Optimizing Quantum Social Welfare in Non-collaborative Games

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Summary

- Non-collaborative games involving multiple players exhibit equilibria wherein no player has an incentive to deviate from their strategy
- The quality of an equilibrium can be quantified by its social welfare – the mean payout each player receives
- Access to shared quantum resources may allow better cooperation, and hence better equilibria
- We consider two scenarios: in one, players may make measurements directly on a quantum state, while in the other, they delegate the measurement to a referee
- We study how to optimise the social welfare in these two settings and compare the classes of equilibria obtainable on several games as a function of the bias of the game

Non-collaborative games

A non-collaborative game G between n players is defined by:

- A set of **questions** $T \subseteq \{0,1\}^n$
- A **prior distribution** Π over the questions T
- A set of **valid answers** $A \subseteq \{0,1\}^n$
- A **payout function** u_i for each player i, with $u_i(a,t) \in \mathbb{R}$.
  - We consider payout functions with the form

$$u_i(a,t) = \begin{cases} 0 \text{ if } (a,t) \in \mathcal{L} \\ v_0 \text{ if } a_i = 0 \text{ and } (a,t) \notin \mathcal{L} \\ v_1 \text{ if } a_i = 1 \text{ and } (a,t) \notin \mathcal{L}, \end{cases}$$

  with $v_0, v_1 > 0$ and $\mathcal{L} \subseteq A \times T$ a set of "losing input-output pairs"
  - Ratio v_0/v_1 controls the bias of game

Example: Winning conditions for two 5-player games: $NC_{00}(C_5)$ and $NC_{01}(C_5)$ [1]

Question $t_1t_1t_2t_3t_5$	Winning condition, $NC_{00}(C_5)$
10000	$a_4 \oplus a_0 \oplus a_1 = 0$
01000	$a_0 \oplus a_1 \oplus a_2 = 0$
00100	$a_1 \oplus a_2 \oplus a_3 = 0$
00010	$a_2 \oplus a_3 \oplus a_4 = 0$
00001	$a_3 \oplus a_4 \oplus a_0 = 0$
11111	$a_0 \oplus a_1 \oplus a_2 \oplus a_3 \oplus a_4 = 1$

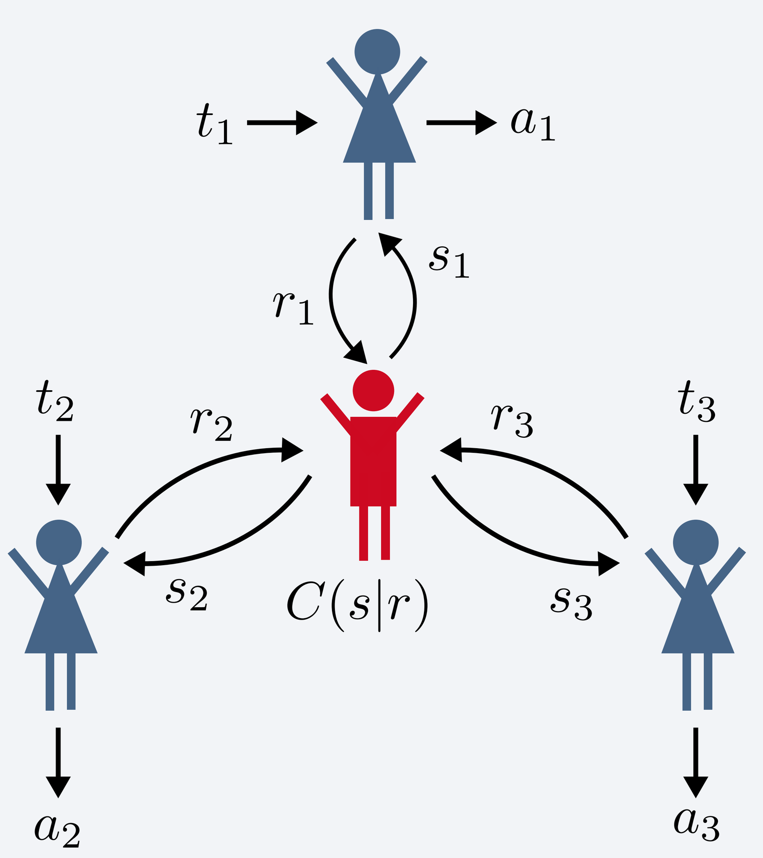
Question $t_1t_1t_2t_3t_5$	Winning condition, $NC_{01}(C_5)$
10100	$a_4 \oplus a_0 \oplus a_1 = 0$
01010	$a_0 \oplus a_1 \oplus a_2 = 0$
00101	$a_1 \oplus a_2 \oplus a_3 = 0$
10010	$a_2 \oplus a_3 \oplus a_4 = 0$
01001	$a_3 \oplus a_4 \oplus a_0 = 0$
11111	$a_0 \oplus a_1 \oplus a_2 \oplus a_3 \oplus a_4 = 1$

