Classical Simulation of Circuits with Realistic Odd-Dimensional Gottesman-Kitaev-Preskill States

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Classically simulating circuits with bosonic codes is challenging due to the prohibitive cost of simulating quantum systems with many, possibly infinite, energy levels. We propose an algorithm to simulate circuits with encoded Gottesman-Kitaev-Preskill (GKP) states, specifically for odd-dimensional encoded qudits. Our approach is tailored to be especially effective in the most challenging but practically relevant regime, where the codeword states exhibit high (but finite) squeezing. Our algorithm leverages the Zak-Gross Wigner function introduced by Davis, Fabre, and Chabaud, which represents infinitely squeezed encoded stabilizer states positively. The run-time of the algorithm scales with the negativity of the Wigner function, allowing for efficient simulation of certain large-scale circuits—namely, input stabilizer GKP states undergoing generalized GKP-encoded Clifford operations followed by modular measurement—with a high degree of squeezing. For stabilizer GKP states exhibiting 12 dB of squeezing, our algorithm can simulate circuits with up to 1000 modes with less than double the number of samples required for a single input mode, which is in stark contrast to existing simulators. Therefore, this approach holds significant potential for benchmarking early implementations of quantum computing architectures utilizing bosonic codes.

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Quantum computers are expected to solve certain types of problems faster than classical computers, a property known as quantum computational advantage [1–4]. Although generally inefficient, classical simulation algorithms that can reproduce the outcome of a quantum computation remain desirable, even at the price of a potentially exponential runtime in the input size. Such algorithms can, for instance, simulate small-size or restricted types of quantum circuits, thereby allowing for benchmarking early implementations of quantum processors [5–8].

A paradigmatic example of such simulation algorithms is the Gottesman-Knill theorem [9–14], allowing for the efficient (i.e., polynomial time) classical simulation of the restricted set of circuits with input qudit stabilizer states, Clifford operations, and Pauli measurements. Other approaches extend these simulation algorithms to address general quantum circuits at the price of run-times higher than polynomial. For instance, one can use finite-dimension quasiprobability distributions such as the Gross Wigner function [15] to provide a sampling algorithm for odd-dimensional systems with a run-time that scales with the negative volume of the quasiprobability distribution [16–19].

For bosonic systems—which hold promise due to their quantum error correction capabilities [20–26] and scalability [27–33]—available simulation algorithms exploit the positivity of quasiprobability distributions [34–36], tensor networks [37,38], the stellar representation of non-Gaussian states [39,40], or decompositions into Gaussian states [41–43]. These approaches have proven effective for certain classes of circuits, yet significant challenges remain for simulating the most experimentally relevant scenarios for quantum computation.

One such scenario involves quantum computations with bosonic encoding, particularly the Gottesman-Kitaev-Preskill (GKP) code [44]. GKP-based bosonic quantum processors are implemented with microwave cavities coupled to superconducting circuits [20,21], trapped ions [22], and photonic platforms [23], and are the object of intense study due to their potential for fault tolerance [45,46]. Such architectures are typically extremely difficult to simulate since GKP codewords are described by infinitely many energy levels, making their brute-force simulation (based

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on expansions in the Fock basis) prohibitive for just a handful of GKP qudits. Also, approaches naively based on quasiprobability distributions are doomed to fail because the negativity of the Wigner function of encoded GKP states is very large, making the run-time of such simulators blow up. Neither are approaches based on decomposing GKP states into superpositions of Gaussian states [41–43] suitable for the practically relevant case of high squeezing GKP states due to the large non-Gaussianity of such states.

While progress has been made, current simulation techniques address only a limited subset of GKP-encoded computations. Specifically, Refs. [19,47–50] provide efficient algorithms for simulating circuits initialized in stabilizer states encoded in ideal (i.e., infinitely squeezed) GKP states, followed by Gaussian operations (including, but not limited to, encoded Clifford operations) and homodyne measurements. However, these methods cannot address the more practically relevant scenario of realistic (i.e., finitely squeezed) GKP states encoding general potentially nonstabilizer states.

Therefore, there is a critical need for new methods to simulate realistic GKP-encoded computations. Here, we propose a simulation algorithm designed to estimate the probability distribution of measurement outcomes for realistic GKP states undergoing encoded Clifford operations and encoded Pauli measurements, specifically for odd-dimensional GKP qudits. Qudit quantum computation has recently gained both theoretical and experimental interest in view, for instance, of enhanced algorithmic efficiency and simulation of high-dimensional systems (see Ref. [51] and references therein). We also provide a weak simulation algorithm for the case of infinitely squeezed GKP states.

We do so by generalizing the Zak-Gross Wigner (ZGW) function, introduced for a single-mode in Ref. [52], to the case of n modes and by showing that it satisfies crucial properties that make it amenable to tackle the simulation of quantum circuits. Such a Wigner function positively represents ideal stabilizer GKP states, while it is negative for magic GKP states and any finitely squeezed GKP states. Inspired by Ref. [16], our simulation algorithm has run-time scaling with the amount of negativity of the ZGW function.

Multimode ZGW function—We generalize the ZGW function [52] to the multimode case. We use the notation $[\mathbf{a},\mathbf{b}] = \mathbf{a}^T\Omega\mathbf{b}$, with $\Omega = \left(\begin{smallmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{smallmatrix}\right)$ the symplectic form. For example, given two vectors of length $2n$,

$$\mathbf{a} = \begin{pmatrix} \mathbf{a_X} \\ \mathbf{a_Z} \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} \mathbf{b_X} \\ \mathbf{b_Z} \end{pmatrix}, \tag{1}$$

we write $[\mathbf{a},\mathbf{b}] = \mathbf{a_X}^T\mathbf{b_Z} - \mathbf{a_Z}^T\mathbf{b_X}$. With the standard quantum optics displacement operator, $\hat{D}(\mathbf{a}) = e^{i\mathbf{a}^T\Omega\hat{\mathbf{r}}}$, whereby $\hat{\mathbf{r}} = (\hat{q}_1,\ldots,\hat{q}_n,\hat{p}_1,\ldots,\hat{p}_n)^T$, we define the operators $\hat{T}_\mathbf{a} = \hat{D}(-\ell\mathbf{a})$ with $\ell = \sqrt{2\pi/d}$, yielding

$$\hat{T}_\mathbf{a} = e^{i\pi\mathbf{a_X}^T\mathbf{a_Z}/d}\hat{T}_{\mathbf{a_X}}\hat{T}_{\mathbf{a_Z}}, \tag{2}$$

with $\hat{T}_{\mathbf{a_X}} = e^{-i\ell\mathbf{a}_X\cdot\hat{\mathbf{p}}}$ and $\hat{T}_{\mathbf{a_Z}} = e^{i\ell\mathbf{a}_Z\cdot\hat{\mathbf{q}}}$. For integer elements of the vector $\mathbf{a}$, these operators can be interpreted as logical Pauli operators on the GKP code space with an additional phase factor. The ideal, non-normalizable GKP states defining the code subspace in dimension d are given by $|j_{\mathrm{GKP}}\rangle = (1/\sqrt{\ell})\sum_n |(dn+j)\ell\rangle$, where the ket is in the position basis and $j \in \{0,\ldots,d-1\}$. We are now equipped to introduce the following definition.

Definition 1: We define the multimode, odd-dimensional ZGW function as

$$W_{\hat{\rho}}(\boldsymbol{\eta}) = \mathrm{Tr}(\hat{\rho}\hat{A}_{\boldsymbol{\eta}}), \tag{3}$$

where the phase point operator $\hat{A}_{\boldsymbol{\eta}}$ is defined as

$$\hat{A}_{\boldsymbol{\eta}} = \frac{1}{(2\pi)^n}\sum_{\mathbf{a}\in\mathbb{Z}^{2n}} e^{i\ell[\mathbf{a},\boldsymbol{\eta}] + i\pi\mathbf{a_X}^T\mathbf{a_Z}}\hat{T}_\mathbf{a}, \tag{4}$$

with $\boldsymbol{\eta} \in [0, d\ell)^{\times 2n}$ and with d a positive odd integer.

For example, for a single mode we can rewrite $\mathbf{a} = (s,t)^T$ and $\boldsymbol{\eta} = (u,v)^T$ such that $\hat{T}_{(s,t)^T} = e^{\pi ist/d}e^{-is\ell\hat{p}}e^{it\ell\hat{q}}$ and

$$W_{\hat{\rho}}\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2\pi}\sum_{s,t\in\mathbb{Z}} \mathrm{Tr}\left(\hat{\rho}e^{i(sv-ut)\ell}e^{i\pi st}\hat{T}_{(s,t)^T}\right), \tag{5}$$

retrieving the definition of Ref. [52]. The multimode ZGW function Eq. (3) satisfies the modified Stratonovich-Weyl axioms [52–54] as in the single-mode case [[55] Sec. A]. Unlike in the case of the standard Wigner function defined for bosonic, i.e., continuous-variable (CV), systems, there is not a one-to-one correspondence between a state and its ZGW function [52].

Efficient simulation of infinitely squeezed GKP states—Here, we provide a weak simulation algorithm, i.e., a simulation algorithm that produces samples from the probability distribution corresponding to the measurement outcomes of the circuit shown in Fig. 1. The simulation method we introduce here only works for circuits initialized with states whose ZGW function is positive at any point,

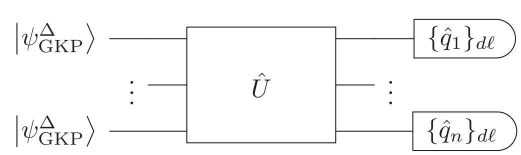

FIG. 1. Circuit class that we consider. It has arbitrary input GKP-encoded qudits with inverse squeezing parameter Δ. The evolution is given by unitaries with corresponding integer symplectic matrix and arbitrary displacement (which include all encoded Clifford operations). The measurements are homodyne measurements modulo $d\ell$.

such as infinitely squeezed stabilizer GKP states. However, we also later provide a method to estimate the probability distribution function for negative ZGW function.

We generalize to the multimode setting the equivalence established in Ref. [52] for a single mode between the ZGW function of a GKP state encoding an odd-dimensional qudit and the Gross Wigner function of the encoded qudit state. The Gross Wigner function [15] is defined for odd-dimensional discrete-variable (DV) states as

$$\bar{W}_{\hat{\rho}}(\mathbf{t}) = \mathrm{Tr}(\hat{\bar{A}}_{\mathbf{t}}\hat{\rho}) \tag{6}$$

in terms of the qudit phase point operator

$$\hat{\bar{A}}_{\mathbf{t}} = \frac{1}{d^n}\sum_{\mathbf{a}\in\mathbb{Z}_d^{2n}} \hat{\bar{T}}_{\mathbf{a}}\omega^{-[\mathbf{t},\mathbf{a}]}, \tag{7}$$

where $\mathbf{t}\in\mathbb{Z}_d^{2n}$ and

$$\hat{\bar{T}}_{\mathbf{a}} = \omega^{2^{-1}\mathbf{a_X}^T\mathbf{a_Z}}\hat{\bar{T}}_{\mathbf{a_X}}\hat{\bar{T}}_{\mathbf{a_Z}} \tag{8}$$

are the DV displacement operators [57]. The operators $\hat{\bar{T}}_{\mathbf{a_X}} = \hat{X}_1^{(\mathbf{a}_X)_1}\otimes\ldots\otimes\hat{X}^{(\mathbf{a_X})_n}$ and $\hat{\bar{T}}_{\mathbf{a_Z}} = \hat{Z}_1^{(\mathbf{a}_Z)_1}\otimes\ldots\otimes\hat{Z}^{(\mathbf{a_Z})_n}$ are defined in terms of the qudit Pauli $\hat{X}$ and $\hat{Z}$ displacements, respectively, using vectors $\mathbf{a_X},\mathbf{a_Z}\in\mathbb{Z}_d^n$ denoting the exponent of the displacement operator at the relevant qudit. Note that $\omega = e^{2\pi i/d}$ is the dth root of unity and $2^{-1} = (d+1)/2$ refers to the inverse in $\mathbb{Z}_d$ [52].

We now introduce a lemma and corollary relating the ZGW function of CV states to the Gross Wigner function of qudit states. We use the notation $\hat{\rho}_L(\mathbf{u}) = \hat{P}_{\mathbf{0}}\hat{T}_{-\mathbf{u}}\hat{\rho}\hat{T}_{\mathbf{u}}\hat{P}_{\mathbf{0}}$ for the CV state $\hat{\rho}$ after GKP error correction [58], whereby $\hat{P}_{\mathbf{0}} = \sum_j |j_{\mathrm{GKP}}\rangle\langle j_{\mathrm{GKP}}|$ is the projector onto the GKP computational subspace and $\mathbf{u}$ is in the domain $\mathbf{u}\in\mathbb{T}^2 = [0,1)^{\times 2}$. Note that after applying the error correction projector, the state is a perfectly encoded subnormalized qudit state $\hat{\bar{\rho}}(\mathbf{u})$, in the GKP basis. The resulting CV state can hence be expressed in terms of the qudit state as $\hat{\rho}_L(\mathbf{u}) = \sum_{j,k}\hat{\bar{\rho}}_{j,k}(\mathbf{u})|j_{\mathrm{GKP}}\rangle\langle k_{\mathrm{GKP}}|$.

Lemma 1 (theorem 2 of Ref. [52]): The ZGW function of a single-mode CV state $\hat{\rho}$ is expressed in terms of the Gross Wigner function as

$$W_{\hat{\rho}}(\boldsymbol{\eta}) = \bar{W}_{\hat{\bar{\rho}}(\mathbf{u})}(\mathbf{t}), \tag{9}$$

where $\boldsymbol{\eta} = \ell(\mathbf{u}+\mathbf{t})$, i.e., $\mathbf{u} = (1/\ell)\boldsymbol{\eta} \bmod 1 \in\mathbb{T}^2 = [0,1)^{\times 2}$ and $\mathbf{t} = (1/\ell)\boldsymbol{\eta} - \mathbf{u}\in\mathbb{Z}_d^2$.

Corollary 1: The ZGW function of a CV GKP state $\hat{\rho}_L$ perfectly encoding a DV state $\hat{\bar{\rho}}$ coincides with the Gross Wigner function of the encoded logical state $\hat{\bar{\rho}}$. Specifically,

$$W_{\hat{\rho}_L}(\boldsymbol{\eta}) \propto \sum_{\mathbf{a}\in\mathbb{Z}^2}\delta(\boldsymbol{\eta}-\ell\mathbf{a})\bar{W}_{\hat{\bar{\rho}}}(\mathbf{a}). \tag{10}$$

In Ref. [[55] Sec. B1], we provide an alternative proof to Lemma 1 to that given in Ref. [52]. In Ref. [[55] Sec. B2], we highlight a connection to the stabilizer subsystem decomposition [49,59]. Finally, in Ref. [[55] Sec. B3], we prove Corollary 1 via Lemma 1. This corollary informs us that positivity of the Gross Wigner function of a qudit state implies positivity of the ZGW function for the perfectly encoded CV state. Hence, all ideally encoded stabilizer GKP states have a positive ZGW function.

