Insights in Graph State Entanglement via r-Local Complementation: Structure and a Quasi-Polynomial Algorithm

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Graph states and r-local complementation

Graph states form a versatile family of entangled quantum states that allow for easy and compact representations. As graph states are used as multi-partite entangled resources in multiple applications, it is crucial to understand when two such states have the same entanglement. According to standard quantum information theory, two quantum states have the same entanglement if they can be transformed into each other using only local operations, where 'local' refers to operations that can be applied to at most one qubit at a time. Indeed, intuitively, such local operations can only decrease the strength of the entanglement of a quantum state, thus if a state can be transformed into another and back to the original one using only local operations, the two states do have the same entanglement. In the case of graph states, having the same entanglement corresponds to being local unitary (LU) equivalent, i.e. there exists $U = U_1 \otimes \cdots \otimes U_n$ that transforms one state into the other, where each U_i is a 1-qubit unitary transformation [2]. Graph states are a sub-family of the well-known stabilizer states. Any stabilizer state can be transformed into a graph state using local Clifford unitaries in a straightforward way [3], as a consequence it is sufficient to consider graph states when working with stabilizer states up to reversible local operations.

As their name suggests, graph states are represented by (simple and undirected) graphs where each vertex corresponds to a qubit. A fundamental property of the graph state formalism is that a local complementation – a graph transformation that consists in complementing the neighbourhood of a given vertex – preserves entanglement: if a graph can be transformed into another by means of local complementations then the two corresponding graph states are LU-equivalent. More precisely the corresponding graph states are local Clifford (LC) equivalent.

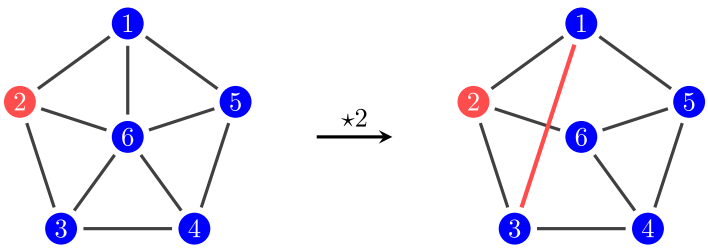

Figure 1: Two graphs related by a local complementation. The two corresponding graph states are LC-equivalent, implying they have the same entanglement.

Van den Nest and others [3]] proved that two graph states are LC-equivalent if and only if the corresponding graphs are related by a sequence of local complementations. LC-equivalence and LU-equivalence of graph states were once conjectured to coincide [4], as they coincide for several families of graph states [5, 6]. However, a counterexample of order 27 has been discovered using computer assisted methods [7] in 2007.

In a recent work [8], presented at QPL 2025, we described a generalization of local complementation, called r-local complementation (where r is a positive integer) that captures LU-equivalence of graph states, and provides, for the first time, a complete graphical theory of the entanglement of graph states.

Contrary to the usual local complementation, which is applied on a vertex, the r-local complementation is applied to a multiset of vertices (i.e. some vertices may be counted twice or more) that is also independent (no edges between vertices of the multiset) and r-incident. r-incidence is a strong condition that ensures r-local complementation can be implemented by local unitary operations on a graph state. For a multiset to be r-incident, $O(n^r)$ linear equations, which depend of the connectivity of the multiset, need to be satisfied.

Definition 1 (r-local complementation, [8]). Given a graph G and an r-incident independent multiset S, let $G \star^r S$ be the graph defined as

$$u \sim_{G\star^r S} v \;\Leftrightarrow\; \left(u \sim_G v \quad \oplus \sum_{w \in N_G(u)\cap N_G(v)} S(w) = 2^{r-1} \bmod 2^r \right)$$

where $N_G(u)$ is the neighbourhood of the vertex u and $S(w)$ is the multiplicity of w in S.

Notice that the 1-local complementation simply denotes the usual local complementation on an independent set of vertices. Equivalence of graphs under r-local complementation (for some fixed $r > 1$) does not correspond to LC-equivalence or LU-equivalence of graph states. Instead it corresponds to an intermediate equivalence: the LC$_r$-equivalence. LC$_r$ denotes the group generated by the local Clifford group augmented with the $Z(\pi/2^r)$ gate. An r-local complementation can be implemented using operations in LC$_r$ and, conversely, if two graph states are LC$_r$-equivalent then the corresponding graphs are related by r-local complementations.

Theorem 1 ([8]). *Given two graphs G_1, G_2, the following statements are equivalent:*

1. *$|G_1\rangle$ and $|G_2\rangle$ are LC$_r$-equivalent.*
2. *G_1 and G_2 are related by r-local complementations.*
3. *G_1 and G_2 are related by a sequence of local complementations along with a single r-local complementation.*

Most importantly, r-local complementation exactly captures the LU-equivalence of graph states.

Theorem 2 ([8]). *If two graph states are LU-equivalent, they are LC$_r$-equivalent for some r, i.e. the corresponding graphs are related by r-local complementations.*

As a direct corollary, LU-equivalence of graph state reduces to equivalence up to local unitary in the Clifford hierarchy. This had been hinted previously in the literature [9].