Strategies and equilibria

- Each player follows a local strategy to produce their answer
- In general, they may also have access to a shared correlation in the form of an advice s_i provided by a mediator with probability $C(s_1 \dots s_n | r_1 \dots r_n)$
- A solution (set of strategies for each player, defined by functions f_i and g_i) induces a distribution

$$P(a|t) = \sum_{\lambda} \Lambda(\lambda) \sum_{s:\forall i, g_i(t_i,s_i,\lambda_i)=a_i} C(s_1 \dots s_n | f(t_1,\lambda_1) \dots f(t_n,\lambda_n))$$

  - We can generally consider just deterministic strategies



- A solution is a **Nash equilibrium** if no player can increase their mean payout by changing their strategy: $\forall i \, \forall t_i, r_i \in T_i \, \forall \mu_i : T_i \times A_i \to A_i$,

$$\sum_{t_{-i},a_{-i}} u_i(a,t) P(a|t) \Pi(t) \geq \sum_{t_{-i},a_{-i}} u_i(\mu_i(t_i,a_i)a_{-i},t) P(a|r_i t_{-i}) \Pi(t)$$

  - Nash equilibria play important roles in applications from economics to engineering
  - Different correlations C lead to different equilibria: Nash (no correlation), Corr (shared randomness), B.I. (belief invariant, or no-signalling), ...
- The **social welfare** of a solution is

$$SW(P) = \sum_{a,t} U(a,t) P(a|t) \Pi(t), \text{ where } U(a,t) = \frac{1}{n} \sum_i u_i(a,t)$$

Two types of quantum strategies

Question: How can quantum resources lead to new equilibria or improve social welfare?

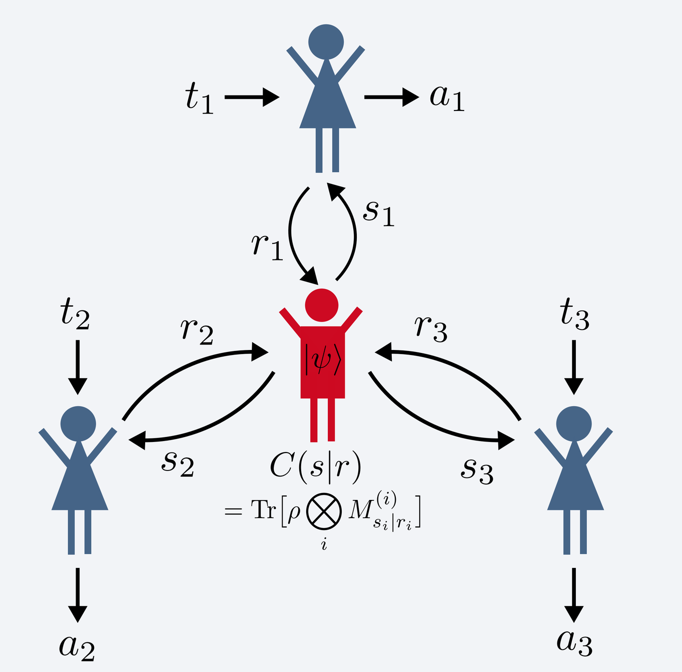
We identify two types of quantum strategy and equilibria:

- Quantum correlated strategies: Advice C is obtained from measurements on a quantum system:

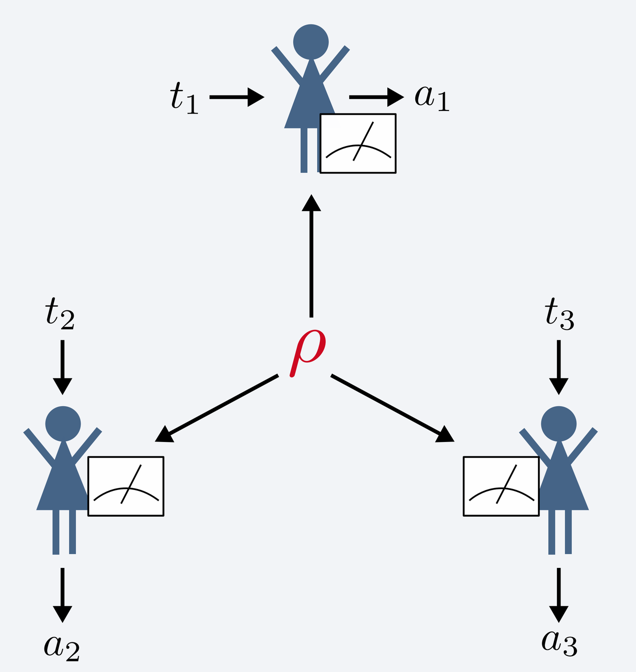
$$C(s|r) = \mathrm{Tr}\left[\rho (M^{(1)}_{s_1|r_1} \otimes \dots \otimes M^{(n)}_{s_n|r_n})\right]$$

  - Measurement delegated to mediator, or performed by parties with quantum "black-boxes"
- Quantum strategies [2]: Each player measures a shared quantum state to determine their output a_i
  - Direct access to quantum resource
  - Notion of equilibria modified: a player can deviate by choosing any other local POVM: $\forall i \, \forall t_i \, \forall N^{(i)} = \{N^{(i)}_{a_i|r_i}\}_{r_i}$

$$\sum_{t_{-i},a} u_i(a,t) \,\mathrm{Tr}\left(\rho \cdot \bigotimes_j M^{(j)}_{a_j|t_j}\right) \Pi(t) \geq \sum_{t_{-i},a} u_i(a,t) \,\mathrm{Tr}\left(\rho \cdot \bigotimes_{j \neq i} M^{(j)}_{a_j|t_j} \otimes N^{(i)}_{a_i|t_i}\right) \Pi(t)$$



