Basis-independent stabilizerness and maximally noisy magic states

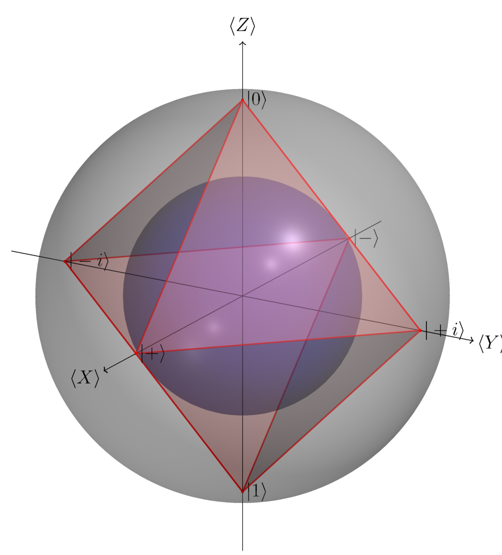
Michael Zurel & Jack Davis

https://arxiv.org/abs/2602.22336

Overview

Stabilizer states are of fundamental importance in quantum computation. They provide the structural foundation upon which much of the field rests, and are indispensable in the sub-fields of quantum error correction and classical simulation [1–8]. Such states define the complementary set of non-stabilizer, or magic states, as those inexpressible as a convex mixture of pure stabilizer states. Classifying, manipulating, and distilling such magic states is essential for achieving universality in the gate-injection model of fault-tolerant quantum computation [9–14]. Consequently, characterizing the structure and geometry of mixed stabilizer states is central for understanding the onset of quantum advantage.

The property of a state being stabilizer or non-stabilizer is inherently basis-dependent: a stabilizer state with respect to the computational basis can be a magic state in another basis. This is an immediate consequence of the unitary group acting transitively on pure states, with a prototypical example being the non-Clifford T gate mapping the stabilizer state $|+\rangle$ into the magic state $|T\rangle$. While the fault-tolerant implementation of such a unitary is highly constrained by the Eastin-Knill theorem [15], this observation nonetheless raises an important question: which mixed quantum states remain stabilizer under arbitrary unitary conjugation? Such *absolutely stabilizer* states were very recently introduced as a byproduct of fidelity-based magic detection and distillation protocols [16]. Little is currently understood about this important new property, and a characterization for the set of such states is highly desirable as it would allow one to test stabilizerness given only a set of eigenvalues.



In this paper, we give a systematic characterization of such states for multiple qudits of all prime dimensions. We do this by introducing a polytope of allowed spectra, one for each Hilbert space $(\mathbb{C}^d)^{\otimes n}$, and give illustrated examples for the cases of one qubit, two qubits, and one qutrit. In particular, we show that beyond the single-qubit case, the set of absolutely stabilizer states is not the largest inscribed ball of the stabilizer polytope, nor any other Hilbert-Schmidt ball in the space of Hermitian operators. For odd-prime dimensional qudits, we also give a complete characterization of absolutely Wigner-positive states, i.e., states whose Gross-Wigner function [17–19] remains nonnegative after conjugation by any unitary [20]. In so doing, we show there are absolutely Wigner-positive states that are not absolutely stabilizer, which can be seen as a unitarily-invariant version of bound magic [21]. We then study the radii of the largest balls contained in the sets of absolutely stabilizer states and absolutely Wigner-positive states. These radii respectively tell us the lowest possible purity of nonstabilizer and Wigner-negative states. Conversely, we also find the radius of the smallest ball containing the set of absolutely Wigner-positive states, giving a tight purity-based necessary condition thereof.

Preliminaries

The stabilizer and Λ polytopes. The stabilizer polytope, STAB, is the convex hull of pure stabilizer states while the Λ polytope is essentially the polar dual thereof, replacing the stabilizer polytope facets with Λ vertices [22–24]. For example, the single-qubit stabilizer polytope is the octahedron and the Λ polytope is the cube.