Next, we show that the evolution of CV states can be described using the ZGW function when restricting to symplectic operations described by integer matrices [60]. These include all encoded Clifford operations as shown in Ref. [[55] Sec. C1].

Theorem 1: Consider the unitary evolution of a state $\hat{\rho}$ under the action of a Gaussian unitary operation $\hat{U}_S$ described by an integer symplectic matrix S. The ZGW function of the evolved state is given by

$$W_{\hat{U}_S\hat{\rho}\hat{U}_S^\dagger}(\boldsymbol{\eta}) = W_{\hat{\rho}}(S\boldsymbol{\eta}-\mathbf{t}), \tag{11}$$

where $\mathbf{t}$ can be computed efficiently from S.

The proof of this theorem and an explanation of how to find $\mathbf{t}$ efficiently is given in Ref. [[55] Sec. C2]. Note that this also implies that if the ZGW function is initially positive, the positivity is preserved by integer symplectic operations, and in particular by all encoded Clifford operations. This is because the symplectic operation can be interpreted as a change of coordinates, hence simply transforming the positive ZGW distribution to another positive ZGW distribution.

We can also trivially account for all displacements due to the covariance property of the modified Stratonovich-Weyl axioms, i.e., $W_{\hat{T}_{\mathbf{b}}\hat{C}\hat{T}_{\mathbf{b}}^\dagger}(\boldsymbol{\eta}) = W_{\hat{C}}(\boldsymbol{\eta}+\ell\mathbf{b})$ for all $\mathbf{b}\in\mathbb{R}^{2n}$, as shown in Ref. [[55] Sec. A]. Therefore, we can consider operations of the form $\hat{T}_{\mathbf{c}}$ for all $\mathbf{c}\in\mathbb{R}^{2n}$. We also note that $\hat{U}_{S,\mathbf{c}} \equiv \hat{T}_{\mathbf{c}}\hat{U}_S = \hat{U}_S\hat{T}_{S\mathbf{c}}$ [61]; therefore we can consider any integer symplectic operations interlaced with arbitrary real displacements.

Finally, we consider modular measurements in the position basis $\hat{M}_Z(\mathbf{s}) = (1/d\ell)\sum_{\mathbf{n}} e^{-i\ell\mathbf{s}\cdot\mathbf{n}}\hat{T}_{(0,\mathbf{n})}$, which is equivalent to measuring $\hat{\mathbf{q}} \bmod d\ell$. Given perfectly encoded GKP states, the measurement values of $\mathbf{s}$ can be used to infer the logical computational basis information, yielding encoded Pauli measurements. In other words, for the GKP encoding, this is equivalent to measuring the logical operator $\hat{Z}_L^{\otimes n}$. Such modular measurements can be implemented either by coupling the mode to an auxiliary ideal GKP state [44] or by performing phase estimation with an auxiliary two-level system [62–64]. Alternatively, if the circuit is nonadaptive, the value of modular measurements can be inferred directly from standard homodyne measurements by taking the value of the output modulo $d\ell$. See Ref. [[55] Sec. C3] for more details on modular measurements.

To provide an algorithm that samples from the output probability distribution, we introduce the following theorem.

Theorem 2: The measurement of the logical operator $\hat{Z}_L^{\otimes n}$ represented by the projector $\hat{M}_Z(\mathbf{s})$ has a probability distribution function for the state $\hat{\rho}$ of the form

$$\mathrm{Tr}(\hat{\rho}\hat{M}_Z(\mathbf{s})) = \int \mathrm{d}\boldsymbol{\eta}_{\mathbf{X}} W_{\hat{\rho}}\begin{pmatrix} \boldsymbol{\eta}_{\mathbf{X}} \\ \mathbf{s} \end{pmatrix}. \tag{12}$$

This theorem is proven in Ref. [[55] Sec. C3], based on the fact that $W_{\hat{M}_Z(\mathbf{s})}(\boldsymbol{\eta}) \propto \delta(\boldsymbol{\eta}_{\mathbf{Z}} - \mathbf{s})$.

Therefore, for ideal stabilizer GKP states undergoing encoded Clifford operations and arbitrary displacements $\hat{U}_{S,\mathbf{c}}$, followed by modular measurements, we propose an efficient simulation algorithm (akin to those given in Refs. [34,35]) as follows. Consider the input state $\hat{\rho}_0 = |0_{\mathrm{GKP}}\rangle\langle 0_{\mathrm{GKP}}|^{\otimes n}$. Its ZGW function $W_{\hat{\rho}_0}(\boldsymbol{\eta})$ can be derived from the ZGW function of a single GKP state by noting that for the tensor product of two states $\hat{\rho}$ and $\hat{\sigma}$, we have [[55] Sec. D]

$$W_{\hat{\rho}\otimes\hat{\sigma}}(\boldsymbol{\eta}) = W_{\hat{\rho}}(\boldsymbol{\eta}^{(1)}) W_{\hat{\sigma}}(\boldsymbol{\eta}^{(2)}), \tag{13}$$

as explicitly shown in [[55] Sec. D]. Sample a vector $\boldsymbol{\eta}$ from the effective probability distribution $W_{\hat{\rho}_0}(\boldsymbol{\eta})$. Next, transform the vector under the symplectic matrix S and the corresponding displacement vector $\mathbf{t}$ described in Ref. [[55] Sec. C2] as $S\boldsymbol{\eta} - \mathbf{t}$. Next, transform the vector by $\mathbf{c}$ according to the linear displacement operator such that the vector becomes $S\boldsymbol{\eta} - \mathbf{t} + \mathbf{c}$. Finally, we see from Eq. (12) that the measurement result is given by the second half of this vector.

This result is strictly included in the results provided in Refs. [48–50]. Indeed, we are dealing here with encoded Clifford operations, a subset of the Gaussian operations associated with rational symplectic matrices shown to be simulatable in [50]. Also, we are considering modular measurements, which can be inferred from the homodyne measurement in [50]. Finally, we are considering here a weaker notion of simulation, i.e., weak simulation, while strong simulation was addressed in [50]. However, the key novelty of the current method is that we are now in a position to tackle the simulation of realistic GKP states, a crucial advancement given their practical significance in experimental quantum computing and error correction.

Simulation with negative ZGW functions—We provide a simulation algorithm that reproduces the statistics of the circuit shown in Fig. 1 for arbitrary squeezing level Δ of the input GKP states. To do so, we consider a different type of simulation, in particular, estimating the probability of each outcome. To provide meaningful estimates of probabilities from the probability density function of the continuous outcomes of the modular measurement $\hat{M}_Z(\mathbf{s})$, we discretize the outcomes into a finite number of bins. Since the modular measurement outcome s_j lies within the interval $(0, \ell d]$, we partition this range into K bins. Let $s_{j,k} = k\ell d/K$ represent the endpoints of these bins and define the corresponding discretized projection operator as

$$\hat{M}_{\mathbf{z}} = \int_{s_{1,z_1}}^{s_{1,z_1}+\ell d/K} \cdots \int_{s_{m,z_m}}^{s_{m,z_m}+\ell d/K} \mathrm{d}\mathbf{x} M_Z(\mathbf{x}), \tag{14}$$

where $\mathbf{z} \in \mathbb{Z}_K^m$ is an m vector corresponding to the choice of bin in each mode for each of the $m \leq n$ measured modes. Next, consider measurements of $\hat{M}_{\mathbf{z}}$ for each $\mathbf{z}$. The probabilities of measuring each projector $\hat{M}_{\mathbf{z}}$ are [[55] Sec. C3]

$$p_{\mathbf{z}} = \mathrm{Tr}(\hat{\rho}\hat{M}_{\mathbf{z}}) = \int \mathrm{d}\boldsymbol{\eta}\, W_{\hat{\rho}}(\boldsymbol{\eta}) W_{\hat{M}_{\mathbf{z}}}(\boldsymbol{\eta}). \tag{15}$$

The probability distribution can be estimated using the procedure of Ref. [16]. We first sample from the evolved ZGW quasiprobability distribution according to

$$\mathrm{Pr}(\boldsymbol{\eta}) = \frac{1}{\mathcal{M}_{\hat{U}_{S,\mathbf{c}}\hat{\rho}_0\hat{U}_{S,\mathbf{c}}^\dagger}} |W_{\hat{U}_{S,\mathbf{c}}\hat{\rho}_0\hat{U}_{S,\mathbf{c}}^\dagger}(\boldsymbol{\eta})|, \tag{16}$$

where the negativity $\mathcal{M}_{\hat{\rho}}$ of an arbitrary state $\hat{\rho}$, or *negative volume* of the ZGW function [52], is defined as

$$\mathcal{M}_{\hat{\rho}} = \int \mathrm{d}\boldsymbol{\eta} |W_{\hat{\rho}}(\boldsymbol{\eta})|. \tag{17}$$

Note that Eq. (13) implies that the negativity is multiplicative. Also, note that encoded Clifford operations do not change the total amount of negativity. Furthermore, we can use the property of covariance under Clifford operations to identify $|W_{\hat{U}_{S,\mathbf{c}}\hat{\rho}_0\hat{U}_{S,\mathbf{c}}^\dagger}(\boldsymbol{\eta})| = |W_{\hat{\rho}_0}(S\boldsymbol{\eta} - \mathbf{t} + \mathbf{c})|$. Therefore, we can sample from the distribution in Eq. (16) by first choosing $\boldsymbol{\eta}$ from

$$\mathrm{Pr}_0(\boldsymbol{\eta}) = \frac{1}{\mathcal{M}_{\hat{\rho}_0}} |W_{\hat{\rho}_0}(\boldsymbol{\eta})| \tag{18}$$

and then evolving the sample vector under the action $\boldsymbol{\eta} \mapsto S\boldsymbol{\eta} - \mathbf{t} + \mathbf{c}$, as done for the case of infinitely squeezed states. This is equivalent to sampling from the distribution in Eq. (16). Based on a single sample, we estimate the probability density function as

$$\tilde{p}_{\mathbf{z}}(\boldsymbol{\eta}) = \mathcal{M}_{\hat{\rho}_0} \mathrm{sign}(W_{\hat{\rho}_0}(S\boldsymbol{\eta} - \mathbf{t} + \mathbf{c})) W_{\hat{M}_{\mathbf{z}}}(\boldsymbol{\eta}). \tag{19}$$

The expectation value of this estimator is given by

$$\begin{aligned} \langle \tilde{p}_{\mathbf{z}} \rangle &= \int \mathrm{d}\boldsymbol{\eta} \tilde{p}_{\mathbf{z}}(\boldsymbol{\eta}) \,\mathrm{Pr}(\boldsymbol{\eta}) \\ &= \int \mathrm{d}\boldsymbol{\eta}\, \mathrm{sign}(W_{\hat{\rho}}(\boldsymbol{\eta})) |W_{\hat{\rho}}(\boldsymbol{\eta})| W_{\hat{M}_{\mathbf{z}}}(\boldsymbol{\eta}) \\ &= \int \mathrm{d}\boldsymbol{\eta}\, W_{\hat{\rho}_0}(S\boldsymbol{\eta} - \mathbf{t} + \mathbf{c}) W_{\hat{M}_{\mathbf{z}}}(\boldsymbol{\eta}). \end{aligned} \tag{20}$$

Our probability estimator takes values between $\pm\mathcal{M}_{\hat{\rho}_0}$, i.e., $\tilde{p}_{\mathbf{z}} \in [-\mathcal{M}_{\hat{\rho}_0}, \mathcal{M}_{\hat{\rho}_0}]$, and therefore the Hoeffding's inequality [65] implies that we require

$$N = \frac{2}{\epsilon^2}\mathcal{M}^2_{\hat{\rho}_0}\log(2/\delta) \tag{21}$$

samples to obtain an approximation of the probability $p_{\mathbf{z}}$ with additive error ϵ and success probability $1-\delta$. Therefore, the computational complexity of the simulation algorithm scales quadratically in the negative volume of the multimode ZGW function. Specifically, in the case of identical inputs with negative ZGW function $\hat{\rho}_0 = \hat{\rho}_{\mathrm{in}}^{\otimes n}$, the number of samples required grows exponentially with the size of the circuit as $\mathcal{M}_{\hat{\rho}_0} = \mathcal{M}^n_{\hat{\rho}_{\mathrm{in}}}$, by virtue of Eq. (13).

Simulation of realistic GKP states—Realistic GKP states $|\psi^{\Delta}_{\mathrm{GKP},j}\rangle$ with Gaussian peaks and a Gaussian envelope over the position basis are described by the wave function

$$\psi^{\Delta}_{\mathrm{GKP},j}(x) \propto \sum_{k\in\mathbb{Z}} e^{-\frac{1}{2}\Delta^2(j\ell+d\ell k)^2} e^{-\frac{1}{2\Delta^2}(x-j\ell-d\ell k)^2}, \tag{22}$$

where Δ determines the squeezing of the peaks; the lower Δ, the larger the squeezing. Such states are described by ZGW functions with negativities [52]; hence, they must be simulated nonefficiently with the algorithm that we have just described. Here, we explicitly evaluate the ZGW function of realistic GKP states and demonstrate that its negativity, and hence also the simulation time, scales inversely with the squeezing of the initial GKP basis states $|\psi^{\Delta}_{\mathrm{GKP},j}\rangle$. In other words, the more squeezed the input GKP basis states are, the faster the classical simulation—in contrast with previously addressed simulators for GKP circuits [41,42].

The ZGW function of a single realistic 0-logical GKP state is given by [[55] Sec. E]

$$W_{\mathrm{GKP}}\begin{pmatrix} u \\ v \end{pmatrix} \propto \vartheta(\Gamma; \mathbf{z}), \tag{23}$$

where $\mathbf{z} = [v/(d\ell), -u/(d\ell), 0, 0]^T$ and

$$\Gamma = \frac{1}{2}\begin{pmatrix} \frac{i}{d\Delta^2} & 1 & -\frac{i}{\Delta^2} & \frac{i}{\Delta^2} \\ 1 & \frac{i\Delta^2}{d} & 1 & 1 \\ -\frac{i}{\Delta^2} & 1 & \frac{i(2d\pi+4d\pi\Delta^4)}{2\pi\Delta^2} & -\frac{id}{\Delta^2} \\ \frac{i}{\Delta^2} & 1 & -\frac{id}{\Delta^2} & \frac{id(\pi+2\pi\Delta^4)}{\pi\Delta^2} \end{pmatrix}. \tag{24}$$

Note that ϑ is the Siegel theta function defined for an $m\times m$ matrix Γ and m-vector $\mathbf{z}$ as

$$\vartheta(\Gamma, \mathbf{z}) = \sum_{\mathbf{t}\in\mathbb{Z}^m} e^{i\pi\mathbf{t}^T\Gamma\mathbf{t}+2\pi i\mathbf{t}^T\mathbf{z}}. \tag{25}$$

In Fig. 2 we plot the ZGW logarithmic negativity $\log\mathcal{M}_{\hat{\rho}_0^{\Delta}}$, where log refers to the natural logarithm, with respect to the level of squeezing Δ of a single realistic 0-logical GKP qutrit state. This result has immediate implications for the simulation of circuits of practical interest. Let us consider, for example, the case of $\Delta = 0.25$, corresponding to 12 dB of squeezing [66] and representing a level of squeezing estimated to be necessary to achieve (for generic GKP states) fault-tolerant quantum computation [67]. Since $\mathcal{M}_{\hat{\rho}_0^{\Delta}} \approx e^{3\times10^{-4}}$ at this level of squeezing, and due to the multiplicativity of the negativity, simulating 1000 input modes requires only a minor overhead (less than twice as many samples) with respect to simulating one input mode with the same squeezing level. This scaling is orders of magnitude more efficient than existing methods in the literature. The total simulation cost will depend on additional factors, such as the size of the symplectic matrix S and the computation of the vector $\mathbf{t}$, as shown in Theorem 1. However, this extra cost will still scale polynomially with the number of modes.

