Admissible Causal Structures and Causal Inequalitites

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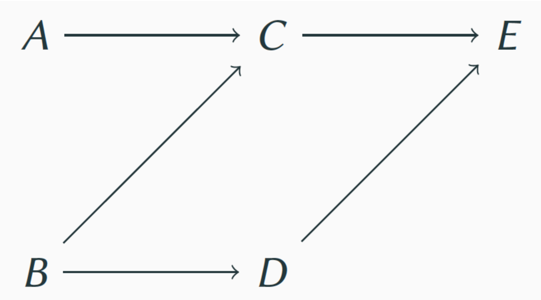
Abstract

Being at the core of many physical theories, often as a self-evident presupposition, the notion of *global time* carries through to the theory implicit assumptions, *e.g.*, that events are causally ordered. More specifically, in quantum theory time appears as an intrinsic parameter from which the causal ordering of the events follows.
In a theory where only the *local* validity of quantum theory is assumed[1] the nature of "space-time" is encoded in the signalling relations among the different *local* labs, where the agents perform quantum experiments. The object describing it is called *process*. The *local* labs can be thought of as the *local* space-time regions. But what are the possible signalling relations among them if quantum mechanics is (locally) valid in each?
We find a graph-theoretic criterion that singles out all such causal structures for unitarily extensible processes [2]. Moreover, we show that quantum theory *does not* allow for causal structures impossible with classical-deterministic theories[3]. This result can be understood as a generalization of the fact that measurements on quantum systems yield non-signaling correlations to arbitrary scenarios. Finally, we provide two graph-theoretic criteria from which, in the classical-deterministic case, non-violations and violations of causal inequalities follow.

1. Causal Models



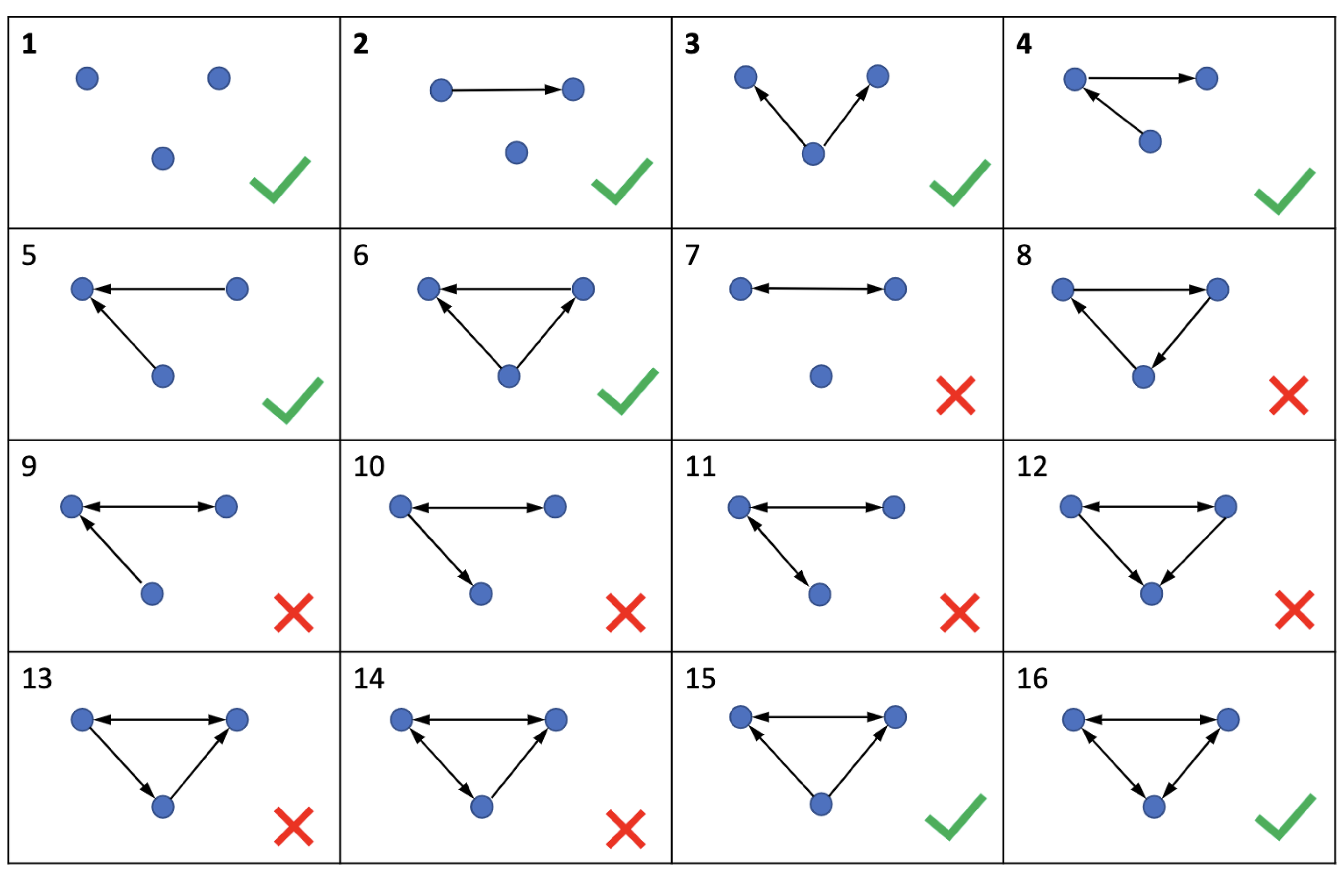
$$\{P_{I_A}, P_{I_B}, P_{I_C|O_A,O_B}, P_{I_D|O_B}, P_{I_E|O_C,O_D}\}$$

