Deriving the Generalised Born Rule from First Principles

Gaurang Agrawal, Matt Wilson

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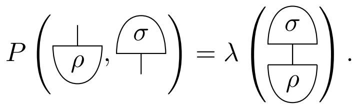
1 Introduction

Within process theories [1] and operational probabilistic theories [2], where physical processes are modelled using symmetric monoidal categories (SMCs) [3, 4], the generalised Born rule stands as a cornerstone principle. It posits a direct connection between the compositional structure of a theory and its observable statistics: the probability of measuring some state ρ and returning result σ is given up to homomorphism λ by the composition rule [2, 3, 5, 6]



(1)

But how fundamental is this principle? Is its prevalence a mere convenience found in certain well-behaved theories, or is it a derivable feature of any attempt to equip a process theory with a probabilistic interpretation [7, 8]? In this article we put forward a derivation of the generalised Born rule from basic principles. We show that any process theory equipped with a reasonable probabilistic interpretation, which we refer to as *probabilistic process theories* can be systematically transformed into an operationally equivalent one in which the generalised Born rule holds up to an injective monoid homomorphism λ from the scalars of the process theory to the monoid of positive Reals under multiplication. The result is first presented for a restricted class of simplified probabilistic process theories as a pedagogical introduction to the main result, and then for the general case.

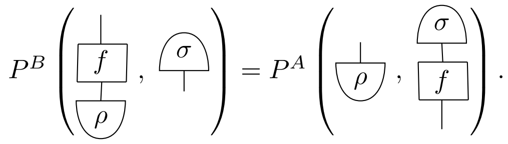
We further introduce noise to probabilistic process theories as formal positive linear combinations of processes. Further quotienting by probabilistic equivalence, gives a way to construct probabilistic process theories in which the generalised Born rule is strengthened, i.e., it holds up-to an injective semiring homomorphism. In the case of noisy quantum theory, this allows us to derive the exact quantum mechanical Born rule $\mathrm{Tr}[\rho\sigma]$

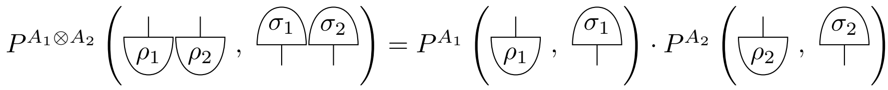
2 Probabilistic Process Theories

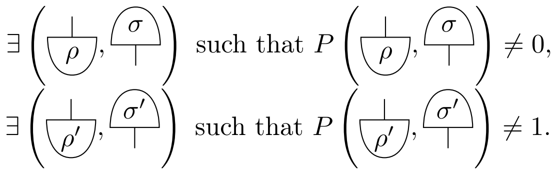
Definition 2.1 (Probabilistic Process Theory). A probabilistic process theory is a tuple $(\mathcal{D} \subseteq \mathcal{C}, \{\mathcal{S}^A\}, \{\mathcal{E}^A\}, \{P^A |\ A \in \mathrm{Ob}(\mathcal{C})\})$ consisting of a pair of symmetric monoidal categories $\mathcal{D} \subseteq \mathcal{C}$ along with families of 'physical' states, effects, and probability functions for each object, satisfying the following:

- Pair of physical states constitute a physical state $(s_A, s_B) \in \mathcal{S}^A \times \mathcal{S}^B \implies s_A \otimes s_B \in \mathcal{S}^{A\otimes B}$
- Pair of physical effects constitute a physical effect $(e_A, e_B) \in \mathcal{E}^A \times \mathcal{E}^B \implies e_A \otimes e_B \in \mathcal{E}^{A\otimes B}$
- For all physical transformations $f \in \mathcal{D}(A, B)$, $\rho \in \mathcal{S}^A$, and $\sigma \in \mathcal{E}^B$: $f \circ \rho \in \mathcal{S}^B$ and $\sigma \circ f \in \mathcal{E}^A$
- The identity on monoidal unit i_I is both a physical state and effect $i_I \in \mathcal{S}^I$, $i_I \in \mathcal{E}^I$
- There exists a probability function $P^A : \mathcal{S}^A \times \mathcal{E}^A \to \mathbb{R}_{\geq 0}$

subject to three constraints:

