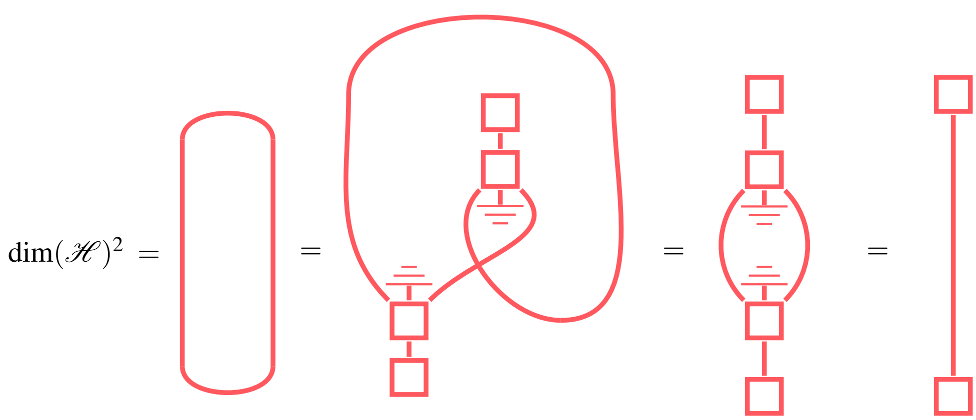
We now proceed to show that $\lambda_{i,j} = \delta_{ij}$, using the left snake equation. The proof that $\rho_{i,j} = \delta_{ij}$ is analogous, using the right snake equation. Taking the trace on both sides of the snake equation, we obtain the following equation, where $\mathscr{H}$ is the underlying Hilbert space for the comonoid (we get $\dim(\mathscr{H})^2$ because we are working in the CPM category, where the scalar is doubled):



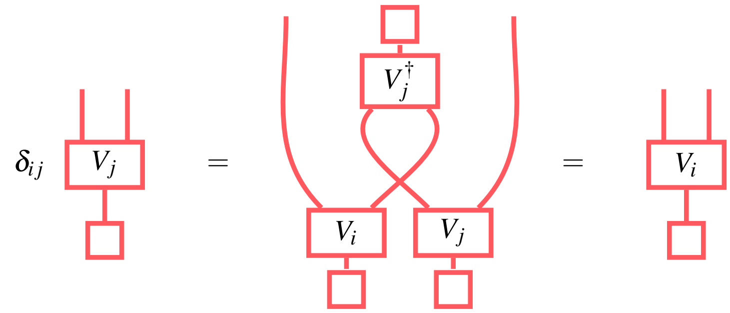
Taking the trace on the LHS of the snake equation and using the fact that $V_j^\dagger V_i = \delta_{ij}\mathbb{1}$, we get a different equation:



The rightmost equality follows from isometry and the snake equation:

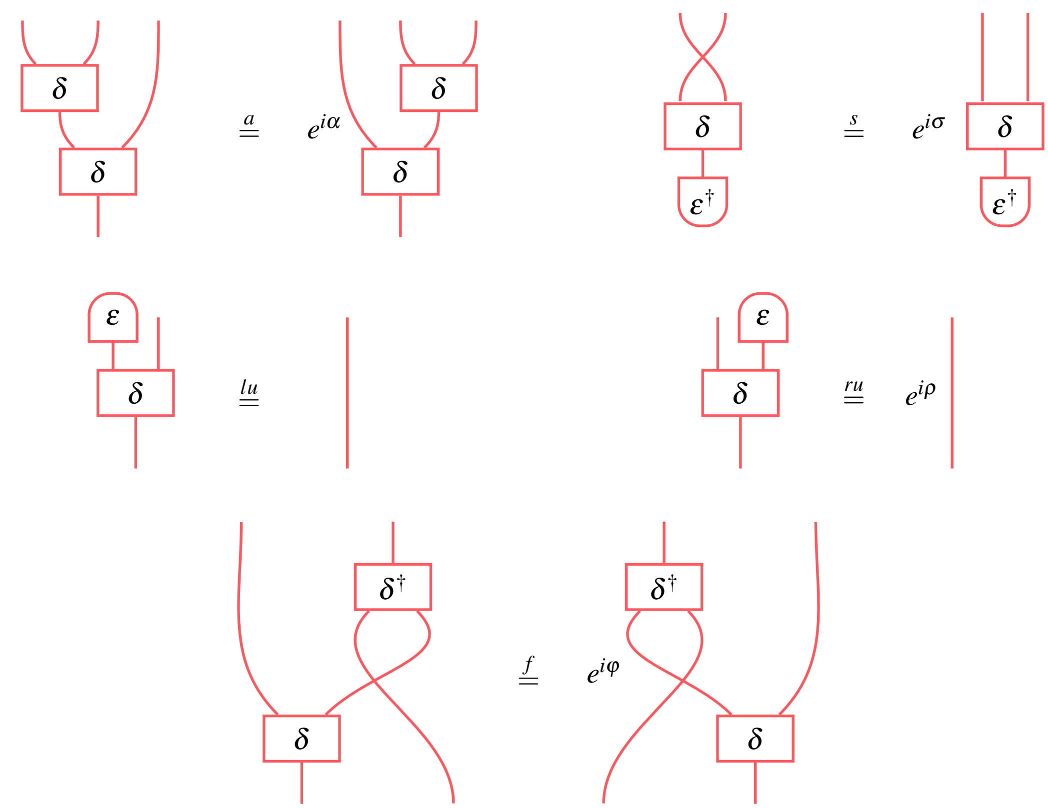


Having established that $\lambda_{i,j} = \delta_{ij} = \rho_{i,j}$, we now show that $n \geq 2$ leads to a contradiction unless $\dim(\mathscr{H}) = 0$ (in which case the statement of this theorem is trivial). Indeed, we have the following equation for all i, j:



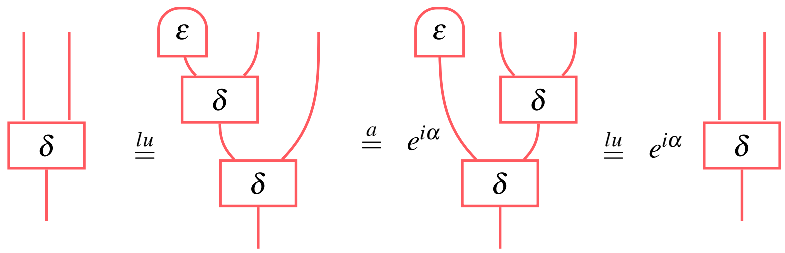
If we have $n \geq 2$, then we can set $i \neq j$ in the equation above, concluding that the RHS state is the zero state. However, we also know that V_i is an isometry and that the adjoint of the counit is a state of norm $\dim(\mathscr{H})^2$, hence the RHS state also has norm $\dim(\mathscr{H})^2$. So either $n = 1$, in which case the comultiplication is pure, or $n \geq 2$ and $\dim(\mathscr{H}) = 0$, in which case the comultiplication is the zero map, which is also pure. □

Lemma 6. *Let $\delta : \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$ and $\varepsilon : \mathscr{H} \to \mathbb{C}$ be morphisms in* fHilb *which satisfy the associativity law up to a phase $e^{i\alpha}$, the symmetry law up to a phase $e^{i\sigma}$, the left unit law exactly, the right unit law up to a phase $e^{i\rho}$ and the Frobenius law up to a phase $e^{i\varphi}$:*

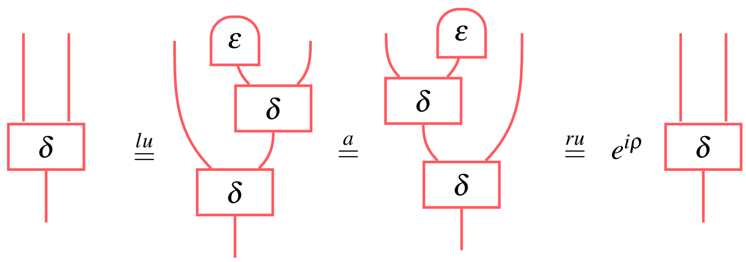


Then $(\delta, \varepsilon, \delta^\dagger, \varepsilon^\dagger)$ is a symmetric †-Frobenius algebra in fHilb*, i.e. $\alpha = \rho = \sigma = \varphi = 0 \bmod 2\pi$.*

