The resource theory of causal influence and knowledge of causal influence

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Understanding and quantifying causal relationships between variables is essential for reasoning about the physical world. In this work, we develop a resource-theoretic framework to do so. Here, we focus on the simplest nontrivial setting— two (classical) variables that are causally ordered, meaning that the first has the potential to influence the second. First, we introduce the resource theory that directly quantifies causal influence of a functional dependence in this setting and show that the problem of deciding convertibility of resources and identifying a complete set of monotones has a straightforward solution. Then, we introduce the resource theory that arises naturally when one has uncertainty about the functional dependence. We describe a linear program for deciding the question of whether one resource (i.e., probability distribution over functions) can be converted to another. Moreover, for the case where the variables are binary, we identify a triple of monotones that are complete. We provide an interpretation of these monotones and show that resourcefulness consists both of knowledge of the functional relations and of the degree of causal connectivity of the function relating the variables. Finally, we discuss how our resource theory connects to standard methods for quantifying cause-effect relations and Shannon theory, as well as its possible quantum generalization.

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Motivation

We begin with a motivating example that demonstrates the application of our resource-theoretic framework. Imagine that Alice and Bob share a communication channel – Alice inputs a message X to the channel, and Bob receives a potentially different message Y. This communication channel leaks to the environment a *flag variable* which encodes information about which function it is applying to X in each round. In the case where X and Y are binary, this flag variable takes four possible values, associated to the four possible deterministic functions that the channel can apply: Identity: $\mathtt{I}$ $(y=x)$, Flip: $\mathtt{F}$ $(y=x\oplus 1)$, Reset to zero: $\mathtt{R}_0$ $(y=0)$, Reset to one: $\mathtt{R}_1$ $(y=1)$.

The possibility of Alice and Bob to characterize the channel – and use it for communication in the future – depends on how much information they have about the flag variable. There are three different cases to consider:

(1) *They do not know anything about the flag variable.*
In this case, the best characterization of the communication channel is given by the conditional probability distribution $\mathbb{P}_{Y|X}$. For example, Alice and Bob can learn that the channel is completely randomizing, that is, that there

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is a probability of $1/2$ that $Y = 0$ and a probability of $1/2$ that $Y = 1$, regardless of the value of X. A channel characterized via this distribution is useless for communication.

(2) *They know everything about the flag variable.*
In this case, they know exactly which function ($\mathtt{I}$, $\mathtt{F}$, $\mathtt{R}_0$ or $\mathtt{R}_1$) was applied in each round. If Bob learns that the function applied was $\mathtt{F}$, he needs to take the negation of the value of Y to learn the value of X, which enables perfect communication. On the other hand, if Bob learns that in a specific round the function applied was $\mathtt{R}_0$, then he knows that there is nothing he can do to learn X.

(3) *They have partial information about the flag variable.*
For example, they might learn that the probability that the communication channel implements $\mathtt{I}$ is $1/2$ and that it implements $\mathtt{F}$ is $1/2$. Then, their description of the channel is given by a *probability distribution over functions*: $\mathbb{P}_B^1 = \frac{1}{2}[\mathtt{I}] + \frac{1}{2}[\mathtt{F}]$. It is not hard to see that the probability distribution over functions $\mathbb{P}_B^1$ gives rise to the conditional probability distribution $\mathbb{P}_{Y|X}$ of a completely randomizing channel. There are, however, *other* probability distributions over functions that *also* give rise to the same $\mathbb{P}_{Y|X}$. For example, it could be obtained by a channel that always ignores X, and resets Y to 0 in half of the runs and to 1 in the other half; it can be described via $\mathbb{P}_B^2 = \frac{1}{2}[\mathtt{R}_0] + \frac{1}{2}[\mathtt{R}_1]$. We see that the characterization of the channel as a probability distribution over functions is *strictly more informative* than the characterization of the channel as a conditional probability distribution.

A resource theory [1] enables one to quantify *resourcefulness* of such channel. Describing these three situations in a resource-theoretic framework requires three distinct resource theories. The relevant resource theory for the case (1) is Shannon theory [1,6]. In this work, we define resource theories for the cases (2) and (3).

Two resource theories

We introduce two distinct resource theories: the Resource Theory of Causal Influence, denoted by **RTCaus**, where the resources are functional dependences, and the Resource Theory of Knowledge of Causal Influence, denoted by **RTKnowCaus**, where the resources are probability distributions over functional dependences. We analyze the simplest nontrivial scenario: a diagram [3,4] with two observed variables, X and Y, with X potentially influencing Y and no confounding. In both resource theories, the free operations are chosen such that they do not introduce new possibilities for causal influence from X to Y.