(I) Associativity: 

(II) Product: 

(III) Non-triviality: 

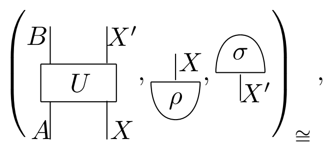
The simplified case then boils down to a theory $(\mathcal{C} \subseteq \mathcal{C}, \mathcal{C}(I,A), \mathcal{C}(A,I), \{P^A\})$ which is a symmetric monoidal category equipped with probability functions on all of its state-effect pairs satisfying axioms **(I), (II)** and **(III)**. For these we use notation $(\mathcal{C}, \{P^A\})$. Our goal is to prove that any probabilistic process theory is operationally equivalent to one in which the generalised Born rule holds.

3 Derivation of the Generalised Born Rule

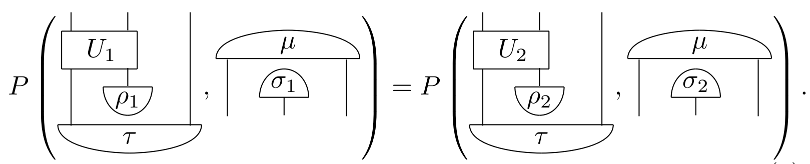
The generalised Born rule is introduced by the quotienting procedure.

Definition 3.1 (**Quotienting a probabilistic process theory).** Given a probabilistic process theory $(\mathcal{D} \subseteq \mathcal{C}, \{\mathcal{S}^A\}, \{\mathcal{E}^A\}, \{P^A\})$, we define its associated quotiented process theory $\mathcal{G}(\mathcal{D})$ as follows:

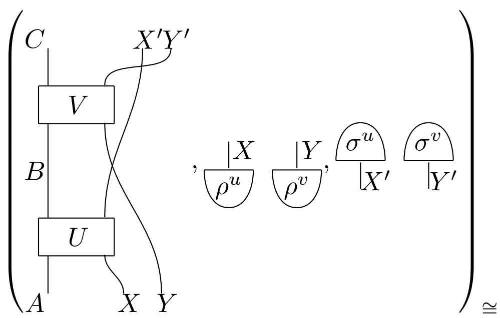
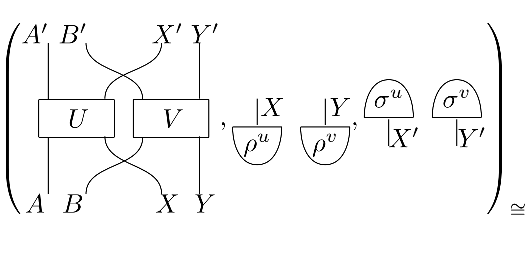
- Objects: same as $\mathcal{D}$
- Morphisms: $\mathcal{G}(\mathcal{D})(A,B) := \{(U,\rho,\sigma)_{\cong}\},\ U \in \mathcal{D}(A \otimes X, B \otimes X'), \rho \in \mathcal{S}^X, \sigma \in \mathcal{E}^{X'}$ Diagrammatically, a morphism is an equivalence class of a tuple



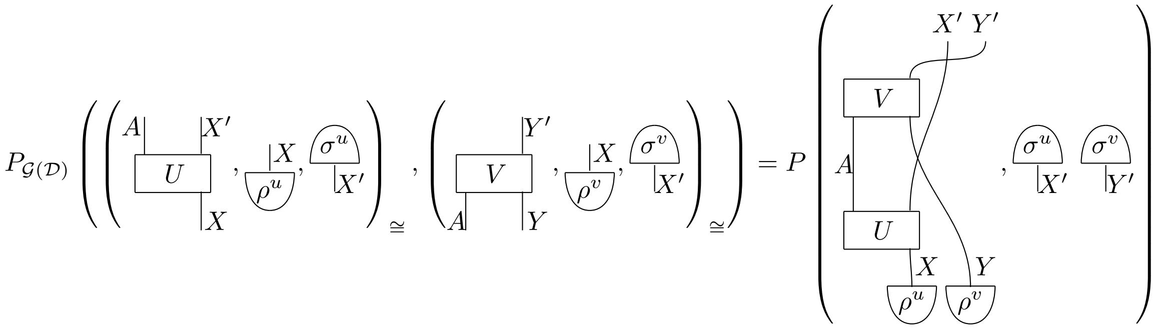
where we say that $(U_1,\rho_1,\sigma_1) \cong (U_2,\rho_2,\sigma_2)$ if and only if