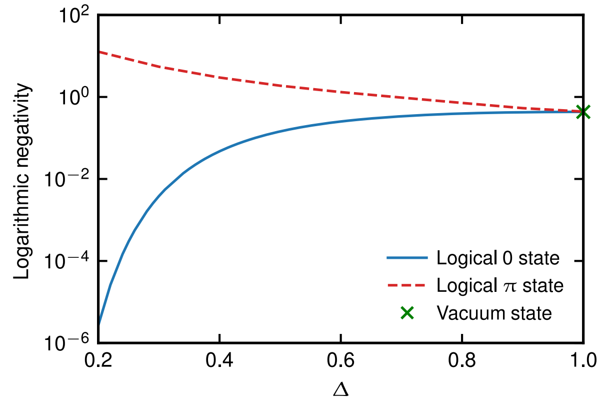

FIG. 2. ZGW logarithmic negativity for the 0-logical and π-logical GKP state at different levels of squeezing Δ. The cross represents the negativity of the vacuum state, coinciding with the negativity of the zero-squeezing limit ($\Delta \to 1$) of both 0- and π-logical GKP states.

We also plot the ZGW logarithmic negativity for the π state $|\psi_\pi\rangle = (1/\sqrt{3})(|0_L\rangle + |1_L\rangle - |2_L\rangle)$, which encodes a "magic" nonstabilizer state, enabling non-Clifford operations [68] (Fig. 2). As illustrated, significant negativity persists across all values of Δ, indicating a substantial overhead when simulating large circuits compared to smaller ones, as expected.

While we have explicitly considered the case of encoded Clifford operations, our method can also in principle tackle the simulation of circuits with encoded non-Clifford operations, as these can be implemented by injecting suitable nonstabilizer states, such as the π state $|\psi_\pi\rangle$ for qutrits. Such states can be pushed to the input of the computation, leaving the rest of the circuit as Clifford-only. Finally, although we have focussed here on finite squeezing, losses on GKP states have been shown to induce non-Pauli channels, and therefore could also increase the negativity of the Zak-Gross Wigner function [69].

Concluding remarks—In summary, we have introduced an algorithm for efficiently simulating GKP-encoded Clifford circuits acting on ideal stabilizer GKP states, leveraging the ZGW function. We then extended the algorithm to the practical setting of realistic, finitely squeezed GKP states, encoding arbitrary logical states. The complexity of the simulation is directly linked to the negativity of the ZGW function, thereby identifying the latter as a computational resource. For stabilizer GKP inputs, it decreases with increasing squeezing. By direct calculation, we have shown that this, in turn, implies only minor simulation overhead in the large squeezing regime. Therefore, our algorithm enables the simulation of large-scale circuits in the regime relevant to experiments targeting fault-tolerant quantum computing, positioning it as a potentially valuable tool for validating bosonic platforms designed for this purpose.

A relevant open question is whether our results can be extended to the case of GKP qubits. Furthermore, our framework has the potential to inspire novel techniques for simulating circuits based on bosonic codes beyond GKP. A promising direction involves the development of a comprehensive quasiprobability framework specifically tailored for bosonic codes. This could pave the way for versatile simulation algorithms, offering reduced computational costs compared to existing methods.

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Data availability—The data that support the findings of this Letter are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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Supplemental Materials: Classical simulation of circuits with realistic odd-dimensional Gottesman-Kitaev-Preskill states

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Appendix A: Modified Stratonovich-Weyl Axioms

Here, we extend the modified Stratonovich-Weyl axioms, as introduced in Ref. [1], to the case of multiple modes. Within the original, unmodified version of these axioms, there is a one-to-one correspondence between the Wigner function and the set of operators [1–3]. The weaker, modified axioms that were considered in Ref. [1] and that we consider here do not guarantee one-to-one correspondence. However, they do provide useful properties such as traciality that we will use to prove simulatability.

We will make use of the properties $\hat{D}(\mathbf{a}+\mathbf{b}) = \hat{D}(\mathbf{a})\hat{D}(\mathbf{b})e^{i[\mathbf{a},\mathbf{b}]/2}$. Hence we see that $\hat{D}(\mathbf{a})\hat{D}(\mathbf{b}) = \hat{D}(\mathbf{b})\hat{D}(\mathbf{a})$ when $e^{i[\mathbf{a},\mathbf{b}]/2} = e^{i[\mathbf{b},\mathbf{a}]/2}$. This is when $[\mathbf{a},\mathbf{b}] = 0 \mod 2\pi$. Furthermore, $\hat{T}_{\mathbf{a}}, T_{\mathbf{b}}$ commute when $[\mathbf{a},\mathbf{b}] = 0 \mod 2\pi/\ell^2$. In the following, we will also extensively use the fact that $\mathbf{a}_{\mathbf{X}}^T\mathbf{a}_{\mathbf{Z}} = \mathbf{a}_{\mathbf{Z}}^T\mathbf{a}_{\mathbf{X}}$ due to the fact that it is a dot product.

Definition 1. *The modified Stratonovich-Weyl axioms are [1]*

1. ***Linearity***. $W_{\hat{C}+\hat{D}}(\boldsymbol{\eta}) = W_{\hat{C}}(\boldsymbol{\eta}) + W_{\hat{D}}(\boldsymbol{\eta})$
2. ***Reality***. $W_{\hat{C}^\dagger}(\boldsymbol{\eta}) = (W_{\hat{C}}(\boldsymbol{\eta}))^*$
3. ***Standardization***. $\int \mathrm{d}\boldsymbol{\eta}\, W_{\hat{C}}(\boldsymbol{\eta}) = \mathrm{Tr}\left(\hat{C}\right)$
4. ***Covariance***. $W_{\hat{T}_{\mathbf{b}}\hat{C}\hat{T}_{\mathbf{b}}^\dagger}(\boldsymbol{\eta}) = W_{\hat{C}}(\boldsymbol{\eta}+\ell\mathbf{b})$ *for all* $\mathbf{b}\in\mathbb{R}^{2n}$.
5. ***Traciality***. $\int \mathrm{d}\boldsymbol{\eta}\, W_{\hat{C}}(\boldsymbol{\eta})W_{\hat{D}}(\boldsymbol{\eta}) = \mathrm{Tr}\left(\mathcal{E}(\hat{C})\hat{D}\right) = \mathrm{Tr}\left(\hat{C}\mathcal{E}(\hat{D})\right) = \mathrm{Tr}\left(\mathcal{E}(\hat{C})\mathcal{E}(\hat{D})\right)$, *for the twirling map*

$$\mathcal{E}(\hat{C}) = \int \mathrm{d}\mathbf{s}\hat{P}_{\mathbf{s}}\hat{C}\hat{P}_{\mathbf{s}}. \tag{A1}$$

where we define $\hat{P}_{\mathbf{s}} = \hat{T}_{\mathbf{s}}\hat{P}_{\mathbf{0}}\hat{T}_{-\mathbf{s}}$ *and* $\hat{P}_{\mathbf{0}}$ *is the GKP projector defined as*

$$\hat{P}_{\mathbf{0}} = \left(\sum_j |j_{GKP}\rangle\langle j_{GKP}|\right)^{\otimes n}. \tag{A2}$$

Lemma 1. *The multimode ZGW function satisfies the modified version of the Stratonovich–Weyl axioms.*

Proof. **Linearity**. The ZGW function satisfies linearity thanks to the linearity of the trace in its definition Eq. (3), i.e.,

$$W_{\hat{C}+\hat{D}}(\boldsymbol{\eta}) \propto \mathrm{Tr}\left((\hat{C}+\hat{D})\hat{A}_{\boldsymbol{\eta}}\right) = \mathrm{Tr}\left(\hat{C}\hat{A}_{\boldsymbol{\eta}}\right) + \mathrm{Tr}\left(\hat{D}\hat{A}_{\boldsymbol{\eta}}\right). \tag{A3}$$

However, it is not a one-to-one map since we can define different Wigner functions depending on the choice of d.

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Reality. We see this from the fact that we can substitute $\mathbf{a} \to -\mathbf{a}$ in the summation and that $\mathbf{a_X}^T\mathbf{a_Z}$ is always an integer. Hence,

$$\begin{aligned}
W_{\hat{C}^\dagger}(\boldsymbol{\eta}) =& \frac{1}{(2\pi)^n} \sum_{\mathbf{a}\in\mathbb{Z}^{2n}} \mathrm{Tr}\Big(\hat{C}^\dagger e^{i\ell[\mathbf{a},\boldsymbol{\eta}]+i\pi\mathbf{a_X}^T\mathbf{a_Z}}\hat{T}_{\mathbf{a}}\Big) \\
=& \frac{1}{(2\pi)^n} \sum_{\mathbf{a}\in\mathbb{Z}^{2n}} \mathrm{Tr}\Big(\hat{C}^\dagger e^{-i\ell[\mathbf{a},\boldsymbol{\eta}]-i\pi\mathbf{a_X}^T\mathbf{a_Z}}\hat{T}_{-\mathbf{a}}\Big) \\
=& \frac{1}{(2\pi)^n} \sum_{\mathbf{a}\in\mathbb{Z}^{2n}} \mathrm{Tr}\bigg(\Big(\Big(e^{i\ell[\mathbf{a},\boldsymbol{\eta}]+i\pi\mathbf{a_X}^T\mathbf{a_Z}}\hat{T}_{\mathbf{a}}\hat{C}\Big)^T\Big)^*\bigg) \\
=& \frac{1}{(2\pi)^n} \sum_{\mathbf{a}\in\mathbb{Z}^{2n}} \mathrm{Tr}\Big(e^{i\ell[\mathbf{a},\boldsymbol{\eta}]+i\pi\mathbf{a_X}^T\mathbf{a_Z}}\hat{T}_{\mathbf{a}}\hat{C}\Big)^* \\
=& \frac{1}{(2\pi)^n} \sum_{\mathbf{a}\in\mathbb{Z}^{2n}} \mathrm{Tr}\Big(\hat{C}e^{i\ell[\mathbf{a},\boldsymbol{\eta}]+i\pi\mathbf{a_X}^T\mathbf{a_Z}}\hat{T}_{\mathbf{a}}\Big)^* \\
=& \big(W_{\hat{C}}(\boldsymbol{\eta})\big)^*
\end{aligned} \tag{A4}$$

where we substituted $\mathbf{a}$ for $-\mathbf{a}$ in the second line, we rewrite the third line using the Hermitian conjugate, then in the fourth line, we use that the trace of the transpose of an operator is equal to the trace of the operator.

Standardization. We see that

$$\begin{aligned}
\int_{\mathbb{T}^{2n}_{d\ell}} \mathrm{d}\boldsymbol{\eta}\, W_{\hat{\rho}}(\boldsymbol{\eta}) =& \frac{1}{(2\pi)^n}\int_{\mathbb{T}^{2n}_{d\ell}} \mathrm{d}\boldsymbol{\eta} \sum_{\mathbf{a}\in\mathbb{Z}^{2n}} \mathrm{Tr}\Big(\hat{\rho}e^{i\ell[\mathbf{a},\boldsymbol{\eta}]+i\pi\mathbf{a_X}^T\mathbf{a_Z}}\hat{T}_{\mathbf{a}}\Big) \\
=& \sum_{\mathbf{a}\in\mathbb{Z}^{2n}} \delta_{\mathbf{a},\mathbf{0}}\,\mathrm{Tr}\Big(\hat{\rho}e^{i\pi\mathbf{a_X}^T\mathbf{a_Z}}\hat{T}_{\mathbf{a}}\Big) \\
=& \mathrm{Tr}(\hat{\rho})
\end{aligned} \tag{A5}$$

where we have used that

$$\begin{aligned}
\frac{1}{(2\pi)^n}\int \mathrm{d}_{\mathbb{T}^{2n}_{d\ell}} d\boldsymbol{\eta} e^{i\ell[\mathbf{a},\boldsymbol{\eta}]} =& \frac{1}{(2\pi)^n}\int \mathrm{d}_{\mathbb{T}^{2n}_{2\pi/\ell}} d\boldsymbol{\eta} e^{i\ell\boldsymbol{\eta}^T\Omega\mathbf{a}} \\
=& \frac{1}{(2\pi)^n}\int \mathrm{d}_{\mathbb{T}^{2n}_{2\pi}} d\boldsymbol{\eta} e^{i\mathbf{a}^T\Omega\boldsymbol{\eta}} \\
=& \delta_{\mathbf{a}^T\Omega,\mathbf{0}} \\
=& \delta_{\mathbf{a},\mathbf{0}}.
\end{aligned} \tag{A6}$$

Covariance. For any choice of $\mathbf{b} \in \mathbb{R}^{2n}$ we have

$$\begin{aligned}
W_{\hat{T}_{\mathbf{b}}\hat{\rho}\hat{T}^\dagger_{\mathbf{b}}}(\boldsymbol{\eta}) =& \frac{1}{(2\pi)^n} \sum_{\mathbf{a}\in\mathbb{Z}^{2n}} \mathrm{Tr}\Big(\hat{T}_{\mathbf{b}}\hat{\rho}\hat{T}^\dagger_{\mathbf{b}} e^{i\ell[\mathbf{a},\boldsymbol{\eta}]+i\pi\mathbf{a_X}^T\mathbf{a_Z}}\hat{T}_{\mathbf{a}}\Big) \\
=& \frac{1}{(2\pi)^n} \sum_{\mathbf{a}\in\mathbb{Z}^{2n}} \mathrm{Tr}\Big(\hat{\rho} e^{i\ell[\mathbf{a},\boldsymbol{\eta}]+i\pi\mathbf{a_X}^T\mathbf{a_Z}}\hat{T}^\dagger_{\mathbf{b}}\hat{T}_{\mathbf{a}}\hat{T}_{\mathbf{b}}\Big) \\
=& \frac{1}{(2\pi)^n} \sum_{\mathbf{a}\in\mathbb{Z}^{2n}} \mathrm{Tr}\Big(\rho e^{i\ell[\mathbf{a},\boldsymbol{\eta}]+i\pi\mathbf{a_X}^T\mathbf{a_Z}} e^{2\pi i[\mathbf{a},\mathbf{b}]/d}\hat{T}_{\mathbf{a}}\hat{T}^\dagger_{\mathbf{b}}\hat{T}_{\mathbf{b}}\Big) \\
=& \frac{1}{(2\pi)^n} \sum_{\mathbf{a}\in\mathbb{Z}^{2n}} \mathrm{Tr}\Big(\hat{\rho} e^{i\ell[\mathbf{a},\boldsymbol{\eta}+\mathbf{b}\ell]+i\pi\mathbf{a_X}^T\mathbf{a_Z}}\hat{T}_{\mathbf{a}}\Big) \\
=& W_{\hat{\rho}}(\boldsymbol{\eta}+\ell\mathbf{b}).
\end{aligned} \tag{A7}$$

Note that this proves covariance under continuous displacements but not under the action of Clifford operations, which will be shown later.

