Diagrammatic Reasoning with Control as a Constructor, Applications to Quantum Circuits

Noé Delorme and Simon Perdrix

Université de Lorraine, CNRS, INRIA, LORIA, 54000 Nancy, France

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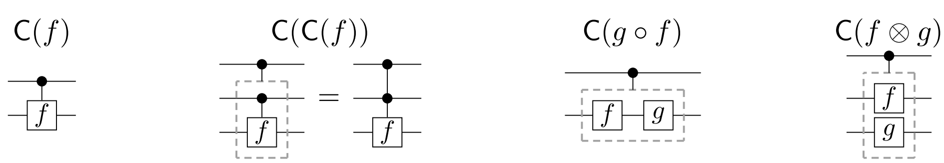
1 Controlled props

Diagrammatic reasoning has been very successful in recent years across various domains. These include quantum computing [11,16,14,2,1,9], linear algebra [19,5], control theory [4,3], natural language processing [10,20] and symbolic dynamics [15] to cite a few. The natural mathematical framework for diagrammatic reasoning is that of symmetric monoidal categories, and more precisely, props. This formalism allows representing processes and how they interact in space-time via sequential and parallel compositions.

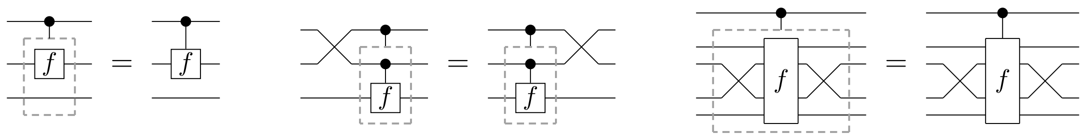
Control is a central concept in quantum and reversible computational models. Roughly speaking, a transformation can be conditioned on an additional system such that, depending on its state, the transformation is either applied or not. For instance, controlling a NOT gate yields the well-known CNOT gate, and adding another control leads to the Toffoli gate. Notice that various quantum programming languages [13,12,17,18,6] provide primitives that allow controlling gates or even circuits.

In this paper, we introduce the formalism of *controlled prop* featuring control as a constructor. Every diagram f with n inputs and n outputs yields a new diagram $\mathsf{C}(f)$ with $1 + n$ inputs and $1 + n$ outputs, representing its controlled version. Graphically, $\mathsf{C}(f)$ is depicted as a bullet point on the additional wire connected to the diagram f, enclosed within a dotted box. This dotted box is omitted when the diagram is made of a single diagram or when there are nested controls.

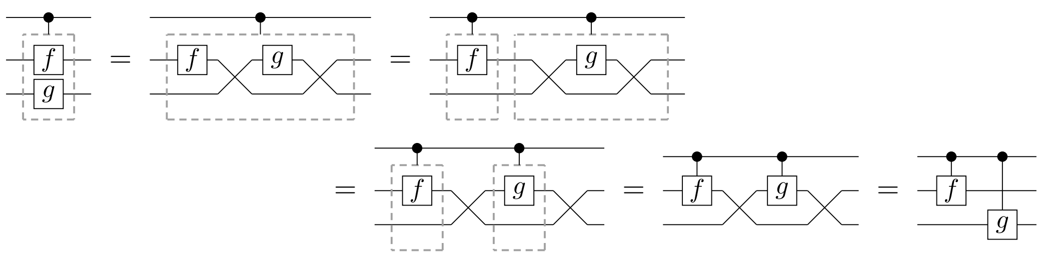


The controlled prop formalism is a versatile language capturing the fundamental properties of control and facilitates diagrammatic reasoning on processes. For instance, if an equation $f = g$ is derivable, so is its controlled version $\mathsf{C}(f) = \mathsf{C}(g)$, whereas the latter can be cumbersome to derive when using the standard prop formalism. Controlled props rely on three constructors – sequential composition, parallel composition, and control – and specify how they interact with a simple axiomatisation.

Definition 1 (controlled prop). *A* controlled prop *is a prop equipped with a* control functor C*, i.e. a functor mapping any object $n \in \mathbb{N}$ to the object $1+n$ and mapping any diagram $f : n \to n$ to a diagram $\mathsf{C}(f) : (1+n) \to (1+n)$ while satisfying the following three equations.*

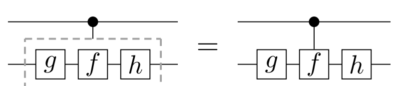


Notice that the functoriality of C additionally implies $\mathsf{C}(g \circ f) = \mathsf{C}(g) \circ \mathsf{C}(f)$ (whenever $g \circ f$ is defined), and $\mathsf{C}(\mathrm{id}_n) = \mathrm{id}_{1+n}$. However, a *control functor* does not preserve parallel composition, i.e. $\mathsf{C}(f \otimes g) \neq \mathsf{C}(f) \otimes \mathsf{C}(g)$ since the latter acts on one more wire. However, it distributes in a natural way, as explained in the following derivation.