Entanglement of small graph states

A new, crucial structural result on r-local complementation is that to capture LU-equivalence of graph states, is it enough to consider r-local complementations where r is logarithmic in the number of qubits, as stated in the following strengthening of Theorem 2.

Theorem 3. *If two graph states on n qubits are LU-equivalent, then they are LC$_r$-equivalent for some $r \leqslant \lceil \log_2(\frac{n+1}{8}) \rceil$, i.e. the corresponding graphs are related by r-local complementations.*

An alternative formulation of Theorem 3 is the following:

Theorem 4. *If two graphs of order at most $2^{r+3} - 1$ are LU-equivalent, they are LC$_r$-equivalent.*

As mentioned above, there exists a pair of 27-qubit graph states that are LU-equivalent but not LC-equivalent [7]. Conversely, it was previously known that LU=LC (meaning that LU-equivalence and LC-equivalence coincide) for graph states defined on at most 8 qubits [10]. When r is chosen to be 1, Theorem 4 already implies a substantial improvement over this bound: LU=LC holds for graphs of order up to 15. Furthermore, when r is chosen to be 2, Theorem 4 implies that for graphs of order up to 31, LU=LC$_2$, i.e. if two graphs of order up to 31 are LU-equivalent, they are LC$_2$-equivalent. Thus, asking whether LU=LC holds for graphs of order up to 26 is equivalent to asking whether LC_2=LC holds for graphs of order up to 26. One direction is to study when a 2-local complementation can be implemented by local complementations. If this were to be the case for every graph of order up to 26, it would show that the 27-vertex counterexample is minimal in number of vertices. By studying the structure of 2-local complementation, and with the help of some computer-assisted proofs, we are able to state LU=LC for a wide range of small graph states.

Proposition 1. *LU=LC holds for graph states on up to 19 qubits.*

This is a big step towards proving (or disproving) that the 27-qubits counter-example to the LU=LC conjecture is minimal.

Deciding LU-equivalence of graph states in quasi-polynomial time

There exists a polynomial-time algorithm that decides LC-equivalence of graph states [11]. Indeed, in the early 90's Bouchet introduced an efficient algorithm that decides when two graph states are related by a sequence of local complementations [12]. For LU-equivalence however, previously only potentially exponential-time algorithms were known [13]. Theorem 3 makes the LU-equivalence problem computationally tractable. Indeed, we describe an algorithm that decides LC_r-equivalence of graph states essentially in time $O(n^r)$, thus deciding LC_r-equivalence of graph states can actually be done in polynomial time when r is fixed.

Theorem 5. *There exists an algorithm that decides if two graph states are LC_r-equivalent with runtime $O(rn^{r+2.38} + n^{6.38})$, where n is the number of qubits.*

Along with Theorem 3, this algorithm constitutes the first quasi-polynomial algorithm for deciding LU-equivalence of graph states.

Theorem 6. *There exists an algorithm that decides if two graph states are LU-equivalent with runtime $n^{\log_2(n)+O(1)}$, where n is number of qubits.*

We now describe informally the algorithm. Thanks to Theorem 1, if G_1 is LC_r-equivalent to G_2, there exists a single r-local complementation, together with usual local complementations, that transforms G_1 into G_2. The algorithm builds such a sequence of generalized local complementations, in essentially four stages:

(i) Both G_1 and G_2 are brought into some standard forms, yielding G_1' and G_2', by means of (usual) local complementations.

(ii) We then focus on the single r-local complementation: all the possible actions of a single r-local complementation over G_1' are described as a vector space, for which we compute a basis $\mathcal{B}$.

(iii) It remains to find, if it exists, the r-local complementation to apply on G_1' that leads to G_2' up to some additional usual local complementations. With an appropriate construction depending on G_1', G_2' and $\mathcal{B}$, we reduce this problem to deciding whether two graphs are LC-equivalent under some additional requirements on the sequence of local complementations to apply. These requirements can be expressed as linear constraints.

(iv) Finally, to find such a sequence of local complementations, we apply a variant of Bouchet's algorithm, generalized to accommodate the additional linear constraints.

Stages (i), (iii) and (iv) can be performed in polynomial time in the order n of the graphs. Indeed, a graph can be brought into standard form efficiently, the construction of step (iii) is straightforward, and the variant of Bouchet's algorithm is proved to be efficient for the graphs we obtain in step (iii). Stage (ii) has essentially an $O(n^r)$ time complexity, it amounts to solving a linear system of $O(n^r)$ equations.

It remains an open question whether there exists an algorithm that decides if two graph states are LU-equivalent with runtime polynomial in the number of qubits. More surprisingly, for now not even an efficient algorithm that *verifies* that two graph states are LU-equivalent have been found; one reason is that verifying that an r-local complementation is valid (by checking the r-incidence condition) requires verifying $O(n^r)$ equations. One solution could be the design of *quantum* algorithms for deciding or verifying LU-equivalence of graph states.

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