Differential Geometry of Contextuality

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Summary

Here we identify states, effects, and transformations in the framework of generalized contextuality as vectors living in a tangent space and the non-contextual conditions as discrete closed paths implying null vertical phases. Two equivalent interpretations hold: the geometrical view where flat space is imposed, implying that the contextual behavior becomes equivalent to the curvature and thus a modification of the valuation function; and the topological view where the valuation functions are preserved, implying that the contextual behavior must be translated as topological failures. Such a formalism allows the study of a set of applications.

Generalized contextuality

Contextuality is the property of a physical system in some mathematical structure that cannot be represented by another compatible structure called classical. In the generalized contextuality approach, the probability with state P, effect E and transformation T is given by a representation

$$p(E|T,P)=\sum_{\lambda,\lambda'}\xi(E|\lambda')\Gamma(\lambda',T,\lambda)\mu(\lambda|P)$$

codified by valuation functions ξ, Γ, μ on the set of ontic variables.

1-form representation

One can rewrite the non-contextual condition of operational equivalence preservation as the preservation of a closed discrete loop in the tangent space by the respective valuation function,

$$\begin{tikzcd}
\sum_r b_r^{(\beta)}E_r=0 \arrow[r, Longrightarrow] \arrow[d] & \sum_r b_r^{(\beta)}\xi(E_r|\lambda')=0,\forall\lambda' \arrow[d] \\
\gamma^{(\beta)} \arrow[r, Longrightarrow] & \phi^{(\beta)}=\left\langle\xi_{\lambda'}\middle|\gamma^{(\beta)}\right\rangle=0,\forall\lambda'
\end{tikzcd}$$

and analogously to states and transformations. The valuation functions are represented as differential forms acting on the effects.

Geometrical view

Here, all loops are just boundaries, $\gamma=\partial S$, and non-contextuality conditions can be rewritten by Stokes theorem to define the coboundary operator as

$$\langle\xi_{\lambda'}|\partial S_\beta\rangle=\langle d\xi_{\lambda'}|S_\beta\rangle=\langle ddc_{\lambda'}|S_\beta\rangle+\langle d\omega_{\lambda'}|S_\beta\rangle.$$

Theorem: *Non-contextuality for measurements (transformations; states) is equivalent to a null contextual curvature* $0=F_{\lambda'}=d\xi_{\lambda'}$ ($0=F_{\lambda'\lambda}=d\Gamma_{\lambda'\lambda}$; $0=F_\lambda=d\mu_\lambda$) *for all hidden variables.*

Topological view

Here, one refuses the use of a curvature, thus $F=0$ and contextuality is not a correction in the valuation due to a hidden object.

Theorem: *If $F=0$, then contextuality is equivalent to monodromy.*

Two equivalent views

We can codify what is going on with a diagram:

$$\begin{tikzcd}
\mathscr{E} \arrow[rr, "\xi_{\lambda'}"] \arrow[dr, "i"'] & & {[0,1]} \\
& \mathscr{S} \arrow[ur] &
\end{tikzcd}.$$

The system $\mathscr{E}$ given the set of effects, the classical representation $\mathscr{S}$, and the target for valuation $[0,1]$, are all fixed, as is the system valuation map $\xi_{\lambda'}$. Due to contextuality the inclusion in (topological view) or the valuation of (geometrical view) the classical representation fails. Both notions are equivalent; it is just a matter of representation of a deeper phenomenon.

Applications

Contextual fraction: Let's restrict the theory to a measurement scenario with a fixed state and satisfy the conditions for applying contextual fraction with a finite number of outcomes for each measurement. One can write the probability as a decomposition

$$\text{NCF}=\sum_r\int_\Lambda\mu(\lambda)\langle dc_\lambda|E_r\rangle \text{ and } \text{CF}=\sum_r\int_\Lambda\mu(\lambda)\langle\omega_\lambda|E_r\rangle,$$

with the non-contextual fraction NCF and the contextual fraction CF.

Interference: In generalized measure theory, interference is a correction to the standard measure theory based on the Kolmogorov axioms. But any correction to the valuation follows from the connection ω. One can see this by noting that $dc_{\lambda'}$ satisfies Kolmogorov axioms, so for disjoint effects,

$$I_2(E,E')=\int d\mu\left(\langle\omega|E\vee E'\rangle-\langle\omega|E\rangle-\langle\omega|E'\rangle\right)$$

so interference is the failure of ω to satisfy the disjoint axiom.

Signed-measures: The violation of the third Kolmogorov axiom leads to the necessity of signed measures. This result gives a different notion of what the curvature means; it codifies the negative part of the valuation. It also explains why we cannot access negative probabilities: they can be seen as a topological failure of the theory.

Embedding: The embedding into a classical model, an equivalent notion of contextuality, can also be understood: in the geometrical view, it cannot have a non-trivial curvature to correct the valuation, and any theory that has such a curvature cannot be represented by a classical theory; in the topological view, a classical theory shows no topological failure, so monodromy is impossible, and a theory with monodromy cannot be represented by a classical theory.

Voroby'ev theorem: Measurement contextuality follows from a loop γ in the effect algebra. As any loop in a Boolean algebra satisfies $\omega=0$, only loops defined through different Boolean algebras can show any contextuality. Voroby'ev result identifies the fact that without such loops, no contextual behavior appears. One can thus generalize the Voroby'ev theorem: an model is non-contextual if its first de Rham cohomological group is trivial when we impose $F=0$.

Disturbance: The geometrical view has a direct way to deal with disturbance, as the triviality of the transition maps $t_{r,r'}$ on intersections of Boolean algebras. The holonomy transformation will be

$$Hol(\partial S)=\prod_r exp\left[\langle\omega|(b_r|E_r\rangle)\right]\prod_r t_{r,r-1}.$$

We can thus define $\eta_{r,r-1}=\langle\eta|(b_r|E_r\rangle)$, and rewrite the valuation function as $\xi=dc+\omega+\eta$, where the disturbance is on the same footing as the contextuality.

For more details

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