Example: bit-to-bit scenario

In **RTCaus**, it is intuitive to see that the most resourceful functions are those that enable perfect communication (in the bit-to-bit case, it is simply $\mathtt{I}$ and $\mathtt{F}$) and the free functions are those for which Y is independent of the value of X (in the bit-to-bit case, it is $\mathtt{R}_0$ and $\mathtt{R}_1$). The situation is more subtle in **RTKnowCaus**. Here we show what *resourcefulness* means in **RTKnowCaus** by examining the bit-to-bit scenario.

In the bit-to-bit scenario, there are four possible functions from X to Y: $\mathtt{I}$, $\mathtt{F}$, $\mathtt{R}_0$ and $\mathtt{R}_1$. The general form of a resource (probability distribution over functions) can be written in terms of three parameters as

$$\mathbb{P}_B = \beta\left(\frac{1-\alpha}{2}[\mathtt{I}] + \frac{1+\alpha}{2}[\mathtt{F}]\right) + (1-\beta)\left(\frac{1-\gamma}{2}[\mathtt{R}_0] + \frac{1+\gamma}{2}[\mathtt{R}_1]\right), \tag{1}$$

where $\beta \in [0,1]$, $\alpha \in [-1,1]$, and $\gamma \in [-1,1]$. In this parametrization, β represents the weight on the sector of functions that carry causal influence, while α represents bias towards the $\mathtt{F}$ function within this sector. Similarly,

γ represents bias towards $\mathtt{R}_1$ in the sector that does not carry causal influence. One can readily see the set of free resources is the one corresponding to $\beta = 0$, containing convex combinations of $[\mathtt{R}_0]$ and $[\mathtt{R}_1]$.

The free operations are all combinations of a probabilistic pre- and a post-processing, where each is drawn from the set of four functions $\mathtt{I}$, $\mathtt{F}$, $\mathtt{R}_0$, and $\mathtt{R}_1$ and where the choice of function may be correlated by a common cause. As a result, there are 16 extremal free operations. Examining the extremal free operations allows us to derive a set of three monotones that fully characterize the pre-order of resources in this scenario:

$$M_\beta(\mathbb{P}_B) := \beta\,, \tag{2}$$

$$M_{|\alpha|}(\mathbb{P}_B) := |\alpha|\,, \tag{3}$$

$$M_{|\gamma|,\beta}(\mathbb{P}_B) := \begin{cases} \frac{\beta}{1-|\gamma|(1-\beta)} & \text{if} \quad \beta \neq 1\,, \\ 1 & \text{if} \quad \beta = 1\,. \end{cases} \tag{4}$$

The interpretation of these monotones can be summarized as follows:

- The M_β monotone quantifies the probability of causal connection in $\mathbb{P}_B$, i.e., the weight assigned to functions that describe causal connection.
- The $M_{|\alpha|}$ monotone quantifies the bias between $\mathtt{I}$ and $\mathtt{F}$ in $\mathbb{P}_B$.
- The $M_{|\gamma|,\beta}$ monotone captures the ratio of the decrease in probability of causal connection to the increase in the bias between $\mathtt{R}_0$ and $\mathtt{R}_1$ under a free operation that maximizes the latter.

This example shows that it is not straight-forward to define measures of causal influence when one deals with probability distribution over functions. We ultimately see that resourcefulness consists *both* of having knowledge of the function relating one's causal relata *and* for the actual function relating the causal relata to be one with large causal connectivity.

Main technical results

Result 1: Full characterization of RTCaus. We define the natural class of free resources and free operations and fully characterize the conditions under which one function can be transformed into another for variables with arbitrary finite cardinality.

Result 2: Resource convertibility in RTKnowCaus can be decided using a linear program. We show this by again identifying the natural definition of the free resources and the free operations for arbitrary finite cardinality variables.

Result 3: Full characterization of the bit-to-bit scenario in RTKnowCaus. We fully characterize the structure of this resource theory by introducing three resource monotones and proving that they form a complete set. In the full draft, we provide an extensive visual interpretation of our results. Moreover, we show the precise relation between the average causal effect (ACE) [2, 5] and measures of causal influence in the bit-to-bit case (the ACE is a lower bound on the probability of causal connection for a given conditional probability distribution).

Final remarks

One motivation for understanding the two resource theories is to ultimately develop analogous resource theories in the context of quantum causal influence. A first lesson of our work is that one must be careful even about what kind of thing is taken to constitute a resource in the context of understanding causal influence.

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