Absolutely stabilizer states and absolutely Wigner-positive states. A state ρ of n qudits with dimension d is called *absolutely stabilizer* if $U\rho U^\dagger \in$ STAB for all unitaries $U \in \mathrm{U}(d^n)$. This property was introduced very recently in [16] and little is known about it. We denote the set of such states as ASTAB. Analogously, a state of odd-prime dimensional qudits is called *absolutely Wigner-positive* if its Gross-Wigner function [17–19] remains non-negative, $W_{U\rho U^\dagger}(v) \geq 0$, for all unitaries $U \in \mathrm{U}(d^n)$. We denote the set of all such states AWP. Similar states have been explored using other kinds of Wigner functions [25, 26] but never with the Gross-Wigner function, which differs significantly in structure and is most prominent in quantum computation [21, 27, 28].

Closed and non-contextual (CNC) framework [29]. A *CNC set* for a multi-qudit system with prime local dimension is essentially a collection of Paulis that don't contain a Mermin square [30]. A *CNC operator* is an operator built from a consistent value-assignment over a given CNC set. Simple examples include stabilizer-state projectors and, in odd-prime dimensions, the Gross-Wigner phase space point operators. The structure of CNC operators changes significantly between $d = 2$ and odd-prime dimensions, but many are in fact vertices of the Λ polytope associated with their dimension. Such vertices are called *CNC-type vertices*.

Results on spectral polytopes

We provide a complete characterization of ASTAB as a polytope in the space of state spectra. We call such objects *absolutely stabilizer spectral polytopes*, or *muggle polytopes* for short. In other words, any n-qudit state with local prime dimension d having a spectrum within the muggle polytope is absolutely stabilizer. The facets of these objects are defined by the spectrum of the vertices of the Λ polytope. In particular, **Theorem 3** states that

$$\rho \in \mathrm{ASTAB} \qquad \iff \qquad \sum_{k=1}^{d^n} \lambda_k^{\uparrow}(\rho)\lambda_k^{\downarrow}(A) \geq 0 \qquad \forall A \in \mathrm{vert}(\Lambda), \tag{1}$$

where $\lambda_k^{\uparrow}(\rho)$ denotes the k-th entry of the vector of eigenvalues of ρ rearranged into non-decreasing order (and similarly for $\lambda_k^{\downarrow}(A)$ but in non-increasing order). Intuitively, each vertex of Λ is associated with a facet of STAB, and (1) forces the entire unitary orbit of ρ (via the eigenvalue re-ordering) to be under the facet.

While this is a conceptually clean description, it contains some redundancy as not all vertices of Λ are needed. This motivates a search for minimal subsets of $\mathrm{vert}(\Lambda)$ that are sufficient to characterize the absolutely stabilizer spectral polytope. Specializing to multi-qubit systems ($d = 2$) we observe that the subset of CNC-type vertices appear to be such a sufficient subset. We formalize this more precisely in the technical document as **Conjecture 1** and analytically verify it in the cases of one and two qubits. In the two-qubit case, this refinement is provably optimal and realizes the muggle polytope as having exactly 18 facets, a reduction in redundancy from 22,320 vertices of Λ.

Supporting numerical evidence is given for higher numbers of qubits in the appendix. Moreover, the expression (1) simplifies dramatically due to the eigenvalue structure of CNC-type operators (see technical manuscript for details).

For multi-qudit systems with odd-prime local dimension, we use the same framework to give a complete and facet-minimal description of the spectra of absolutely Wigner-positive states. Such *AWP polytopes* are simpler than their muggle counterparts due to the smaller number of Gross phase space point operators and the relatively easier description of their spectra. In particular, **Theorem 6** shows that

$$\rho \in \mathrm{AWP} \qquad \Longleftrightarrow \qquad \sum_{k=1}^{(d^n-1)/2} \lambda_k^{\downarrow}(\rho) \leq \frac{1}{2} \tag{2}$$

Comparing the muggle polytope with the AWP polytope, the former is in general found to be a subset of the latter; see Fig. 1 to the side. This implies the existence of mixed states that are non-stabilizer (i.e. magic) yet are absolutely Wigner-positive. This can be seen as a strengthening of the notion of bound magic introduced in [21], where it was shown that the set of Wigner-positive states was a strict superset of the stabilizer polytope.