$\forall \tau \in \mathcal{S}^{A\otimes Z}, \mu \in \mathcal{E}^{B\otimes Z}:$  (2)

- Sequential composition and parallel composition are defined as

 and  respectively.

- The probability function $P^A_{\mathcal{G}(\mathcal{D})}$: States $\times$ effects $\to \mathbb{R}_{\geq 0}$ is given by



In the full paper, we verify that $\mathcal{G}(\mathcal{D})$ equipped with sequential and parallel compositions forms a consistent simplified probabilistic process theory. Furthermore, we prove the following theorems among other results.

Theorem 3.2 (Generalised Born rule for probabilistic process theories)**.** *The quotient $\mathcal{G}$ of a probabilistic process theory satisfies the generalised Born rule Eq.(1), in other words, there exists a function $\lambda_{\mathcal{G}(\mathcal{D})} : \mathcal{D}(I, I) \to \mathbb{R}_{\geq 0}$ which is an injective monoid homomorphism mapping scalars composed of state-effect pairs exactly to their probability range.*

We characterise the quotiented normalised quantum theory, proving that quotienting via $\mathcal{G}$ on (Unitaries $\subseteq$ **FHilb**, $\{|\psi\rangle_{\text{normal}}\}, \{\langle\phi|_{\text{normal}}\}, |\langle - | - \rangle|^2$) yields a theory isomorphic to rank-1 CPTNI maps, i.e., Kraus operators $(\mathcal{K}, \mathrm{Tr}[- \circ -])$ where *Unitaries* is the SMC unitaries, and $\mathcal{K}$ is the SMC of rank-1 CPTNI maps.

4 Noise and Semiring Isomorphisms

With the aim of strengthening the correspondence between scalars and probabilities and furthermore recovering the completely positive maps from textbook quantum theory, we consider adding noise to probabilistic process theories. We do this by considering weighted sets of morphisms to build a new Summed Category $\mathcal{S}(\mathcal{C})$, giving a semiring structure where λ function elevates to semiring homomorphism from scalars to to sums of probabilities. To eliminate redundancy between statistically equivalent sum mixtures (e.g. distinct Kraus decompositions), we perform a final quotient.

Definition 4.1 (Noisy Category $\mathcal{N}(\mathcal{C})$). We define the noisy category $\mathcal{N}(C)$ as $\mathcal{Q}(\mathcal{S}(\mathcal{C}))$, where $\mathcal{Q}$ is the quotient with respect to statistically equivalent morphisms analogous to $\mathcal{G}$. In other words, its morphisms are the equivalence classes of morphisms from $\mathcal{S}(\mathcal{C})$ under the probabilistic equivalence relation similar to Eq.(2).

Theorem 4.2 (Born rule as semiring isomorphism in noisy theories). *For every probabilistic process theory there exists an associated simplified probabilistic process theory with a generalised Born rule that is additive. Concretely, there exists a semiring isomorphism $\lambda_{\mathcal{N}(\mathcal{C})}$ between the scalars $(\mathcal{N}(\mathcal{C})(I, I), 1_I, \otimes_{nc}, 0_{nc}, \cup_{nc})$ and the range of probabilities.*