$$P_{I_A,I_B,I_C,I_D,I_E|O_A,O_B,O_C,O_D} = P_{I_A}P_{I_B}P_{I_C|O_A,O_B}P_{I_D|O_B}P_{I_E|O_C,O_D}$$

Definition 1 (Causal model [4]) *An n-party* causal model *is a directed graph $G = (\mathbb{Z}_n, E)$ (causal structure) equipped with $\{\rho_{k|\mathrm{Pa}(k)}\}_{k\in\mathbb{Z}_n}$ (model parameters). In the* classical-deterministic *case, the model parameters $\rho_{k|\mathrm{Pa}(k)}$ are functions $\mathcal{O}_{\mathrm{Pa}(k)} \to \mathcal{I}_k$ and define a* classical map *$\omega := (\rho_{k|\mathrm{Pa}(k)})_{k\in\mathbb{Z}_n}$. In the* quantum *case, the model parameters $\rho_{k|\mathrm{Pa}(k)}$ are the Choi operators of completely positive trace-preserving maps $\mathcal{L}(\mathcal{O}_{\mathrm{Pa}(k)}) \to \mathcal{L}(\mathcal{I}_k)$, such that $\forall i,j \in \mathbb{Z}_n : [\rho_{i|\mathrm{Pa}(i)}, \rho_{j|\mathrm{Pa}(j)}] = 0$, and define a* quantum map *$W := \prod_{k\in\mathbb{Z}_n} \rho^{k|\mathrm{Pa}(k)}$. We call the causal model* consistent *if ω, W, is an n-party classical-deterministic or quantum process, respectively.*

2. Admissible causal structures

Theorem 1 (Admissible causal structure (quantum)) *Let $G = (\mathbb{Z}_n, E)$ be a directed graph. There exists a faithful and consistent quantum causal model with causal structure G if and only if each directed cycle in G has siblings (nodes with common parents).*



3. Equivalence of Quantum and Classical causal models

Theorem 2 *Let $G = (V, E)$ be a directed graph. There exists a quantum faithful causal model with causal structure G if and only if there exists a classical-deterministic faithful causal model with causal structure G*

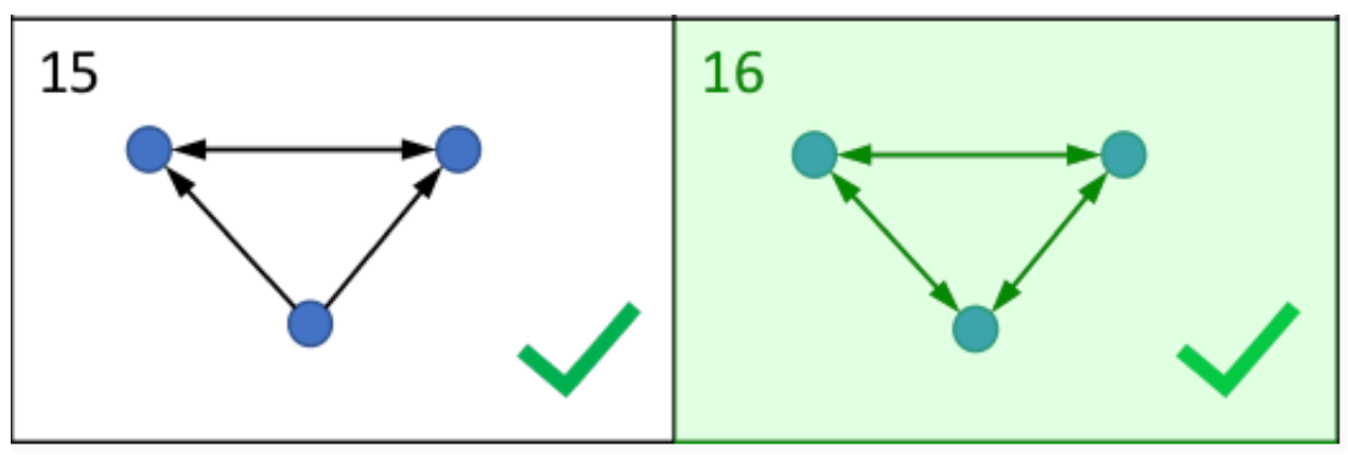
4. Causal Inequalities

Definition 2 (Causal Game) *A random input $l \in L$, $|L| = |V|$, is distributed to each party $i \in V$ and a random bit $s \in S = \{0,1\}$ to each party $j \in V \setminus l$. The parties win the* non-causal game *if party i_l correctly guesses the random variable s.*

The following result on (non-)violation of causal inequalities relies on the nature of the directed cycles in the respective causal structure. A directed *induced* cycle $C = (v_0, v_1, \ldots, v_k, v_0)$ in a graph $G = (V, E)$ is a directed cycle without directed sub-cycles.

Theorem 3 (Classical deterministic processes and causal inequalities) *Let ω be a classical-deterministic process with causal structure $G = (V, E)$.*

1. *If all directed cycles C in G are induced, then ω does not violate any causal inequality.*
2. *If G contains a non-induced directed cycle C where $\forall k, \ell \in C : \mathrm{Pa}(k) \cap \mathrm{Pa}(\ell) \in C$, then ω violates a causal inequality.*



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