Traciality. We wish to demonstrate that

$$\int \mathrm{d}\boldsymbol{\eta}\, W_{\hat{C}}(\boldsymbol{\eta})W_{\hat{D}}(\boldsymbol{\eta}) = \mathrm{Tr}\Big(\mathcal{E}(\hat{C})\hat{D}\Big) = \mathrm{Tr}\Big(\hat{C}\mathcal{E}(\hat{D})\Big) = \mathrm{Tr}\Big(\mathcal{E}(\hat{C})\mathcal{E}(\hat{D})\Big). \tag{A8}$$

First, we will prove that

$$\mathrm{Tr}\Big(\mathcal{E}(\hat{C})\hat{D}\Big) = \mathrm{Tr}\Big(\hat{C}\mathcal{E}(\hat{D})\Big). \tag{A9}$$

We have

$$\begin{aligned}\mathrm{Tr}\Big(\mathcal{E}(\hat{C})\hat{D}\Big) &= \mathrm{Tr}\Big(\int \mathrm{d}\mathbf{s}\hat{P}_{\mathbf{s}}\hat{C}\hat{P}_{\mathbf{s}}\hat{D}\Big)\\ &= \int \mathrm{d}\mathbf{s}\,\mathrm{Tr}\Big(\hat{P}_{\mathbf{s}}\hat{C}\hat{P}_{\mathbf{s}}\hat{D}\Big)\\ &= \int \mathrm{d}\mathbf{s}\,\mathrm{Tr}\Big(\hat{C}\hat{P}_{\mathbf{s}}\hat{D}\hat{P}_{\mathbf{s}}\Big)\\ &= \mathrm{Tr}\Big(\hat{C}\mathcal{E}(\hat{D})\Big).\end{aligned} \tag{A10}$$

We also have

$$\begin{aligned}\mathrm{Tr}\Big(\mathcal{E}(\hat{C})\mathcal{E}(\hat{D})\Big) &= \mathrm{Tr}\Big(\int \mathrm{d}\mathbf{s}\int \mathrm{d}\mathbf{s}'\hat{P}_{\mathbf{s}}\hat{C}\hat{P}_{\mathbf{s}}\hat{P}_{\mathbf{s}'}\hat{D}\hat{P}_{\mathbf{s}'}\Big)\\ &= \int \mathrm{d}\mathbf{s}\int \mathrm{d}\mathbf{s}'\,\mathrm{Tr}\Big(\hat{P}_{\mathbf{s}'}\hat{P}_{\mathbf{s}}\hat{C}\hat{P}_{\mathbf{s}}\hat{P}_{\mathbf{s}'}\hat{D}\Big)\\ &= \int \mathrm{d}\mathbf{s}\,\mathrm{Tr}\Big(\hat{P}_{\mathbf{s}}\hat{C}\hat{P}_{\mathbf{s}}\hat{D}\Big)\\ &= \mathrm{Tr}\Big(\mathcal{E}(\hat{C})\hat{D}\Big)\end{aligned} \tag{A11}$$

where we have used that $\hat{P}_{\mathbf{s}}\hat{P}_{\mathbf{s}'} = \delta_{\mathbf{s},\mathbf{s}'}\hat{P}_{\mathbf{s}}$. Next, we use Lemma 1 which gives an expression for $W_{\hat{\rho}}(\boldsymbol{\eta}) = \bar{W}_{\hat{\rho}_L(\mathbf{u})}(\mathbf{t})$. Therefore, we can write

$$\begin{aligned}\int \mathrm{d}\boldsymbol{\eta}\, W_{\hat{C}}(\boldsymbol{\eta})\hat{A}_{\boldsymbol{\eta}} &= \int \mathrm{d}\mathbf{u}\sum_{\mathbf{t}\in\mathbb{Z}^{2n}} \bar{W}_{\hat{C}_L(\mathbf{u})}(\mathbf{t})A_{\mathbf{u}+\mathbf{t}}\\ &= \int \mathrm{d}\mathbf{u}\sum_{\mathbf{t}\in\mathbb{Z}^{2n}} \bar{W}_{\hat{C}_L(\mathbf{u})}(\mathbf{t})T_{\mathbf{u}}\hat{T}_{\mathbf{t}}A_{\mathbf{0}}\hat{T}_{-\mathbf{t}}\hat{T}_{-\mathbf{u}}\\ &= \int \mathrm{d}\mathbf{u}T_{\mathbf{u}}\hat{C}_L(\mathbf{u})T_{-\mathbf{u}}\\ &=\mathcal{E}(\hat{C}),\end{aligned} \tag{A12}$$

which allows us to derive the main result, i.e.,

$$\begin{aligned}\int \mathrm{d}\boldsymbol{\eta}W_{\hat{C}}(\boldsymbol{\eta})W_{\hat{D}}(\boldsymbol{\eta}) &= \int \mathrm{d}\boldsymbol{\eta}W_{\hat{C}}(\boldsymbol{\eta})\,\mathrm{Tr}\Big(\hat{D}\hat{A}_{\boldsymbol{\eta}}\Big)\\ &= \mathrm{Tr}\Big(\hat{D}\int \mathrm{d}\boldsymbol{\eta}W_{\hat{C}}(\boldsymbol{\eta})\hat{A}_{\boldsymbol{\eta}}\Big)\\ &= \mathrm{Tr}\Big(\hat{D}\mathcal{E}(\hat{C})\Big).\end{aligned} \tag{A13}$$

□

Appendix B: Connection to Gross Wigner function

As an example, we begin by calculating the Gross Wigner function of the single-qubit computational basis state $|j\rangle$ as

$$\begin{aligned}\bar{W}_{|j\rangle\langle j|}(\mathbf{t}) =& \frac{1}{d}\sum_{\mathbf{a}\in\mathbb{Z}_d^2}\omega^{2^{-1}a_X a_Z}\omega^{-(t_X a_Z - t_Z a_X)}\,\mathrm{Tr}\left(\hat{\bar{T}}_{a_X}\hat{\bar{T}}_{a_Z}\,|j\rangle\langle j|\right)\\ =& \frac{1}{d}\sum_{\mathbf{a}\in\mathbb{Z}_d^2}\omega^{2^{-1}a_X a_Z}\omega^{-(t_X a_Z - t_Z a_X)}\omega^{j a_Z}\,\mathrm{Tr}\left(|j+a_X\rangle\langle j|\right)\\ =& \frac{1}{d}\sum_{\mathbf{a}\in\mathbb{Z}_d^2}\omega^{2^{-1}a_X a_Z}\omega^{-(t_X a_Z - t_Z a_X)}\omega^{j a_Z}\delta_{a_X,0}\\ =& \frac{1}{d}\sum_{a_Z\in\mathbb{Z}_d}\omega^{-t_X a_Z}\omega^{j a_Z}\\ =& \delta_{t_X,j},\end{aligned} \tag{B1}$$

where we use Eq. (8) in the first line.

The DV parity operator is defined as $\hat{\bar{\Pi}}_d = (\sum_{j=0}^{d-1}|-j\rangle\langle j|)^{\otimes n}$. We define the logical parity operator as $\hat{\Pi}_d^L = \left(\sum_{j=0}^{d-1}|-j_{\mathrm{GKP}}\rangle\langle j_{\mathrm{GKP}}|\right)^{\otimes n}$.

Lemma 2. *The phase point operator $A_{\boldsymbol{\eta}}$ at $\boldsymbol{\eta}=\mathbf{0}$ coincides with the logical DV parity operator.*

Proof. The phase point operator defined in Eq. (4), i.e.,

$$\hat{A}_{\boldsymbol{\eta}} = \frac{1}{(2\pi)^n}\sum_{\mathbf{a}\in\mathbb{Z}^{2n}}e^{i\ell[\mathbf{a},\boldsymbol{\eta}]+i\pi\mathbf{a_X}^T\mathbf{a_Z}}\hat{T}_{\mathbf{a}}, \tag{B2}$$

at $\boldsymbol{\eta}=\mathbf{0}$ is given by

$$\hat{A}_{\mathbf{0}} = \frac{1}{(2\pi)^n}\sum_{\mathbf{a}\in\mathbb{Z}^{2n}}e^{i\pi\mathbf{a_X}^T\mathbf{a_Z}}\hat{T}_{\mathbf{a}}. \tag{B3}$$

We compute its action at $\boldsymbol{\eta}=\mathbf{0}$ on $|\mathbf{x}\rangle$

$$\begin{aligned}A_{\mathbf{0}}|\mathbf{x}\rangle =& \frac{1}{(2\pi)^n}\sum_{\mathbf{a}\in\mathbb{Z}^{2n}}e^{i\pi\mathbf{a_X}^T\mathbf{a_Z}}\hat{T}_{\mathbf{a}}|\mathbf{x}\rangle\\ =& \frac{1}{(2\pi)^n}\sum_{\mathbf{a}\in\mathbb{Z}^{2n}}e^{i\pi\mathbf{a_X}^T\mathbf{a_Z}}e^{i\mathbf{a_X}^T\mathbf{a_Z}\ell^2/2}\hat{T}_{\mathbf{a_X}}\hat{T}_{\mathbf{a_Z}}|\mathbf{x}\rangle\\ =& \frac{1}{(2\pi)^n}\sum_{\mathbf{a}\in\mathbb{Z}^{2n}}e^{\pi i(d+1)\mathbf{a_Z}^T\mathbf{a_X}/d}e^{i\ell\mathbf{a_Z}^T\mathbf{x}}|\mathbf{x}+\ell\mathbf{a_X}\rangle,\end{aligned} \tag{B4}$$

where we have used Eq. (2) in the second line and that $\hat{T}_{\mathbf{a_Z}}|\mathbf{x}\rangle = e^{i\ell\mathbf{a_Z}^T\mathbf{x}}|\mathbf{x}\rangle$ and that $\ell^2/2=\pi/d$ in the third line.

Now, we rewrite the integer vector $\mathbf{a}=\mathbf{u}+d\mathbf{t}$ in terms of a part that is a multiple of d and a remainder part where $\mathbf{u}\in\mathbb{Z}_d^{2n}$

and $\mathbf{t} \in \mathbb{Z}^{2n}$. We also use the definition of $\omega = e^{2\pi i/d}$ such that

$$
\begin{aligned}
\hat{A}_{\mathbf{0}} \left|\mathbf{x}\right\rangle =& \frac{1}{(2\pi)^n} \sum_{\mathbf{u}\in\mathbb{Z}_d^{2n}} \sum_{\mathbf{t}\in\mathbb{Z}^{2n}} \omega^{\frac{(d+1)}{2}(\mathbf{u}+d\mathbf{t})_Z^T(\mathbf{u}+d\mathbf{t})_X} e^{i\ell(\mathbf{u}+d\mathbf{t})_Z^T\mathbf{x}} \left|\mathbf{x}+\ell(\mathbf{u}+d\mathbf{t})_X\right\rangle \\
=& \frac{1}{(2\pi)^n} \sum_{\mathbf{u}\in\mathbb{Z}_d^{2n}} \sum_{\mathbf{t}\in\mathbb{Z}^{2n}} \omega^{2^{-1}(\mathbf{u}+d\mathbf{t})_Z^T(\mathbf{u}+d\mathbf{t})_X} e^{i\ell(\mathbf{u}+d\mathbf{t})_Z^T\mathbf{x}} \left|\mathbf{x}+\ell(\mathbf{u}+d\mathbf{t})_X\right\rangle \\
=& \frac{1}{(2\pi)^n} \sum_{\mathbf{u}\in\mathbb{Z}_d^{2n}} \sum_{\mathbf{t}\in\mathbb{Z}^{2n}} \omega^{2^{-1}\mathbf{u}_Z^T\mathbf{u}_X} e^{i\ell\mathbf{u}_Z^T\mathbf{x}} e^{i\ell d\mathbf{t}_Z^T\mathbf{x}} \left|\mathbf{x}+\ell(\mathbf{u}+d\mathbf{t})_X\right\rangle \\
=& \frac{1}{(2\pi)^n} \sum_{\mathbf{u}\in\mathbb{Z}_d^{2n}} \sum_{\mathbf{t}\in\mathbb{Z}^{2n}} \omega^{2^{-1}\mathbf{u}_Z^T\mathbf{u}_X} e^{i\ell\mathbf{u}_Z^T\mathbf{x}} e^{2\pi i\mathbf{t}_Z^T\mathbf{x}/\ell} \left|\mathbf{x}+\ell(\mathbf{u}+d\mathbf{t})_X\right\rangle \\
=& \frac{\ell^n}{(2\pi)^n} \sum_{\mathbf{u}\in\mathbb{Z}_d^{2n}} \sum_{\mathbf{t}\in\mathbb{Z}^{2n}} \omega^{2^{-1}\mathbf{u}_Z^T\mathbf{u}_X} e^{i\ell\mathbf{u}_Z^T\mathbf{x}} \delta(\mathbf{x}-\ell\mathbf{t}_\mathbf{Z}) \left|\mathbf{x}+\ell(\mathbf{u}+d\mathbf{t})_X\right\rangle ,
\end{aligned}
\tag{B5}
$$

whereby in the third line we have used that $\omega^{(d+1)da/2} = 1$ for any choice of $a \in \mathbb{Z}$. Hence, the expression is only non-zero when $\mathbf{x} = \ell\mathbf{t}_\mathbf{Z}$. Hence, given that ${\mathbf{u}_\mathbf{Z}}^T\mathbf{u}_\mathbf{X}$ and ${\mathbf{u}_\mathbf{Z}}^T\mathbf{t}_\mathbf{Z}$ are integers we have

$$
\begin{aligned}
\hat{A}_{\mathbf{0}} \left|\ell\mathbf{t}_\mathbf{Z}\right\rangle =& \frac{1}{(\ell d)^n} \sum_{\mathbf{u}\in\mathbb{Z}_d^{2n}} \sum_{\mathbf{t}\in\mathbb{Z}^{2n}} \omega^{\mathbf{u}_Z^T(2^{-1}\mathbf{u}_X+\mathbf{t}_\mathbf{Z})} \left|\ell\mathbf{t}_\mathbf{Z}+\ell(\mathbf{u}+d\mathbf{t})_X\right\rangle \\
=& \frac{1}{(\ell)^n} \sum_{\mathbf{u}_\mathbf{X}\in\mathbb{Z}_d^{n}} \sum_{\mathbf{t}\in\mathbb{Z}^{2n}} \delta_{\mathbf{u}_\mathbf{X},-2\mathbf{t}_\mathbf{Z}} \left|\ell\mathbf{t}_\mathbf{Z}+\ell(\mathbf{u}+d\mathbf{t})_X\right\rangle \\
=& \frac{1}{\ell^n} \sum_{\mathbf{t}_\mathbf{X},\mathbf{t}_\mathbf{Z}\in\mathbb{Z}^n} \left|\ell(d\mathbf{t}_\mathbf{X}-\mathbf{t}_\mathbf{Z})\right\rangle .
\end{aligned}
\tag{B6}
$$