Results on their inscribed balls

We also study the largest inscribed ball sitting within ASTAB as well as in AWP. These are important objects to determine as their radii represent the lowest purity that a non-stabilizer state (respectively, Wigner negative state) can have. In other words, they are a tight purity-based sufficient condition for membership in their respective polytope. We find that for odd-prime dimensions these two radii in fact coincide (**Theorems 8 and 9**); see Fig. 1 for an illustration. Moreover, we determine the circumradius of the AWP polytope, i.e. the radius of the smallest ball containing all AWP states. Hence, for AWP states we have found the optimal purity-based necessary and sufficient conditions for testing membership (**Theorem 9**). These two radii, together with the facet-minimal description of the spectral polytope via Eq. (2), effectively solves the classification problem of absolutely Wigner-positive states.

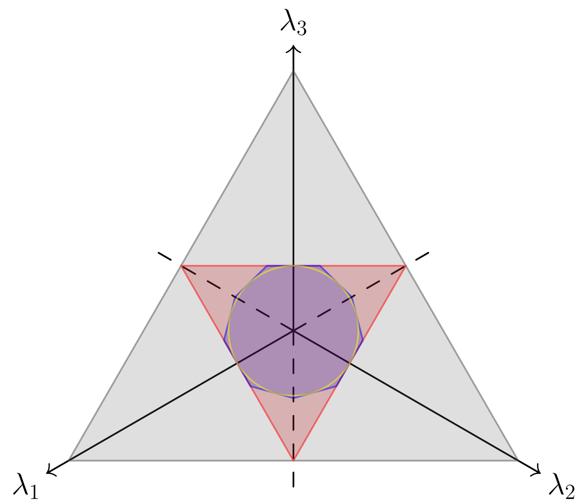


Figure 1: Eigenvalue simplex plot of the single qutrit muggle polytope (blue), AWP polytope (red), and the circumradius of their common inscribed ball (yellow).

In the case of multi-qubit systems we have derived, based again on the study of the Λ polytope and its vertices, what we believe to be the radius of the largest inscribed ball inside the muggle polytope. This radius (our **Conjecture 2**) is notable because is matches exactly the main conjecture of the recent work [16] which introduced the Triangle Criterion, a fidelity-based witness for detecting magic. This intriguing connection offers a novel geometric perspective of this radius and the structure of maximally noisy magic states.

References

[1] D. Gottesman, "Stabilizer codes and quantum error correction", PhD thesis (California Institute of Technology, 1997).

[2] D. Gottesman, "Fault-Tolerant Quantum Computation with Higher-Dimensional Systems", in Nasa international conference on quantum computing and quantum communications, edited by C. P. Williams (1999), pp. 302–313.

[3] S. Aaronson and D. Gottesman, "Improved simulation of stabilizer circuits", Physical Review A **70**, 052328 (2004).

[4] E. T. Campbell and D. E. Browne, "On the Structure of Protocols for Magic State Distillation", in *Theory of quantum computation, communication, and cryptography*, edited by A. Childs and M. Mosca (Springer Berlin Heidelberg, Berlin, Heidelberg, 2009), pp. 20–32.

[5] S. Bravyi, G. Smith, and J. A. Smolin, "Trading Classical and Quantum Computational Resources", Physical Review X **6**, 021043 (2016).

[6] S. Bravyi and D. Gosset, "Improved Classical Simulation of Quantum Circuits Dominated by Clifford Gates", Physical Review Letters **116**, 250501 (2016).

[7] S. Bravyi, D. Browne, P. Calpin, E. Campbell, D. Gosset, and M. Howard, "Simulation of quantum circuits by low-rank stabilizer decompositions", Quantum **3**, 181 (2019).

[8] J. R. Seddon, B. Regula, H. Pashayan, Y. Ouyang, and E. T. Campbell, "Quantifying Quantum Speedups: Improved Classical Simulation From Tighter Magic Monotones", PRX Quantum **2**, 010345 (2021).