We also point out a fundamental property – the *conjugation law* – which is central when dealing with reversing and control. Given f and an left-invertible g, one can conjugate f by g with the *compute-uncompute* pattern $h \circ f \circ g$, where h denotes a left inverse of g. This pattern is widely used in algorithm design, e.g. for a basis change, for ancilla preparation and release, for oracle implementation, and also for circuit optimisation [12]. Controlling such a pattern can be reduced to controlling only f, leading to the notion of *conjugated prop*.

Definition 2 (conjugated prop). *A* conjugated prop *is a controlled prop satisfying the following additional equation whenever $h \circ g = \mathrm{id}_n$.*



2 Quantum circuits

An important motivation for our framework comes from quantum computing, where control is extensively used. Complete sets of relations for quantum circuits have been established in earlier works using the formalism of props [9,8,7]. These complete equational theories do not rely on control as a constructor, but however point out the pivotal role of controlled operations in diagrammatic reasoning, and motivate the introduction of controlled props. As our main application in this present paper, we define the *controllable quantum circuits* and give a complete set of relations for it (see Figure 1). In its simplest form, where controllable quantum circuits are seen as a conjugated prop, this complete set of relations is made of four basic equations (Equations (1) to (4)), together

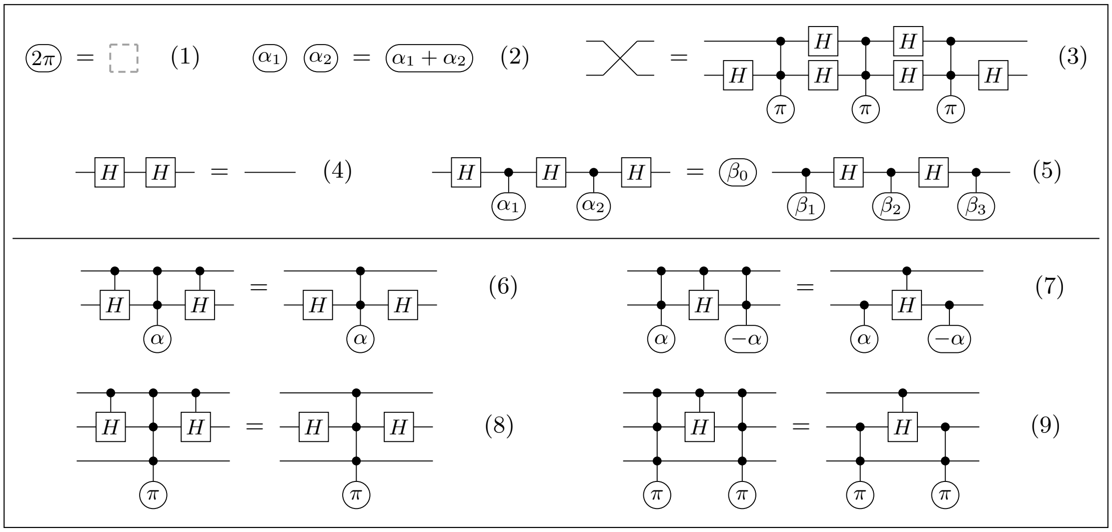


Fig. 1: Complete set of relations for controllable quantum circuits generated by the Hadamard gate H together with the global phase gates α for any $\alpha \in \mathbb{R}$.

with the standard Euler decomposition (Equation (5)). Notice that this equational theory is provably minimal. We also provide a complete set of equations for controllable circuits seen as a controlled prop, with only a few additional relations (Equations (6) to (9)) corresponding to particular instances of the conjugation law.

Remarkably, all relations involve circuits acting on at most three qubits, whereas it has been shown in [7] that any complete set of relations for (unitary) quantum circuits, defined as a prop, requires at least one relation involving circuits acting on n qubits for any $n \in \mathbb{N}$. Hence, controlled props circumvent the unboundedness issue of props, making them particularly well suited to diagrammatic reasoning.

As another application, we consider controllable quantum circuits with auxiliary qubits, a.k.a. *ancillae*, for which we also introduce a simple complete set of relations. It illustrates that controlled props are not necessarily composed only of diagrams with matching numbers of inputs and outputs (even if the control functor applies only to such diagrams). Finally, the versatility of the formalism allows, in particular, to express control in qudit systems. In these settings, there are several control functors (one per states), and we review some notable properties that multiple coexisting control functors may satisfy.

Beyond quantum computing, we expect controlled props to be useful in other domains where conditional behavior plays a central role, such as classical reversible computing. Also, we notice that the notion of control functor can be straightforwardly generalised to arbitrary symmetric monoidal categories and also to weaker structures such as braided monoidal categories.

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