This is the same as the action of the logical parity operator on $\left|\mathbf{x}\right\rangle$, which can be seen by inspecting

$$
\begin{aligned}
\hat{\Pi}_d^L \left|\mathbf{x}\right\rangle =& \left(\sum_{j=0}^{d-1} \left|-j_{\text{GKP}}\right\rangle \left\langle j_{\text{GKP}}\right|\right)^{\otimes n} \left|\mathbf{x}\right\rangle \\
=& \bigotimes_{k=1}^{n} \sum_{j=0}^{d-1} \left|-j_{\text{GKP}}\right\rangle \left\langle j_{\text{GKP}} | x_k\right\rangle \\
=& \bigotimes_{k=1}^{n} \sum_{j=0}^{d-1} \sum_{n,n'} \left|(-j+dn')\ell\right\rangle \left\langle (j+dn)\ell | x_k\right\rangle \\
=& \bigotimes_{k=1}^{n} \sum_{j=0}^{d-1} \sum_{n,n'} \left|(-j+dn')\ell\right\rangle \delta(x_k-(j+dn)\ell).
\end{aligned}
\tag{B7}
$$

Hence, this expression is only non-zero when $x_k = \ell t_k$ for some choice of $t_k \in \mathbb{Z}$. We therefore have

$$
\begin{aligned}
\hat{\Pi}_d^L \left|\mathbf{x}\right\rangle =& \bigotimes_{k=1}^{n} \sum_{j=0}^{d-1} \sum_{n,n'} \sum_{t_k} \left|(-j+dn')\ell\right\rangle \delta(\ell t_k-(j+dn)\ell)\delta(x_k-\ell t_k) \\
=& \bigotimes_{k=1}^{n} \sum_{n,n'} \sum_{t_k} \left|(-t_k+dn+dn')\ell\right\rangle \delta(x_k-\ell t_k).
\end{aligned}
\tag{B8}
$$

Therefore, if we limit to the non-zero case where $\mathbf{x} = \ell\mathbf{t}_\mathbf{Z}$ and we substitute the vector $\mathbf{n}+\mathbf{n}'$ with $\mathbf{t}_\mathbf{X}$ we find

$$
\begin{aligned}
\hat{\Pi}_d^L \left|\ell\mathbf{t}_\mathbf{Z}\right\rangle =& \sum_{\mathbf{t}_\mathbf{X}\in\mathbb{Z}^n} \bigotimes_{k=1}^{n} \left|(-(\mathbf{t}_\mathbf{Z})_k+d(\mathbf{t}_\mathbf{X})_k)\ell\right\rangle \\
=& \sum_{\mathbf{t}_\mathbf{X}\in\mathbb{Z}^n} \left|(-\mathbf{t}_\mathbf{Z}+d\mathbf{t}_\mathbf{X})\ell\right\rangle .
\end{aligned}
\tag{B9}
$$

Hence $\hat{A}_{\mathbf{0}} = \hat{\Pi}_d^L$. □

1. Proof of Lemma 1

We now prove Lemma 1. We provide an alternative proof to that given in Ref. [1]. First, note that we can write

$$\begin{aligned}\hat{T}_{\boldsymbol{\eta}/\ell}\hat{A}_{\mathbf{0}}\hat{T}_{-\boldsymbol{\eta}/\ell} =&\frac{1}{(2\pi)^n}\sum_{\mathbf{a}\in\mathbb{Z}^{2n}} e^{i\pi\mathbf{a_X}^T\mathbf{a_Z}}\hat{T}_{\boldsymbol{\eta}/\ell}\hat{T}_{\mathbf{a}}\hat{T}_{-\boldsymbol{\eta}/\ell}\\ =&\frac{1}{(2\pi)^n}\sum_{\mathbf{a}\in\mathbb{Z}^{2n}} e^{i\pi\mathbf{a_X}^T\mathbf{a_Z}}e^{i[\mathbf{a},\boldsymbol{\eta}/\ell]\ell^2}\hat{T}_{\mathbf{a}}\hat{T}_{\boldsymbol{\eta}/\ell}\hat{T}_{-\boldsymbol{\eta}/\ell}\\ =&\frac{1}{(2\pi)^n}\sum_{\mathbf{a}\in\mathbb{Z}^{2n}} e^{i\ell[\mathbf{a},\boldsymbol{\eta}]+i\pi\mathbf{a_X}^T\mathbf{a_Z}}\hat{T}_{\mathbf{a}}\\ =&\hat{A}_{\boldsymbol{\eta}}.\end{aligned} \tag{B10}$$

Proof. Note that from Lemma 3, we see that $\hat{P}_{\mathbf{0}}\hat{A}_{\mathbf{0}}\hat{P}_{\mathbf{0}} = \hat{A}_{\mathbf{0}}$, where $\hat{P}_{\mathbf{0}}$ is the logical GKP projection operator defined in Eq. (A2). Also, note that $\boldsymbol{\eta}\in\mathbb{T}_d^{2n} = [0,\ell d)^{\times 2n}$. We write $\boldsymbol{\eta} = \ell(\mathbf{u}+\mathbf{t})$ where $\mathbf{u} = \frac{1}{\ell}\boldsymbol{\eta} \mod 1 \in \mathbb{T}^{2n} = [0,1)^{\times 2n}$ and $\mathbf{t} = \frac{1}{\ell}\boldsymbol{\eta} - \mathbf{u}\in\mathbb{Z}_d^{2n}$. Hence, we have that

$$\begin{aligned}W_{\hat{\rho}}(\boldsymbol{\eta}) =&\mathrm{Tr}(\hat{T}_{\boldsymbol{\eta}/\ell}\hat{A}_{\mathbf{0}}\hat{T}_{-\boldsymbol{\eta}/\ell}\hat{\rho})\\ =&\mathrm{Tr}(\hat{T}_{\mathbf{u}+\mathbf{t}}\hat{A}_{\mathbf{0}}\hat{T}_{-(\mathbf{u}+\mathbf{t})}\hat{\rho})\\ =&\mathrm{Tr}(\hat{T}_{\mathbf{u}}\hat{T}_{\mathbf{t}}\hat{A}_{\mathbf{0}}\hat{T}_{-\mathbf{t}}\hat{T}_{-\mathbf{u}}\hat{\rho})\\ =&\mathrm{Tr}(\hat{T}_{\mathbf{u}}\hat{T}_{\mathbf{t}}\hat{P}_{\mathbf{0}}\hat{A}_{\mathbf{0}}\hat{P}_{\mathbf{0}}\hat{T}_{-\mathbf{t}}\hat{T}_{-\mathbf{u}}\hat{\rho}).\end{aligned} \tag{B11}$$

Next, we use that $[\hat{P}_{\mathbf{0}},\hat{T}_{\mathbf{t}}] = 0$ for $\mathbf{t}\in\mathbb{Z}^n$. Hence,

$$\begin{aligned}W_{\hat{\rho}}(\boldsymbol{\eta}) =&\mathrm{Tr}(\hat{T}_{\mathbf{t}}\hat{A}_{\mathbf{0}}\hat{T}_{-\mathbf{t}}\hat{P}_{\mathbf{0}}\hat{T}_{-\mathbf{u}}\hat{\rho}\hat{T}_{\mathbf{u}}\hat{P}_{\mathbf{0}})\\ =&\mathrm{Tr}(\hat{T}_{\mathbf{t}}\hat{A}_{\mathbf{0}}\hat{T}_{-\mathbf{t}}\hat{\rho}_L(\mathbf{u}))\\ =&\mathrm{Tr}(\hat{A}_{\mathbf{t}}\hat{\rho}_L(\mathbf{u}))\\ =&\bar{W}_{\hat{\bar{\rho}}(\mathbf{u})}(\mathbf{t}),\end{aligned} \tag{B12}$$

where $\hat{\bar{\rho}}(\mathbf{u})$ is the qudit state that is perfectly encoded by the logical GKP state $\hat{\rho}_L(\mathbf{u})$. □

2. Connection to stabilizer subsystem decomposition

Note that the subsystem decomposition can be defined for qudits as [4, 5]

$$\hat{\rho}_L = \int_R \mathrm{d}\mathbf{u}\hat{\rho}_L(\mathbf{u}), \tag{B13}$$

over the region $R = [-\frac{1}{2},\frac{1}{2})^{\times 2}$. Hence, we see that the Gross Wigner function of the logical qudit subsystem decomposition state $\hat{\rho}_L$ derived from a general CV state $\hat{\rho}$ can be evaluated via the ZGW function as

$$\begin{aligned}\bar{W}_{\hat{\rho}_L}(\mathbf{t}) =& \int_R \mathrm{d}\mathbf{u}\bar{W}_{\hat{\rho}_L(\mathbf{u})}(\mathbf{t})\\ =& \int_R \mathrm{d}\mathbf{u}W_{\hat{\rho}}(\mathbf{u}+\mathbf{t}),\end{aligned} \tag{B14}$$

where $W_{\hat{\rho}}(\mathbf{u}+\mathbf{t})$ is the ZGW function of the state $\hat{\rho}$, and where we have made use of (B12).

3. Proof of Corollary 1

In this subsection, we prove Corollary 1 using Lemma 1.

Proof. First, note that for a single mode, we have

$$\begin{aligned}
\hat{P}_{\mathbf{0}}\hat{T}_{\mathbf{u}}\hat{P}_{\mathbf{0}} &= \sum_{j,j'} |j_{\text{GKP}}\rangle \langle j_{\text{GKP}}| e^{i\ell^2 u_X u_Z/2}\hat{T}_{u_X}\hat{T}_{u_Z} |j'_{\text{GKP}}\rangle \langle j'_{\text{GKP}}| \\
&= \sum_{j,j'} |j_{\text{GKP}}\rangle \langle j_{\text{GKP}}| e^{i\ell^2 u_X u_Z/2} e^{-i\hat{p}\ell u_X} e^{i\ell\hat{q}u_Z} |j'_{\text{GKP}}\rangle \langle j'_{\text{GKP}}| \\
&= \sum_{m,m',k,k'} \sum_{j,j'} |\ell(dk+j)\rangle \langle \ell(dk'+j)| e^{i\ell^2 u_X u_Z/2} e^{-i\hat{p}\ell u_X} e^{i\ell\hat{q}u_Z} |\ell(dm+j')\rangle \langle \ell(dm'+j')| \\
&= \sum_{m,m',k,k'} \sum_{j,j'} e^{i\ell^2 u_X u_Z/2} e^{i\ell^2(dm+j')u_Z} |\ell(dk+j)\rangle \langle \ell(dk'+j)|\ell(dm+j'+u_X)\rangle \langle \ell(dm'+j')| \,.
\end{aligned} \tag{B15}$$

Now, consider that the indices in the sum over j, j' can take values $0, \ldots, d-1$ while the integers dk', dm are multiples of d and $u_X \in [0, 1)$. Therefore, the bra-ket in the last line will therefore only be non-zero if $u_X = 0$ and $j = j'$ and $m = k'$. We can express this as

$$\begin{aligned}
\hat{P}_{\mathbf{0}}\hat{T}_{\mathbf{u}}\hat{P}_{\mathbf{0}} = & \sum_{m,m',k,k'} \sum_{j,j'} e^{i\ell^2 u_X u_Z/2} e^{i\ell^2(dm+j')u_Z} \delta_{m,k'}\delta_{j,j'}\delta(\ell u_X) |\ell(dk+j)\rangle \langle \ell(dm'+j')| \\
&= \sum_{m,m',k} \sum_{j} e^{i\ell^2(dm+j)u_Z} \delta(\ell u_X) |\ell(dk+j)\rangle \langle \ell(dm'+j)| \\
&= \sum_{m,m',k} \sum_{j} e^{i\ell^2 j u_Z} \delta(u_Z - m)\delta(\ell u_X) |\ell(dk+j)\rangle \langle \ell(dm'+j)| \,.
\end{aligned} \tag{B16}$$

However, note that $\mathbf{u} \in [0, 1)^{\times 2}$ and so we must have $\mathbf{u} = \mathbf{0}$. Therefore,

$$\begin{aligned}
\hat{P}_{\mathbf{0}}\hat{T}_{\mathbf{u}}\hat{P}_{\mathbf{0}} = & \sum_{m,m',k} \sum_{j} e^{i\ell^2 j u_Z} \delta(u_Z - m)\delta(\ell u_X) |\ell(dk+j)\rangle \langle \ell(dm'+j)| \\
=& \frac{1}{\ell} \sum_{m',k} \sum_{j} \delta(\mathbf{u}) |\ell(dk+j)\rangle \langle \ell(dm'+j)| \\
=& \frac{1}{\ell}\delta(\mathbf{u})\hat{P}_{\mathbf{0}}.
\end{aligned} \tag{B17}$$

Given a perfectly encoded GKP state $\hat{\rho} = \sum_{j,k} \hat{\bar{\rho}}_{j,k} |j_{\text{GKP}}\rangle \langle k_{\text{GKP}}|$, such that $\hat{\rho} = \hat{P}_{\mathbf{0}}\hat{\rho}\hat{P}_{\mathbf{0}}$, logically encoding a DV state $\hat{\bar{\rho}}$, we can consider the result of error correction as [6]

$$\begin{aligned}
\hat{\rho}_L(\mathbf{u}) =& \hat{P}_{\mathbf{0}}\hat{T}_{-\mathbf{u}}P_{\mathbf{0}}\hat{\rho}P_{\mathbf{0}}\hat{T}_{\mathbf{u}}\hat{P}_{\mathbf{0}} \\
=& \frac{1}{\ell^2}\delta^2(\mathbf{u})\hat{\rho} \\
=& \frac{d}{2\pi}\delta^2(\mathbf{u})\hat{\rho}_L(\mathbf{0}).
\end{aligned} \tag{B18}$$

where we see that $\hat{\rho}_L(\mathbf{0}) = \hat{\rho}$. The ZGW function is given by

$$W_{\hat{\rho}}(\boldsymbol{\eta}) = \bar{W}_{\hat{\rho}_L(\mathbf{u})}(\mathbf{t}) \tag{B19}$$

$$= \text{Tr}(\hat{A}_{\mathbf{t}}\hat{\rho}_L(\mathbf{u})) \tag{B20}$$

$$= \frac{d}{2\pi}\delta^2(\mathbf{u})\text{Tr}(\hat{A}_{\mathbf{t}}\hat{\rho}_L(\mathbf{0})) \tag{B21}$$

$$= \frac{d}{2\pi}\delta^2(\mathbf{u})\bar{W}_{\hat{\bar{\rho}}}(\mathbf{t}) \tag{B22}$$

$$= \frac{d}{2\pi}\sum_{\mathbf{m}\in\mathbb{Z}_d^2}\delta^2(\boldsymbol{\eta} - \ell\mathbf{m})\bar{W}_{\hat{\bar{\rho}}}(\mathbf{m}), \tag{B23}$$

where in the last line we use that $\mathbf{u} = \mathbf{0}$ and therefore $\boldsymbol{\eta} = \ell\mathbf{t}$ must have elements that are integer multiples of ℓ.