[9] D. Gottesman and I. L. Chuang, "Demonstrating the viability of universal quantum computation using teleportation and single-qubit operations", Nature **402**, 390–393 (1999).

[10] S. Bravyi and A. Kitaev, "Universal quantum computation with ideal Clifford gates and noisy ancillas", Physical Review A **71**, 022316 (2005).

[11] V. Veitch, S. H. Mousavian, D. Gottesman, and J. Emerson, "The resource theory of stabilizer quantum computation", New Journal of Physics **16**, 013009 (2014).

[12] M. Howard and E. Campbell, "Application of a Resource Theory for Magic States to Fault-Tolerant Quantum Computing", Physical Review Letters **118**, 090501 (2017).

[13] J. R. Seddon and E. T. Campbell, "Quantifying magic for multi-qubit operations", Proceedings of the Royal Society A **475**, 20190251 (2019).

[14] A. Heimendahl, M. Heinrich, and D. Gross, "The axiomatic and the operational approaches to resource theories of magic do not coincide", Journal of Mathematical Physics **63**, 112201 (2022).

[15] B. Eastin and E. Knill, "Restrictions on Transversal Encoded Quantum Gate Sets", Physical Review Letters **102**, 110502 (2009).

[16] Z. Liu, T. Haug, Q. Ye, Z.-W. Liu, and I. Roth, *A magic criterion (almost) as nice as PPT, with applications in distillation and detection*, 2025.

[17] W. K. Wootters, "A Wigner-function formulation of finite-state quantum mechanics", Annals of Physics **176**, 1–21 (1987).

[18] D. Gross, "Hudson's theorem for finite-dimensional quantum systems", Journal of Mathematical Physics **47**, 122107 (2006).

[19] X. Wang, M. M. Wilde, and Y. Su, "Quantifying the magic of quantum channels", New Journal of Physics **21**, 103002 (2019).

[20] R. Salazar, J. Czartowski, and A. de Oliveira Junior, *Resource theory of Absolute Negativity*, 2022.

[21] V. Veitch, C. Ferrie, D. Gross, and J. Emerson, "Negative quasi-probability as a resource for quantum computation", New Journal of Physics **14**, 113011 (2012).

[22] A. Heimendahl, "The stabilizer polytope and contextuality for qubit systems", MA thesis (University of Cologne, 2019).

[23] M. Zurel, C. Okay, and R. Raussendorf, "Hidden Variable Model for Universal Quantum Computation with Magic States on Qubits", Physical Review Letters **125**, 260404 (2020).

[24] M. Zurel, C. Okay, R. Raussendorf, and A. Heimendahl, "Hidden variable model for quantum computation with magic states on qudits of any dimension", Quantum **8**, 1323 (2024).

[25] V. Abgaryan, A. Khvedelidze, and A. Torosyan, "The Global Indicator of Classicality of an Arbitrary N-Level Quantum System", Journal of Mathematical Sciences **251**, 301–314 (2020).

[26] J. Denis, J. Davis, R. B. Mann, and J. Martin, "Polytopes of absolutely Wigner bounded spin states", Quantum **8**, 1550 (2024).

[27] A. Mari and J. Eisert, "Positive Wigner Functions Render Classical Simulation of Quantum Computation Efficient", Physical Review Letters **109**, 230503 (2012).

[28] N. Delfosse, C. Okay, J. Bermejo-Vega, D. E. Browne, and R. Raussendorf, "Equivalence between contextuality and negativity of the Wigner function for qudits", New Journal of Physics **19**, 123024 (2017).

[29] R. Raussendorf, J. Bermejo-Vega, E. Tyhurst, C. Okay, and M. Zurel, "Phase-space-simulation method for quantum computation with magic states on qubits", Physical Review A **101**, 012350 (2020).

[30] N. D. Mermin, "Hidden variables and the two theorems of John Bell", Reviews of Modern Physics **65**, 803–815 (1993).