Measuring relational information between quantum states, and applications

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Introduction: projective-unitary invariance

• Which properties of a set of quantum states are invariant under unitary transformations?

Projective-unitary (PU)-invariant properties

Example of PU-invariant: two-state overlap $\Delta_{AB} = |\langle A|B\rangle|^2 = \mathrm{Tr}(\rho_A \rho_B)$

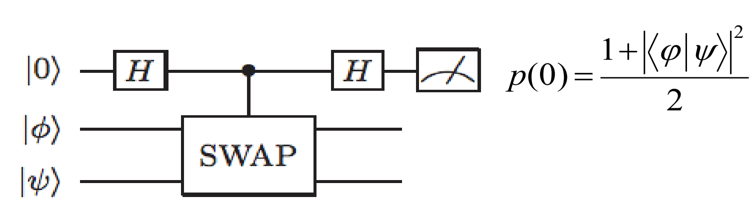
- Results by Chien and Waldron [1] for pure states:
  - All PU-invariant properties are functions of *k*-state Bargmann invariants [2]:

$$\Delta_{ABC\cdots K} = \langle A|B\rangle\langle B|C\rangle\langle C|D\rangle\cdots\langle K|A\rangle$$

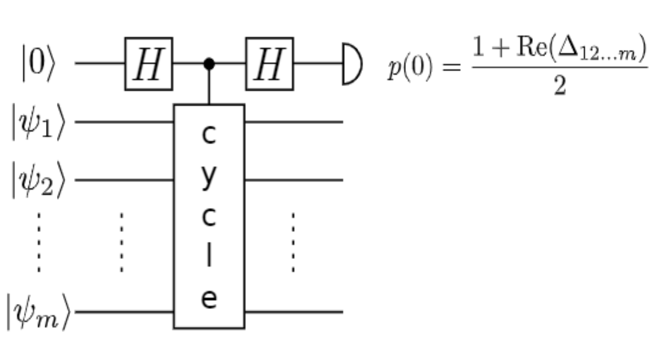
  - For *N* states, we may need up to *N*-state Bargmann invariants.
  - If no pair of states are orthogonal, 3-state invariants suffice for complete PU characterization.
- Bargmann invariants have been discussed in the context of geometric phases, and characterization of multi-photon indistinguishability.

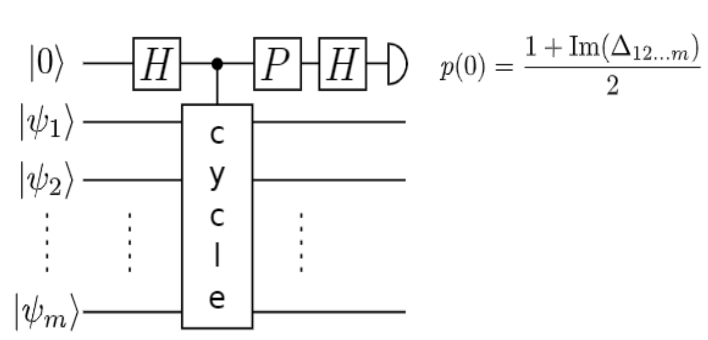
How to measure Bargmann invariants

- The 2-state invariant (overlap) can be measured using the well-known SWAP test:

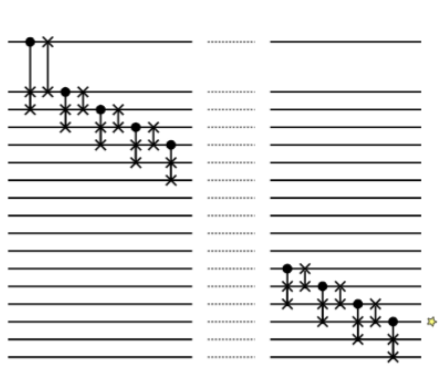


- We propose the **cycle test**, a generalization to measure real and imaginary parts of Bargmann invariants:



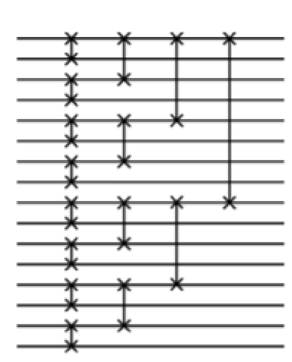


- Circuit on left: used in [3] to evaluate nonlinear functionals of a state.
- Efficient controlled-cycle gate decompositions:



Local gates: linear depth (aux. qubit "moves")

Non-local gates: log-depth (controlled by GHZ aux. register, not shown)

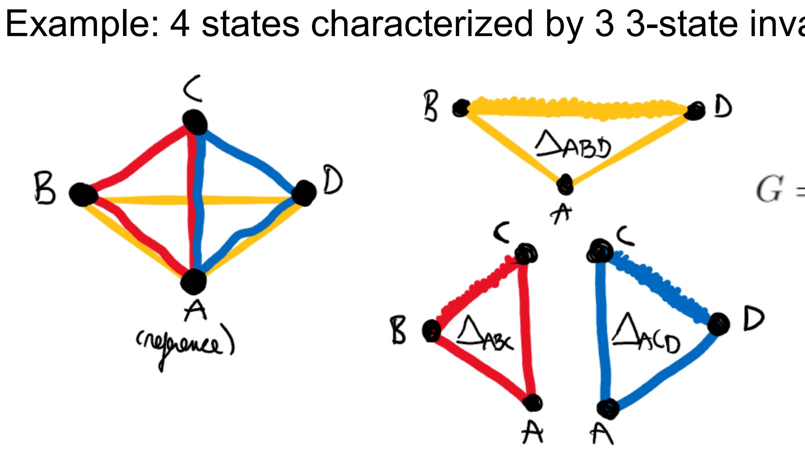


Gram matrix encodes all PU invariants

- PU properties of *N* generic pure states depend only on 3-state invariants of all pairs of states with a single reference state. Representation: **Gram matrix *G*** of inner products