Note that this Wigner function is unnormalizable, but this is to be expected because the state $\hat{\rho}_0$ is also unnormalizable. We can simplify the expression as

$$W_{\hat{\rho}}(\boldsymbol{\eta}) \propto \sum_{\mathbf{m}\in\mathbb{Z}_d^2} \delta(\boldsymbol{\eta} - \ell\mathbf{m})\bar{W}_{\hat{\bar{\rho}}}(\mathbf{m}). \tag{B24}$$

□

Hence, for a logical zero state $\hat{\rho}_0$ we have, using Eq. (B1), that

$$\begin{aligned} W_{\hat{\rho}_0}(\boldsymbol{\eta}) &\propto \sum_{\mathbf{m}\in\mathbb{Z}_d^2} \delta(\boldsymbol{\eta}-\ell\mathbf{m})\bar{W}_{|0\rangle\langle 0|}(\mathbf{m}) \\ &= \sum_{\mathbf{m}\in\mathbb{Z}_d^2} \delta(\boldsymbol{\eta}-\ell\mathbf{m})\delta_{m_X,0} \\ &= \sum_{m\in\mathbb{Z}_d} \delta(\eta_Z-\ell m)\delta(\eta_X). \end{aligned} \tag{B25}$$

Appendix C: ZGW function evolution

1. Decomposition of operations

In the following section, we demonstrate that the set of encoded Clifford operations generates the set of integer symplectic matrices. This implies that any integer symplectic matrix that we consider in our paper can be decomposed as a set of encoded Clifford operations.

First, we define a generating set of Clifford operations. In the set, we must have the ability to implement any Pauli operation and the ability to transform the qudit stabilizer tableau with an integer symplectic matrix defined over $\mathbb{Z}_d$ [7]. The Clifford group is generated by the SUM gate (equivalently the control-Z gate as it can be conjugated by a Fourier transform on the target qudit), Fourier transform, phase gate and $\hat{Z}$ gate.

Now, we consider the set of integer symplectic matrices $S \in \mathrm{Sp}(2n,\mathbb{Z})$ with the aim to show that these are all generated by the qudit Clifford group, and we make use of the following Lemma.

Lemma 3. *(Theorem 2 of Ref. [8].) The integer symplectic group Sp$(2n,\mathbb{Z})$ is generated by matrices*

$$\begin{pmatrix} \mathbb{1} & S \\ 0 & \mathbb{1} \end{pmatrix}, \quad \begin{pmatrix} A & 0 \\ 0 & A^{-T} \end{pmatrix}, \quad \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \tag{C1}$$

where $S \in$ Sym$(n,\mathbb{Z})$ in the set of symmetric integer matrices, A is unimodular, i.e., $\det A = \pm 1$*, and X is a diagonal matrix with entries zero or one while $Y = \mathbb{1} - X$.*

Note that

$$\begin{pmatrix} \mathbb{1} & S \\ 0 & \mathbb{1} \end{pmatrix}\begin{pmatrix} \mathbb{1} & S' \\ 0 & \mathbb{1} \end{pmatrix} = \begin{pmatrix} \mathbb{1} & S+S' \\ 0 & \mathbb{1} \end{pmatrix}, \tag{C2}$$

hence, we can build the matrix with symmetric block S using a summation of generators of the form

$$S_1 = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \tag{C3}$$

and any matrix found by interchanging rows and corresponding columns of S_1, S_2.

The block diagonal matrix consisting of unimodular blocks has block A generated by elements of the form

$$A_1 = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{C4}$$

along with matrices of the form

$$A_3 = \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}^2. \tag{C5}$$

Finally, note that the third matrix in Eq. (C1) with diagonal blocks can be generated using combinations of the Fourier transform matrices.

This allows us to prove the following Lemma.

Lemma 4. *The integer symplectic group $Sp(2n, \mathbb{Z})$ is generated by the symplectic matrices representing the Fourier transform, the SUM gate and the phase gate.*

Proof. We assume that we have access to both the SUM gate $e^{-i\hat{q}_1\hat{p}_2}$, which has a symplectic matrix which takes the form of

$$\begin{pmatrix} A_2 & 0 \\ 0 & A_2^{-T} \end{pmatrix}, \tag{C6}$$

along with $e^{-i\hat{q}_i\hat{p}_j}$ with any choice of $i, j \in \{1, \ldots, n\}$ and $i \neq j$. This means that we can generate symplectic matrices with the top-left block A_2, defined in Eq. (C4), as well as matrices like A_2, where its rows and columns have been exchanged. This allows us to generate, for example, the symplectic matrix with block A_2^T. Furthermore, we also have access to the Fourier transform, which unlocks access to matrices of the form

$$\begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \tag{C7}$$

and thereby also

$$\begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}^2 = \begin{pmatrix} X^2 - Y^2 & 0 \\ 0 & X^2 - Y^2 \end{pmatrix} \tag{C8}$$

where $X^2 - Y^2$ is a diagonal matrix with elements ± 1 along the diagonal. Together, these matrices generate the full set of matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & A^{-T} \end{pmatrix}. \tag{C9}$$

To see why, note that we have A_2 because of the SUM gate, and we can generate the matrix A_1 as follows. Consider the case of a 2×2 block matrix. We are able to produce operations with block matrices of the form of A_2 given in Eq. (C4) along with the operation described with the block matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \tag{C10}$$

which corresponds to $e^{-i\hat{q}_2\hat{p}_1}$. Using the matrix generated by applying two Fourier transforms on the second mode, corresponding to the symplectic matrix given in Eq. (C8), we see that we have access to the symplectic matrix with top-left block

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{C11}$$

Combining the blocks of these symplectic operations, we find that we can build a symplectic matrix which has a top-left block of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{C12}$$

which corresponds to $r_1 \leftrightarrow r_2$. We then repeat this process with the remaining $r_2 \leftrightarrow r_3, \ldots r_{n-1} \leftrightarrow r_n$.

Finally, we note that the matrix S with block S_1 is generated by the phase gate, and the same matrix with block S_2 is generated by the control-Z gate. Interchanging the rows and columns of each matrix corresponds to choosing different modes for which the phase gate or control-Z gate should act on. □

2. Operations

In the following, we show that it is possible to perform Gaussian operations with integer symplectic matrices and that it is possible to find $W_{\hat{U}_S\hat{\rho}\hat{U}_S^\dagger}$ from $W_{\hat{\rho}}$ given a Gaussian unitary $\hat{U}_S$ represented by an integer symplectic matrix S.

We have

$$\begin{aligned}W_{\hat{U}\hat{\rho}\hat{U}^\dagger}(\boldsymbol{\eta}) =&\frac{1}{(2\pi)^n}\sum_{\mathbf{a}\in\mathbb{Z}^{2n}}\mathrm{Tr}\Big(\hat{U}\hat{\rho}\hat{U}^\dagger e^{i\ell[\mathbf{a},\boldsymbol{\eta}]+i\pi\mathbf{a_X}^T\mathbf{a_Z}}\hat{T}_{\mathbf{a}}\Big)\\ =&\frac{1}{(2\pi)^n}\sum_{\mathbf{a}\in\mathbb{Z}^{2n}}\mathrm{Tr}\Big(\hat{\rho}e^{i\ell[\mathbf{a},\boldsymbol{\eta}]+i\pi\mathbf{a_X}^T\mathbf{a_Z}}\hat{T}_{S\mathbf{a}}\Big),\end{aligned}\tag{C13}$$

where we have used that $\hat{U}^\dagger\hat{D}(\mathbf{r})\hat{U}=\hat{D}(S\mathbf{r})$ [9].

For certain symplectic matrices, we find that covariance holds because $(S\mathbf{a})_X^T(S\mathbf{a})_Z=\mathbf{a_X}^T\mathbf{a_Z}$, which allows us to interpret the symplectic operation as a linear transformation of the coordinates. However, this does not hold for general integer symplectic matrices, and we will now show that $(S\mathbf{a})_X^T(S\mathbf{a})_Z=\mathbf{a_X}^T\mathbf{a_Z}+\mathbf{t}^T\mathbf{a} \mod 2$, where $\mathbf{t}$ is a vector derived from the symplectic matrix. Note that we are only interested in the value of the expression modulo two because it is evaluated in the exponential with a factor of π.

First, note that we can explicitly write

$$S\mathbf{a}=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}\mathbf{a_X}\\ \mathbf{a_Z}\end{pmatrix}=\begin{pmatrix}A\mathbf{a_X}+B\mathbf{a_Z}\\ C\mathbf{a_X}+D\mathbf{a_Z}\end{pmatrix}\tag{C14}$$

hence

$$\begin{aligned}(S\mathbf{a})_X^T(S\mathbf{a})_Z=&(A\mathbf{a_X}+B\mathbf{a_Z})^T(C\mathbf{a_X}+D\mathbf{a_Z})\\ =&\mathbf{a_X}^TA^TC\mathbf{a_X}+\mathbf{a_Z}^TB^TD\mathbf{a_Z}+\mathbf{a_X}^TA^TD\mathbf{a_Z}+\mathbf{a}_Z^TB^TC\mathbf{a_X}.\end{aligned}\tag{C15}$$

Next, we use that $A^TD-C^TB=\mathbb{1}$, which is a property of symplectic matrices, such that

$$\begin{aligned}(S\mathbf{a})_X^T(S\mathbf{a})_Z=&\mathbf{a_X}^TA^TC\mathbf{a_X}+\mathbf{a_Z}^TB^TD\mathbf{a_Z}+\mathbf{a_X}^T(\mathbb{1}+C^TB)\mathbf{a_Z}+\mathbf{a}_Z^TB^TC\mathbf{a_X}\\ =&\mathbf{a_X}^TA^TC\mathbf{a_X}+\mathbf{a_Z}^TB^TD\mathbf{a_Z}+\mathbf{a_X}^T\mathbf{a_Z}+\mathbf{a_X}^TC^TB\mathbf{a_Z}+\mathbf{a}_Z^TB^TC\mathbf{a_X}\\ =&\mathbf{a_X}^TA^TC\mathbf{a_X}+\mathbf{a_Z}^TB^TD\mathbf{a_Z}+\mathbf{a_X}^T\mathbf{a_Z}+2\mathbf{a_X}^TC^TB\mathbf{a_Z}.\end{aligned}\tag{C16}$$

We note that since A^TC and B^TD are both symmetric according to the properties of symplectic matrices, all off-diagonal terms will appear twice, i.e., $a_j(A^TC)_{jk}a_k=a_k(A^TC)_{kj}a_j$ and hence in the summation over all these terms the sum of the off-diagonal terms will be a multiple of two and hence will be zero modulo two. Therefore, we only need to consider the diagonal terms of the matrix

$$T=\begin{pmatrix}A^TC & 0\\ 0 & B^TD\end{pmatrix}.\tag{C17}$$

We denote the diagonal elements of T using a vector $\bar{\mathbf{t}}$ such that $\bar{t}_i=T_{ii}$. Note also that since $\bar{t}_i$ and a_i are integers and a_i^2 is even if and only if a_i is even, then $\bar{t}_ia_i^2=\bar{t}_ia_i \mod 2$. Hence we have

$$(S\mathbf{a})_X^T(S\mathbf{a})_Z=\mathbf{a_X}^T\mathbf{a_Z}+\bar{\mathbf{t}}^T\mathbf{a} \mod 2.\tag{C18}$$

We can, therefore, evaluate the expression for the Wigner function as

$$\begin{aligned}W_{\hat{U}\hat{\rho}\hat{U}^\dagger}(\boldsymbol{\eta})=&\frac{1}{(2\pi)^n}\sum_{\mathbf{a}\in\mathbb{Z}^{2n}}\mathrm{Tr}\Big(\hat{\rho}e^{i\ell\mathbf{a}^T\Omega\boldsymbol{\eta}+i\pi((S\mathbf{a})_X^T(S\mathbf{a})_Z-\bar{\mathbf{t}}^T\mathbf{a})}\hat{T}_{S\mathbf{a}}\Big)\\ =&\frac{1}{(2\pi)^n}\sum_{\mathbf{a}\in\mathbb{Z}^{2n}}\mathrm{Tr}\Big(\hat{\rho}e^{i\ell\mathbf{a}^TS^TS^{-T}\Omega\boldsymbol{\eta}+i\pi((S\mathbf{a})_X^T(S\mathbf{a})_Z-\bar{\mathbf{t}}^TS^{-1}S\mathbf{a})}\hat{T}_{S\mathbf{a}}\Big)\\ =&\frac{1}{(2\pi)^n}\sum_{\mathbf{a},\mathbf{b}\in\mathbb{Z}^{2n}}\delta_{\mathbf{b},S\mathbf{a}}\,\mathrm{Tr}\Big(\hat{\rho}e^{i\ell\mathbf{b}^TS^{-T}\Omega\boldsymbol{\eta}+i\pi(\mathbf{b}_X^T\mathbf{b}_Z-\bar{\mathbf{t}}^TS^{-1}\mathbf{b})}\hat{T}_{\mathbf{b}}\Big)\\ =&\frac{1}{(2\pi)^n}\sum_{\mathbf{a},\mathbf{b}\in\mathbb{Z}^{2n}}\delta_{\mathbf{b},S\mathbf{a}}\,\mathrm{Tr}\Big(\hat{\rho}e^{i\ell\mathbf{b}^TS^{-T}\Omega\boldsymbol{\eta}+i\pi(\mathbf{b}_X^T\mathbf{b}_Z-\mathbf{b}^T\Omega\Omega^{-1}S^{-T}\bar{\mathbf{t}})}\hat{T}_{\mathbf{b}}\Big).\end{aligned}\tag{C19}$$

We then use that $S^{-T}\Omega = \Omega S$ and $\Omega^{-1}S^{-T} = S\Omega^{-1}$ and therefore

$$\begin{aligned}W_{\hat{U}\hat{\rho}\hat{U}^\dagger}(\boldsymbol{\eta}) =&\frac{1}{(2\pi)^n}\sum_{\mathbf{a},\mathbf{b}\in\mathbb{Z}^{2n}}\delta_{\mathbf{b},S\mathbf{a}}\operatorname{Tr}\Big(\hat{\rho}e^{i\ell\mathbf{b}^T\Omega S\boldsymbol{\eta}+i\pi(\mathbf{b}_X^T\mathbf{b}_Z-\mathbf{b}^T\Omega S\Omega^{-1}\bar{\mathbf{t}})}\hat{T}_{\mathbf{b}}\Big)\\ =&\frac{1}{(2\pi)^n}\sum_{\mathbf{a},\mathbf{b}\in\mathbb{Z}^{2n}}\delta_{\mathbf{b},S\mathbf{a}}\operatorname{Tr}\Big(\hat{\rho}e^{i\ell[\mathbf{b},S\boldsymbol{\eta}-\frac{\pi}{\ell}S\Omega^{-1}\bar{\mathbf{t}}]+i\pi\mathbf{b}_X^T\mathbf{b}_Z}\hat{T}_{\mathbf{b}}\Big).\end{aligned} \tag{C20}$$

Note that we can simply define $\mathbf{t} = \frac{\pi}{\ell}S\Omega^{-1}\bar{\mathbf{t}}$ and hence

$$\begin{aligned}W_{\hat{U}\hat{\rho}\hat{U}^\dagger}(\boldsymbol{\eta}) =&\frac{1}{(2\pi)^n}\sum_{\mathbf{a},\mathbf{b}\in\mathbb{Z}^{2n}}\delta_{S^{-1}\mathbf{b},\mathbf{a}}\operatorname{Tr}\Big(\hat{\rho}e^{i\ell([\mathbf{b},S\boldsymbol{\eta}-\mathbf{t}]+i\pi\mathbf{b}_X^T\mathbf{b}_Z}\hat{T}_{\mathbf{b}}\Big)\\ =&\frac{1}{(2\pi)^n}\sum_{\mathbf{b}\in\mathbb{Z}^{2n}}\operatorname{Tr}\Big(\hat{\rho}e^{i\ell([\mathbf{b},S\boldsymbol{\eta}-\mathbf{t}]+i\pi\mathbf{b}_X^T\mathbf{b}_Z}\hat{T}_{\mathbf{b}}\Big)\\ =&W_{\hat{\rho}}(S\boldsymbol{\eta}-\mathbf{t}),\end{aligned} \tag{C21}$$

where we use in the first line that $\det S = 1$.