We characterise the noisy theory for $\{$**FHilb**$, |\langle - | - \rangle|^2\}$, proving that it corresponds to completely positive maps **CP** with quantum mechanical Born rule $\mathrm{Tr}[\rho, \sigma]$. Similarly, the noisy theory for normalised quantum theory $\mathcal{G}$(Unitaries $\subseteq$ **FHilb**, $\{|\psi\rangle_{\text{normal}}\}, \{\langle\phi|_{\text{normal}}\}, |\langle - | - \rangle|^2$), is shown to corresponds to (**CP**, $\mathrm{Tr}[\rho, \sigma]$). Hence, we show that the category **CP**, arises naturally in our constructions. The construction is furthermore entirely adjoint-free, in contrast to existing approaches such as the Selinger's **CP** construction [9], the **CP*** construction [10, 11], and the **CP**$^\infty$ construction [12]. This suggests that the structure of quantum processes can be derived from principles of compositionality and statistical distinguishability alone, without a priori assumptions about an underlying normalisation or a built-in notion of adjoint. We also show that when the compositional structure of a noisy theory is given by completely positive maps **CP** the quantum mechanical Born rule is the only available consistent rule since identity forms the unique semiring isormorphism in real numbers.

Corollary 4.3. *In any probabilistic physical theory arising from the construction $\mathcal{N}$, with the underlying SMC being* **CP***, the possible probability functions on that theory are constrained to $P(\rho, \sigma) = \lambda(\sigma \circ \rho) = \mathrm{Tr}[\rho\sigma]$.*

References

[1] Bob Coecke and Aleks Kissinger. *Picturing Quantum Processes: A First Course in Quantum Theory and Diagrammatic Reasoning.* Cambridge University Press, 2017. doi: https://doi.org/10.1017/9781316219317.

[2] Giulio Chiribella, Giacomo Mauro D'Ariano, and Paolo Perinotti. Probabilistic theories with purification. *Phys. Rev. A*, 81(6):062348, June 2010. doi: 10.1103/PhysRevA.81.062348.

[3] Samson Abramsky and Bob Coecke. A categorical semantics of quantum protocols. *arXiv*, February 2004. doi: 10.48550/arXiv.quant-ph/0402130.

[4] J. Baez and M. Stay. Physics, Topology, Logic and Computation: A Rosetta Stone. In *New Structures for Physics*, pages 95–172. Springer, Berlin, Germany, July 2010. ISBN 978-3-642-12821-9. doi: 10.1007/978-3-642-12821-9_2.

[5] Bob Coecke and Aleks Kissinger. Categorical Quantum Mechanics I: Causal Quantum Processes. *arXiv*, October 2015. doi: 10.48550/arXiv.1510.05468.

[6] Bob Coecke and Aleks Kissinger. *Picturing Quantum Processes: A First Course in Quantum Theory and Diagrammatic Reasoning.* Cambridge University Press, Cambridge, England, UK, March 2017. ISBN 978-1-10710422-8. doi: 10.1017/9781316219317.

[7] Jonathan Barrett. Information processing in generalized probabilistic theories. *arXiv*, August 2005. doi: 10.48550/arXiv.quant-ph/0508211.

[8] Martin Plávala. General probabilistic theories: An introduction. *Phys. Rep.*, 1033:1–64, September 2023. ISSN 0370-1573. doi: 10.1016/j.physrep.2023.09.001.

[9] Peter Selinger. Dagger Compact Closed Categories and Completely Positive Maps: (Extended Abstract). *Electron. Notes Theor. Comput. Sci.*, 170:139–163, March 2007. ISSN 1571-0661. doi: 10.1016/j.entcs.2006.12.018.

[10] Bob Coecke, Chris Heunen, and Aleks Kissinger. Categories of Quantum and Classical Channels. *arXiv*, May 2013. doi: 10.48550/arXiv.1305.3821.

[11] Bob Coecke, Chris Heunen, and Aleks Kissinger. Categories of Quantum and Classical Channels (extended abstract). *arXiv*, August 2014. doi: 10.4204/EPTCS.158.1.

[12] Bob Coecke and Chris Heunen. Pictures of complete positivity in arbitrary dimension. *Inform. And Comput.*, 250:50–58, October 2016. ISSN 0890-5401. doi: 10.1016/j.ic.2016.02.007.