Quantum correlated strategy – $Q_{\text{corr}}(G)$



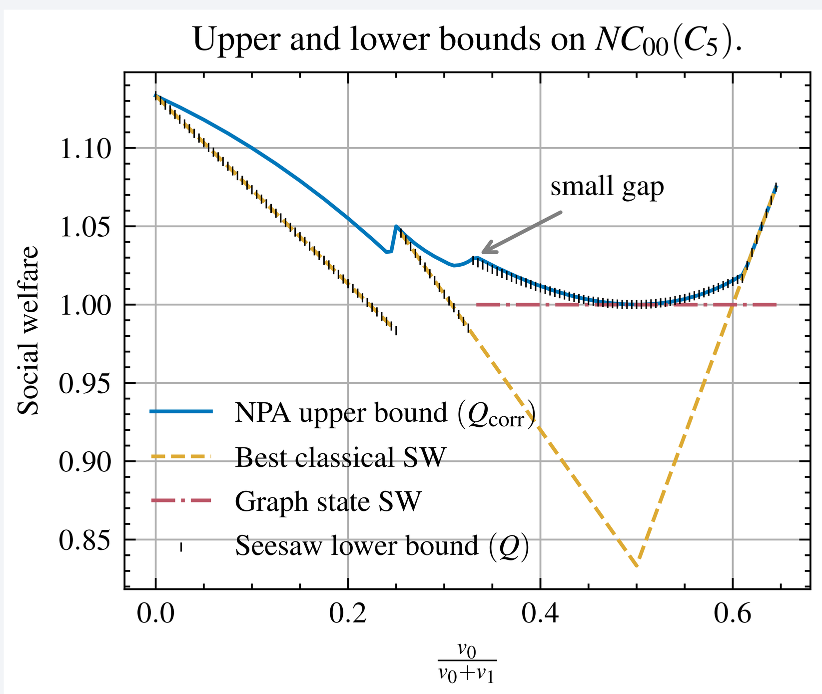
Quantum strategy – $Q(G)$

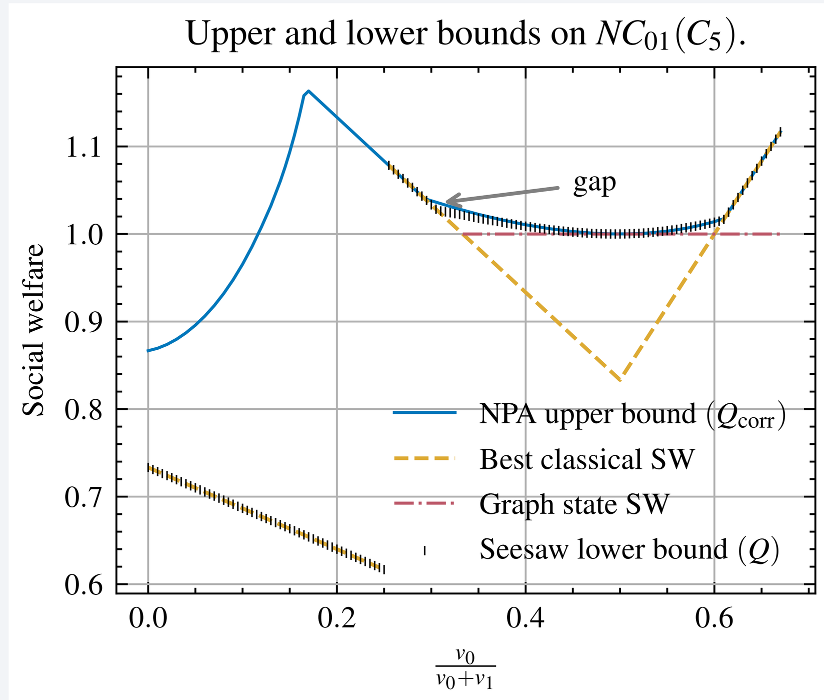
For any game G, the sets of equilibria satisfy

$\text{Nash}(G) \subset \text{conv}(\text{Nash}(G)) \subset \text{Corr}(G) \subset Q(G) \subset Q_{\text{corr}}(G) \subset \text{B.I.}(G) \subset \text{Comm}(G)$

Results: Social welfare of different strategies

- We optimised the social welfare over different strategy classes for three games: $NC_{00}(C_5)$, $NC_{01}(C_5)$, and $NC(C_3)$ (not shown here) [1]
  - **Best classical SW:** computed exactly
  - **Graph state SW:** pseudo-telepathic equilibria using GHZ states [1]
  - **Seesaw lower bound:** numerical optimisation by iterating SDPs to find explicit strategies lower-bounding QSW over $Q(G)$
  - **NPA upper bound:** SDP hierarchy providing dimension-independent upper bound on equilibria in $Q_{\text{corr}}(G)$ [3, 4]





Conclusions and open questions

- Two different ways to use quantum resources lead to distinct classes of equilibria
- Numerical evidence of strict separation between $Q(G)$ and $Q_{\text{corr}}(G)$, but analytic proof still to be found
- Quantum social welfare can be improved beyond pseudo-telepathic strategies
- Methods to directly obtain upper bounds on $Q(G)$ and lower bounds on $Q_{\text{corr}}(G)$?

References and acknowledgments

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