3. Measurements

Using the traciality property of the Stratonovich-Weyl axioms given in Eq. (A1), we see that we can recover the Born rule from the ZGW function whenever one of the operators satisfies $\hat{A} = \mathcal{E}(\hat{A})$. The measurements that we show to be simulatable, i.e., the POVM operators $\hat{M}_Z(\mathbf{s})$, are indeed chosen because they are invariant under such twirling operation.

We now provide a proof that these operators are invariant under the twirling operation. Consider $\hat{T}_{\mathbf{a}}$ such that $[\hat{T}_{\mathbf{a}}, \hat{P}_{\mathbf{0}}] = 0$, where $\hat{P}_{\mathbf{0}}$ is the GKP projector as defined in the main text. We find

$$\begin{aligned}\mathcal{E}(\hat{T}_{\mathbf{a}}) &= \int \mathrm{d}\mathbf{s}\hat{P}_{\mathbf{s}}\hat{T}_{\mathbf{a}}\hat{P}_{\mathbf{s}}\\ &= \int \mathrm{d}\mathbf{s}\hat{T}_{\mathbf{s}}\hat{P}_{\mathbf{0}}\hat{T}_{-\mathbf{s}}\hat{T}_{\mathbf{a}}\hat{T}_{\mathbf{s}}\hat{P}_{\mathbf{0}}\hat{T}_{-\mathbf{s}}\\ &= \int \mathrm{d}\mathbf{s}e^{2\pi i[\mathbf{a},\mathbf{s}]/d}\hat{T}_{\mathbf{s}}\hat{P}_{\mathbf{0}}\hat{T}_{-\mathbf{s}}\hat{T}_{\mathbf{s}}\hat{T}_{\mathbf{a}}\hat{P}_{\mathbf{0}}\hat{T}_{-\mathbf{s}}\\ &= \int \mathrm{d}\mathbf{s}e^{2\pi i[\mathbf{a},\mathbf{s}]/d}\hat{T}_{\mathbf{s}}\hat{P}_{\mathbf{0}}\hat{T}_{-\mathbf{s}}\hat{T}_{\mathbf{s}}\hat{P}_{\mathbf{0}}\hat{T}_{\mathbf{a}}\hat{T}_{-\mathbf{s}}\\ &= \int \mathrm{d}\mathbf{s}\hat{T}_{\mathbf{s}}\hat{P}_{\mathbf{0}}\hat{T}_{-\mathbf{s}}\hat{T}_{\mathbf{a}},\end{aligned} \tag{C22}$$

where in the last step we use that $\hat{T}_{-\mathbf{s}}\hat{T}_{\mathbf{s}} = \mathbb{1}$ and $\hat{P}_{\mathbf{0}}\hat{P}_{\mathbf{0}} = \hat{P}_{\mathbf{0}}$ since it is a projector. Next, to show that this is equal to $\hat{T}_{\mathbf{a}}$, consider

$$\begin{aligned}\int \mathrm{d}\mathbf{s}\hat{T}_{\mathbf{s}}\hat{P}_{\mathbf{0}}\hat{T}_{-\mathbf{s}} &= \int \mathrm{d}s_1\cdots\int \mathrm{d}s_{2n}\hat{T}_{s_1}\dots\hat{T}_{s_{2n}}\hat{P}_{\mathbf{0}}\hat{T}_{-s_{2n}}\dots\hat{T}_{-s_1}\\ &= \int \mathrm{d}s_1\cdots\int \mathrm{d}s_{2n}\hat{T}_{s_1}\dots\hat{T}_{s_{2n}}(\hat{P}_{\mathbf{0}}^{(1)}\otimes\cdots\otimes\hat{P}_{\mathbf{0}}^{(n)})\hat{T}_{-s_{2n}}\dots\hat{T}_{-s_1}\\ &= \int \mathrm{d}s_1\int \mathrm{d}s_2\hat{T}_{s_1}\hat{T}_{s_2}\hat{P}_{\mathbf{0}}^{(1)}\hat{T}_{-s_2}\hat{T}_{-s_1}\otimes\cdots\otimes\int \mathrm{d}s_{2n-1}\int \mathrm{d}s_{2n}\hat{T}_{s_{2n-1}}\hat{T}_{s_{2n}}\hat{P}_{\mathbf{0}}^{(n)}\hat{T}_{-s_{2n}}\hat{T}_{-s_{2n-1}}\end{aligned} \tag{C23}$$

where by a slight abuse of notation we write $\hat{T}_{s_j} = \hat{T}_{(0,\ldots,0,s_j,0,\ldots,0)^T}$. Furthermore, we have

$$
\begin{aligned}
&\int \mathrm{d}s_X \int \mathrm{d}s_Z e^{-is_X\hat{p}} e^{is_Z\hat{q}} \hat{P}^{(1)}_{\mathbf{0}} e^{-is_Z\hat{q}} e^{is_X\hat{p}} \\
=&\int \mathrm{d}s_X \int \mathrm{d}s_Z \sum_{n,n'=-\infty}^{\infty} \sum_{j=0}^{d} e^{-is_X\hat{p}} e^{is_Z\hat{q}} \left|\ell(dn+j)\right\rangle \left\langle \ell(dn'+j)\right| e^{-is_Z\hat{q}} e^{is_X\hat{p}} \\
=&\int \mathrm{d}s_X \int \mathrm{d}s_Z \sum_{n,n'=-\infty}^{\infty} \sum_{j=0}^{d} e^{-is_X\hat{p}} e^{is_Z(\ell(dn+j))} \left|\ell(dn+j)\right\rangle \left\langle \ell(dn'+j)\right| e^{-is_Z(\ell(dn'+j))} e^{is_X\hat{p}} \\
=&\int \mathrm{d}s_X \sum_{n,n'=-\infty}^{\infty} \sum_{j=0}^{d} e^{-is_X\hat{p}} \left|\ell(dn+j)\right\rangle \left\langle \ell(dn'+j)\right| \delta(\ell dn - \ell dn') e^{is_X\hat{p}} \\
=&\int \mathrm{d}s_X \sum_{n=-\infty}^{\infty} \sum_{j=0}^{d} e^{-is_X\hat{p}} \left|\ell(dn+j)\right\rangle \left\langle \ell(dn+j)\right| e^{is_X\hat{p}} \\
=&\mathbb{1},
\end{aligned}
\tag{C24}
$$

where the bra and ket are in the position basis. Combining these expressions, we see that $\mathcal{E}(\hat{T}_{\mathbf{a}}) = \hat{T}_{\mathbf{a}}$.

General logical GKP Pauli operators are defined as $\hat{T}^L_{\mathbf{a}} = e^{\pi i \mathbf{a_X}^T \mathbf{a_Z}} \hat{T}_{\mathbf{a}}$. Note that when restricting to Z-logical operator the phase factor is one. We can consider the measurement of the single-mode logical operator $\hat{Z}_L = e^{i\hat{q}\ell}$ by phase estimation. We can equivalently consider the measurement of this operator using the projector $\hat{M}^{(1)}_Z(s) = \frac{1}{d\ell}\sum_n e^{-ins\ell} e^{in\hat{q}\ell}$. The projector can be expressed as

$$
\begin{aligned}
\hat{M}^{(1)}_Z(s) =& \frac{1}{d\ell} \sum_{n\in\mathbb{Z}} e^{-ins\ell} e^{in\hat{q}\ell} \\
=& \frac{1}{d\ell} \sum_{n} e^{in(\hat{q}-s)\ell} \\
=& \frac{1}{d\ell} \sum_{n} e^{2\pi i n(\hat{q}-s)/(2\pi/\ell)} \\
=& \frac{1}{d\ell} \sum_{n} e^{2\pi i n(\hat{q}-s)/(d\ell)} \\
=& \sum_{n} \delta(\hat{q} - s - \ell dn) \\
=& \sum_{n} \left|s+\ell dn\right\rangle \left\langle s+\ell dn\right| .
\end{aligned}
\tag{C25}
$$

Note that, despite being unnormalizable, this is a valid Kraus operator. This can be seen by integrating the operator over $[0, d\ell)$, which gives the identity. Now, we show that the measurement of the operator can be simulated using our algorithm. We see immediately that $[\hat{Z}^n_L, \hat{P}_{\mathbf{0}}] = 0$ and therefore

$$
\begin{aligned}
\mathcal{E}(\hat{M}^{(1)}_Z(s)) =& \frac{1}{d\ell} \sum_{n} e^{-is\ell n} \mathcal{E}(\hat{Z}^n_L) \\
=& \frac{1}{d\ell} \sum_{n} e^{-is\ell n} \hat{Z}^n_L \\
=& \hat{M}_Z(s).
\end{aligned}
\tag{C26}
$$

This means we can evaluate

$$
\mathrm{PDF}(s) = \mathrm{Tr}\Big(\hat{\rho}\hat{M}^{(1)}_Z(s)\Big) = \frac{1}{d\ell} \sum_{n} e^{-is\ell n}\, \mathrm{Tr}\Big(\hat{\rho}\hat{Z}^n_L\Big). \tag{C27}
$$

Consider the ZGW function of the operator $\hat{M}_Z^{(1)}(s)$. We have

$$
\begin{aligned}
W_{\hat{M}_Z^{(1)}(s)}\begin{pmatrix}u\\v\end{pmatrix} =&\frac{1}{2\pi}\sum_{m,m'\in\mathbb{Z}}\mathrm{Tr}\Big(\hat{M}_Z^{(1)}(s)e^{-i(mv-um')\ell}e^{i\pi mm'}\hat{T}_{(m,m')^T}\Big)\\
=&\frac{1}{2\pi}\frac{1}{d\ell}\sum_{n}e^{-is\ell n}\sum_{m,m'\in\mathbb{Z}}\mathrm{Tr}\Big(\hat{Z}_L^n e^{-i(mv-um')\ell}e^{i\pi mm'}\hat{T}_{(m,m')^T}\Big)\\
=&\frac{1}{2\pi}\frac{1}{d\ell}\sum_{n}e^{-is\ell n}\sum_{m,m'\in\mathbb{Z}}\mathrm{Tr}\Big(e^{i\hat{q}\ell n}e^{-i(mv-um')\ell}e^{i\pi mm'}e^{-i\pi mm'/d}e^{-i\ell m'\hat{p}}e^{i\ell m\hat{q}}\Big)\\
=&\frac{1}{2\pi}\frac{1}{d\ell}\sum_{n}e^{-is\ell n}\sum_{m,m'\in\mathbb{Z}}e^{i\pi mm'(d-1)/d}\int \mathrm{d}x\,\langle x|\,e^{-i(mv-um')\ell}e^{-i\ell m'\hat{p}}e^{i\ell(m+n)\hat{q}}\,|x\rangle\\
=&\frac{1}{2\pi}\frac{1}{d\ell}\sum_{n}e^{-is\ell n}\sum_{m,m'\in\mathbb{Z}}e^{i\pi mm'(d-1)/d}\int \mathrm{d}x\,\langle x|\,e^{-i(mv-um')\ell}e^{i\ell(m+n)x}\,|x+\ell m'\rangle\\
=&\sum_{n}e^{-is\ell n}\sum_{m,m'\in\mathbb{Z}}e^{i\pi mm'(d-1)/d}e^{-i(mv-um')\ell}\delta(\ell(m+n))\delta(\ell m')\\
\propto&\sum_{n}e^{-is\ell n}\sum_{m\in\mathbb{Z}}e^{-imv\ell}\delta(\ell(m+n))\\
\propto&\sum_{n}e^{i\ell n(v-s)}\\
=&\sum_{n}e^{2\pi in(v-s)/(2\pi\ell^{-1})}\\
\propto&\sum_{n}\delta(v-s-2\pi n/\ell)\\
=&\sum_{n}\delta(v-s-\ell dn)\\
\propto&\delta(v-s),
\end{aligned}
\tag{C28}
$$

where in the last line we use that v is defined modulo ℓd. Note that the measurement operator is not normalizable, so we do not expect the operator's Wigner function to be normalizable either. The distribution corresponding to the measurement outcomes for the modular measurement of a single mode single mode is given by

$$
\begin{aligned}
\mathrm{Tr}\Big(\hat{\rho}\hat{M}_Z^{(1)}(s)\Big) &= \int \mathrm{d}u\int \mathrm{d}v W_{\hat{\rho}}\begin{pmatrix}u\\v\end{pmatrix}W_{\hat{M}_Z^{(1)}(s)}\begin{pmatrix}u\\v\end{pmatrix}\\
&\propto \int \mathrm{d}u\int \mathrm{d}v W_{\hat{\rho}}\begin{pmatrix}u\\v\end{pmatrix}\delta(v-s)\\
&= \int \mathrm{d}u W_{\hat{\rho}}\begin{pmatrix}u\\s\end{pmatrix}.
\end{aligned}
\tag{C29}
$$

To sample from multiple modes we define the operator $\hat{M}_Z(\mathbf{s}) = \hat{M}_{Z,1}(s_1)\otimes\cdots\otimes\hat{M}_{Z,n}(s_n)$. This has a Wigner function of the form

$$
W_{\hat{M}_Z(\mathbf{s})}(\boldsymbol{\eta}) \propto \sum_{\mathbf{m}\in\mathbb{Z}^n}\delta(\boldsymbol{\eta}_{\mathbf{Z}}-\mathbf{s}).
\tag{C30}
$$

We, therefore, see that the multimode measurement statistics can be evaluated as

$$
\begin{aligned}
\mathrm{Tr}\Big(\hat{\rho}\hat{M}_Z(\mathbf{s})\Big) &= \int \mathrm{d}\boldsymbol{\eta}\, W_{\hat{\rho}}(\boldsymbol{\eta})W_{\hat{M}_Z(\mathbf{s})}(\boldsymbol{\eta})\\
&\propto \int \mathrm{d}\boldsymbol{\eta}\, W_{\hat{\rho}}(\boldsymbol{\eta})\delta(\boldsymbol{\eta}_{\mathbf{Z}}-\mathbf{s})\\
&= \int \mathrm{d}\boldsymbol{\eta}_{\mathbf{X}}\, W_{\hat{\rho}}\begin{pmatrix}\boldsymbol{\eta}_{\mathbf{X}}\\ \mathbf{s}\end{pmatrix}.
\end{aligned}
\tag{C31}
$$

However, note that this equation represents a probability distribution which is normalized over $\mathbf{z}$. Hence, we see that the right-hand side is correctly normalized since integrating both sides gives unity, as can be seen by the standardization property of Def. 1. We therefore have that

$$\mathrm{Tr}\Big(\hat{\rho}\hat{M}_Z(\mathbf{s})\Big) = \int \mathrm{d}\boldsymbol{\eta}_{\mathbf{X}}\, W_{\hat{\rho}}\begin{pmatrix}\boldsymbol{\eta}_{\mathbf{X}}\\ \mathbf{s}\end{pmatrix}. \tag{C32}$$

Appendix D: Wigner function of the tensor product of two states

In this section, we prove Eq. (13). I.e., we show that the Wigner function of the tensor product of two states is equal to the product of the individual Wigner functions.