$$G_{ij} = \langle i|j\rangle$$

Example: 4 states characterized by 3 3-state invariants, all including reference state A:



$$G = \begin{pmatrix} 1 & \sqrt{\Delta_{12}} & \sqrt{\Delta_{13}} & \sqrt{\Delta_{14}} \\ \sqrt{\Delta_{12}} & 1 & \sqrt{\Delta_{23}}e^{i\phi_{23}} & \sqrt{\Delta_{24}}e^{i\phi_{24}} \\ \sqrt{\Delta_{13}} & \sqrt{\Delta_{23}}e^{-i\phi_{23}} & 1 & \sqrt{\Delta_{34}}e^{i\phi_{34}} \\ \sqrt{\Delta_{14}} & \sqrt{\Delta_{24}}e^{-i\phi_{24}} & \sqrt{\Delta_{34}}e^{-i\phi_{34}} & 1 \end{pmatrix}$$

- Phase of G_{kl} = phase of Δ_{1kl} ← reference state

- **All parameters in *G* are gauge-invariant** and can be measured with cycle tests
- Robustness results: continuity for mixed states with high purity; approximate transformations from one set to another with "close" invariants

Application: Linear independence test

• N states are LI iff det(G) >0

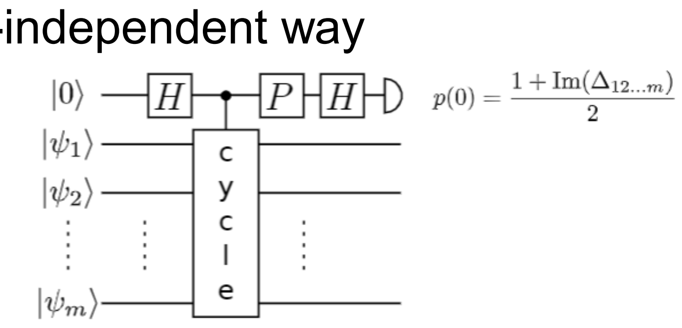
• Example: N=3 states

$$G = \begin{pmatrix} 1 & |\langle\psi_1|\psi_2\rangle| & |\langle\psi_1|\psi_3\rangle| \\ |\langle\psi_1|\psi_2\rangle| & 1 & \langle\psi_2|\psi_3\rangle \\ |\langle\psi_1|\psi_3\rangle| & \langle\psi_2|\psi_3\rangle^* & 1 \end{pmatrix} = \begin{pmatrix} 1 & \sqrt{\Delta_{12}} & \sqrt{\Delta_{13}} \\ \sqrt{\Delta_{12}} & 1 & \sqrt{\Delta_{23}}e^{i\phi_{23}} \\ \sqrt{\Delta_{13}} & \sqrt{\Delta_{23}}e^{-i\phi_{23}} & 1 \end{pmatrix}$$

$$\det(G) > 0 \Leftrightarrow 1 - (\Delta_{12} + \Delta_{13} + \Delta_{23}) + 2\sqrt{\Delta_{12}\Delta_{13}\Delta_{23}}\cos(\phi_{23}) > 0.$$

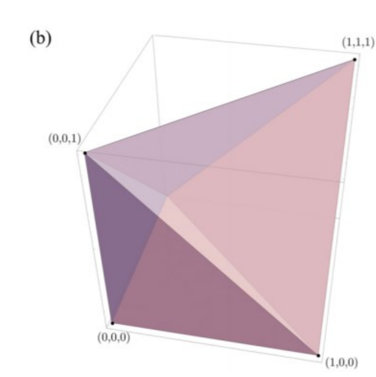
Application: Basis-independent imaginarity witness

- **Imaginarity**: necessity of complex amplitudes in quantum theory [4]
- Imaginary part of invariants due to complex amplitudes in input states, measured in a basis-independent way

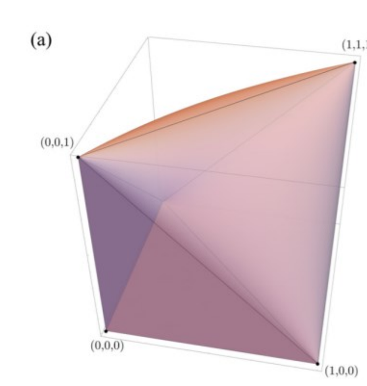


Application: Basis-independent coherence witnesses

- Bargmann invariant measurements can witness coherence in a basis-independent way
- Example: overlaps of coherence-free (diagonal) states satisfy linear bounds, which are violated by (coherent) states [5]



3 overlaps of 3 coherence-free states (diagonal in single basis)



3 overlaps of 3 general states violate coherence-free polytope facets

Conclusion

- We've shown how to measure projective-unitary (Bargmann) invariants of a set of states, encoding a complete set in the Gram matrix G, and applications
- Open questions: algorithmic use in e.g. dimensionality reduction; application in photonic indistinguishability criteria; other foundational relevance (besides witnessing imaginarity and coherence).

References

This work: Oszmaniec, Brod, Galvão, arXiv:2109.10006 [quant-ph].

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