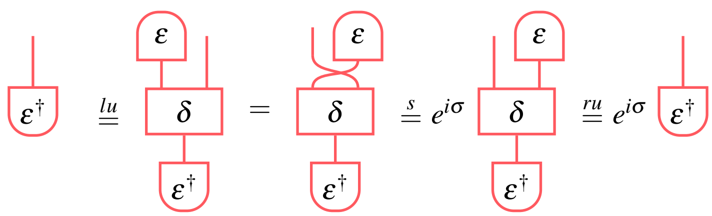
Proof. The associativity law and left unit law prove that $\alpha = 0 \bmod 2\pi$:



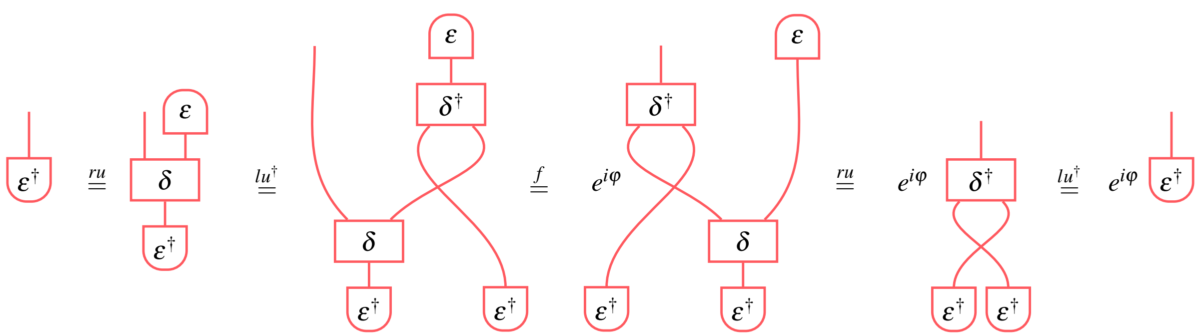
Then, the associativity law and the unit laws prove that $\rho = 0 \bmod 2\pi$:



Then, the symmetry law and the unit laws prove that $\sigma = 0 \bmod 2\pi$:



Finally, the Frobenius law and the right unit law prove that $\varphi = 0 \bmod 2\pi$:



This concludes our proof. □

Corollary 7 (CP †-SSFAs are all canonical)**.**
The †-SSFAs in CPM (fHilb) *are all canonical, i.e. they all arise by doubling of †-SSFAs in* fHilb.

Proof. Now let $(\Delta, E, \Delta^\dagger, E^\dagger)$ be a †-SSFA in CPM (fHilb). Because (Δ, E) is a comonoid which is isometric and satisfies the snake equations, we now know that Δ and E are pure, i.e. that we can find linear maps δ and ε in fHilb such that $\Delta = \mathrm{CPM}(\delta)$ and $E = \mathrm{CPM}(\varepsilon)$ arise by doubling. We now wish to conclude that $(\delta, \varepsilon, \delta^\dagger, \varepsilon^\dagger)$ form a †-SSFA in fHilb, but this doesnt immediately follow from the equations in CPM(fHilb): while δ is certainly an isometry in fHilb, the associativity law, unit laws, symmetry law and Frobenius law for δ and ε are only guaranteed to hold up to phase. In fact, $(\delta, \varepsilon, \delta^\dagger, \varepsilon^\dagger)$ is not, in general, a †-SSFA in fHilb. However, it is easy to show (cf. Lemma 6 below) that $(\delta, e^{-i\lambda}\varepsilon, \delta^\dagger, e^{i\lambda}\varepsilon^\dagger)$ is always a †-SSFA in fHilb, where $e^{i\lambda}$ is the phase associated to the identity in the left unit law. Because $\mathrm{CPM}(e^{-i\lambda}\varepsilon) = \mathrm{CPM}(\varepsilon)$, this is enough to prove our result. □

Acknowledgements

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