Consider two states $\hat{\rho}, \hat{\sigma}$ defined over m and m' modes, with $n = m + m'$ total modes. We have

$$W_{\hat{\rho}\otimes\hat{\sigma}}(\boldsymbol{\eta}) = \mathrm{Tr}((\hat{\rho}\otimes\hat{\sigma})A_{\boldsymbol{\eta}}) \tag{D1}$$

where

$$\begin{aligned}
A_{\boldsymbol{\eta}} =& \frac{1}{(2\pi)^2}\sum_{\mathbf{a}\in\mathbb{Z}^{2n}} e^{i\ell[\mathbf{a},\boldsymbol{\eta}]+\pi\mathbf{a_X}^T\mathbf{a_Z}}\hat{T}_{\mathbf{a}}\\
=& \frac{1}{(2\pi)^2}\sum_{\mathbf{a}\in\mathbb{Z}^{2n}} e^{i\ell(\mathbf{a_X}\cdot\boldsymbol{\eta}_{\mathbf{Z}}-\mathbf{a_Z}\cdot\boldsymbol{\eta}_{\mathbf{X}})+\pi\mathbf{a_Z}\cdot\mathbf{a_X}}\hat{T}_{\mathbf{a}}\\
=& \frac{1}{(2\pi)^2}\sum_{a_1,a_2\in\mathbb{Z}^2} T_{\mathbf{a}^{(1)}}\hat{T}_{\mathbf{a}^{(2)}}e^{i\pi(\mathbf{a_Z}^{(1)}\cdot\mathbf{a_X}^{(1)}+\mathbf{a_Z}^{(2)}\cdot\mathbf{a_X}^{(2)})}\\
&\times e^{i\ell(\mathbf{a_X}^{(1)}\cdot\boldsymbol{\eta}_{\mathbf{Z}}^{(1)}+\mathbf{a_X}^{(2)}\cdot\boldsymbol{\eta}_{\mathbf{Z}}^{(2)}-\mathbf{a_Z}^{(1)}\cdot\boldsymbol{\eta}_{\mathbf{X}}^{(1)}-\mathbf{a_Z}^{(2)}\cdot\boldsymbol{\eta}_{\mathbf{X}}^{(2)})}\\
=& A_{\boldsymbol{\eta}^{(1)}}\otimes A_{\boldsymbol{\eta}^{(2)}}.
\end{aligned} \tag{D2}$$

Therefore

$$\begin{aligned}
W_{\hat{\rho}\otimes\hat{\sigma}}(\boldsymbol{\eta}) =& \mathrm{Tr}\Big((\hat{\rho}\otimes\hat{\sigma})(\hat{A}_{\boldsymbol{\eta}^{(1)}}\otimes\hat{A}_{\boldsymbol{\eta}^{(2)}})\Big)\\
=& \mathrm{Tr}\Big(\hat{\rho}\hat{A}_{\boldsymbol{\eta}^{(1)}}\Big)\,\mathrm{Tr}\Big(\hat{\sigma}\hat{A}_{\boldsymbol{\eta}^{(2)}}\Big)\\
=& W_{\hat{\rho}}(\boldsymbol{\eta}^{(1)})W_{\hat{\sigma}}(\boldsymbol{\eta}^{(2)}).
\end{aligned} \tag{D3}$$

Appendix E: ZGW function of finitely squeezed GKP states

In this Appendix, we derive the ZGW function for realistic, i.e. finitely squeezed, GKP states.

1. Zero-logical state

We use the following form of the GKP state encoding the computational basis state $|j\rangle$ for odd d,

$$\psi^{\Delta}_{\text{GKP},j}(x) = \sum_{k\in\mathbb{Z}} e^{-\frac{1}{2}\Delta^2(j\ell+d\ell k)^2}e^{-\frac{1}{2\Delta^2}(x-j\ell-d\ell k)^2}. \tag{E1}$$

We calculate the ZGW function of a pure state as [1]

$$
\begin{aligned}
W_{\hat{\rho}_0^{\Delta}}(\boldsymbol{\eta}) =& \frac{1}{2\pi}\sum_{\mathbf{a}\in\mathbb{Z}^2}\mathrm{Tr}\Big(\hat{\rho}e^{i\ell[\mathbf{a},\boldsymbol{\eta}]+i\pi a_X a_Z}\hat{T}_{\mathbf{a}}\Big)\\
=& \frac{1}{2\pi}\sum_{\mathbf{a}\in\mathbb{Z}^2}\mathrm{Tr}\Big(\hat{\rho}e^{i\ell[\mathbf{a},\boldsymbol{\eta}]+i\pi a_X a_Z}e^{i\pi a_X a_Z/d}\hat{T}_{\mathbf{a_X}}\hat{T}_{\mathbf{a_Z}}\Big)\\
=& \int \mathrm{d}x\int \mathrm{d}x'\frac{1}{2\pi}\sum_{\mathbf{a}\in\mathbb{Z}^2}\mathrm{Tr}\Big(\psi(x)\psi^*(x')e^{\pi i a_X a_Z/d}\left|\hat{q}=x\right\rangle\left\langle\hat{q}=x'\right|\hat{T}_{a_X}\hat{T}_{a_Z}e^{i\ell[\mathbf{a},\boldsymbol{\eta}]+i\pi a_X a_Z}\Big)\\
=& \int \mathrm{d}x\int \mathrm{d}x'\frac{1}{2\pi}\sum_{\mathbf{a}\in\mathbb{Z}^2}\mathrm{Tr}\Big(\psi(x)\psi^*(x')e^{\pi i a_X a_Z/d}\left|\hat{q}=x\right\rangle\left\langle\hat{q}=x'-\ell a_X\right|e^{i a_Z\ell(x'-\ell a_X)}e^{i\ell[\mathbf{a},\boldsymbol{\eta}]+i\pi a_X a_Z}\Big)\\
=& \int \mathrm{d}x\frac{1}{2\pi}\sum_{\mathbf{a}\in\mathbb{Z}^2}\psi(x)\psi^*(x+\ell a_X)e^{\pi i a_X a_Z/d}e^{i a_Z\ell x}e^{i\ell[\mathbf{a},\boldsymbol{\eta}]+i\pi a_X a_Z}.
\end{aligned}
\tag{E2}
$$

Hence, we can substitute the wavefunction into this expression to get

$$
\begin{aligned}
W_{\hat{\rho}_0^{\Delta}}(\boldsymbol{\eta}) =& \sum_{k,k',a_X,a_Z\in\mathbb{Z}}\frac{1}{(2\pi)}e^{-\frac{1}{2}\Delta^2(j\ell+d\ell k)^2}e^{-\frac{1}{2}\Delta^2(j\ell+d\ell k')^2}e^{i\ell[\mathbf{a},\boldsymbol{\eta}]+\frac{i2\pi}{d}\mathbf{a_X}^T\mathbf{a_Z}\frac{1}{2}(d+1)}\\
&\times\int \mathrm{d}x e^{ixa_Z\ell}e^{-\frac{1}{2\Delta^2}(x-j\ell-d\ell k)^2}e^{-\frac{1}{2\Delta^2}(x+\ell a_X-j\ell-d\ell k')^2}\\
=& \sum_{k,k',a_X,a_Z\in\mathbb{Z}}\frac{1}{(2\pi)}e^{-\frac{1}{2}\Delta^2(j\ell+d\ell k)^2}e^{-\frac{1}{2}\Delta^2(j\ell+d\ell k')^2}e^{i\ell a_X\boldsymbol{\eta}_{\mathbf{Z}}-i\ell a_Z\boldsymbol{\eta}_{\mathbf{X}}+\frac{i2\pi}{d}\mathbf{a_X}^T\mathbf{a_Z}\frac{1}{2}(d+1)}\\
&\times\sqrt{\pi}|\Delta|e^{\frac{1}{4}\left(2ia_Z\ell(-a_X\ell+2j\ell+dk\ell+dk'\ell)-\frac{(a_X\ell+dk\ell-dk'\ell)^2}{\Delta^2}-a_Z^2\ell^2\Delta^2\right)}\\
\propto& \sum_{t\in\mathbb{Z}^4}e^{i\pi t^T\Gamma t+2\pi i t^T r}
\end{aligned}
\tag{E3}
$$

where we have defined $t=(a_X,a_Z,k,k')$ and $r=(\ell\boldsymbol{\eta}_{\mathbf{Z}},-\ell\boldsymbol{\eta}_{\mathbf{X}},0,0)$ and by inspection we see that

$$
i\pi\Gamma=\begin{pmatrix}
-\frac{\ell^2}{4\Delta^2} & \frac{\pi i}{2} & -\frac{1}{4\Delta^2}d\ell^2 & \frac{1}{4\Delta^2}d\ell^2\\
\frac{\pi i}{2} & -\frac{1}{4}\ell^2\Delta^2 & \frac{1}{4}id\ell^2 & \frac{1}{4}i\ell^2 d\\
-\frac{1}{4\Delta^2}d\ell^2 & \frac{1}{4}id\ell^2 & -\frac{1}{2}\Delta^2 d^2\ell^2-\frac{1}{4\Delta^2}d^2\ell^2 & -\frac{1}{4\Delta^2}d^2\ell^2\\
\frac{1}{4\Delta^2}d\ell^2 & \frac{1}{4}id\ell^2 & -\frac{1}{4\Delta^2}d^2\ell^2 & -\frac{1}{2}\Delta^2 d^2\ell^2-\frac{1}{4\Delta^2}d^2\ell^2
\end{pmatrix}.
\tag{E4}
$$

Hence

$$
\begin{aligned}
\Gamma =& \frac{i}{\pi}\begin{pmatrix}
\frac{\ell^2}{4\Delta^2} & -\frac{\pi i}{2} & \frac{1}{4\Delta^2}d\ell^2 & -\frac{1}{4\Delta^2}d\ell^2\\
-\frac{\pi i}{2} & \frac{1}{4}\ell^2\Delta^2 & -\frac{1}{4}id\ell^2 & -\frac{1}{4}i\ell^2 d\\
\frac{1}{4\Delta^2}d\ell^2 & -\frac{1}{4}id\ell^2 & \frac{1}{2}\Delta^2 d^2\ell^2-\frac{1}{4\Delta^2}d^2\ell^2 & \frac{1}{4\Delta^2}d^2\ell^2\\
-\frac{1}{4\Delta^2}d\ell^2 & -\frac{1}{4}id\ell^2 & \frac{1}{4\Delta^2}d^2\ell^2 & \frac{1}{2}\Delta^2 d^2\ell^2-\frac{1}{4\Delta^2}d^2\ell^2
\end{pmatrix}\\
=& \frac{1}{2}\begin{pmatrix}
\frac{i}{d\Delta^2} & 1 & \frac{i}{\Delta^2} & -\frac{i}{\Delta^2}\\
1 & \frac{i\Delta^2}{d} & 1 & 1\\
\frac{i}{\Delta^2} & 1 & \frac{id}{\Delta^2}(1+2\Delta^4) & -\frac{id}{\Delta^2}\\
-\frac{i}{\Delta^2} & 1 & -\frac{id}{\Delta^2} & \frac{id}{\Delta^2}(1+2\Delta^4)
\end{pmatrix}.
\end{aligned}
\tag{E5}
$$

2. Phase state

We consider the phase state $|\psi_\pi\rangle=\frac{1}{\sqrt{3}}(|0_L\rangle+|1_L\rangle-|2_L\rangle)$ which is able to inject a non-Clifford unitary gate and promote the otherwise Clifford circuit to universality [10].

The wavefunction of the realistic state is of the form

$$
\psi^{\Delta}_{\mathrm{GKP},\pi}(x)=\sum_{j=0}^{2}(1-2\delta_{j,2})\sum_{k\in\mathbb{Z}}e^{-\frac{1}{2}\Delta^2(j\ell+d\ell k)^2}e^{-\frac{1}{2\Delta^2}(x-j\ell-d\ell k)^2}.
\tag{E6}
$$

The ZGW function is

$$
\begin{aligned}
W_{\hat{\rho}^{\Delta}_{\pi}}(\boldsymbol{\eta}) = &\int \mathrm{d}x \frac{1}{(2\pi)} \sum_{\mathbf{a}\in\mathbb{Z}^{2n}} \sum_{j=0}^{2}(1-2\delta_{j,2}) \sum_{k\in\mathbb{Z}} e^{-\frac{1}{2}\Delta^2(j\ell+d\ell k)^2} e^{-\frac{1}{2\Delta^2}(x-j\ell-d\ell k)^2} \\
&\times \sum_{j'=0}^{2}(1-2\delta_{j',2}) \sum_{k'\in\mathbb{Z}} e^{-\frac{1}{2}\Delta^2(j'\ell+d\ell k')^2} e^{-\frac{1}{2\Delta^2}(x+\ell a_X-j'\ell-d\ell k')^2} e^{ixa_Z\ell} e^{i\ell[\mathbf{a},\boldsymbol{\eta}]+\frac{i2\pi}{d}\mathbf{a_X}^T\mathbf{a_Z}\frac{1}{2}(d+1)} \\
\propto &\frac{1}{(2\pi)} \sum_{\mathbf{a}\in\mathbb{Z}^{2n}} \sum_{j,j'=0}^{2} (1-2\delta_{j,2})(1-2\delta_{j',2}) \sum_{k,k'\in\mathbb{Z}} e^{-\frac{1}{2}\Delta^2(j\ell+d\ell k)^2} e^{-\frac{1}{2}\Delta^2(j'\ell+d\ell k')^2} e^{i\ell[\mathbf{a},\boldsymbol{\eta}]+\frac{i2\pi}{d}\mathbf{a_X}^T\mathbf{a_Z}\frac{1}{2}(d+1)} \\
&\times e^{\frac{1}{4}\left(2ia_Z\ell^2(-a_X+j+j'+dk+dk')-\frac{\ell^2(a_X+j-j'+dk-dk')^2}{\Delta^2}-a_Z^2\ell^2\Delta^2\right)} \\
\propto &\frac{1}{(2\pi)} \sum_{j,j'=0}^{2} (1-2\delta_{j,2})(1-2\delta_{j',2}) \sum_{t\in\mathbb{Z}^4} e^{i\pi t^T\Gamma^{(j,j')}t+2\pi i t^T r^{(j,j')}} .
\end{aligned}
\tag{E7}
$$

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