A Complete Equational Theory for Real-Clifford+CH Quantum Circuits

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Abstract

We introduce a complete equational theory for the fragment of quantum circuits generated by the real Clifford gates plus the two-qubit controlled-Hadamard gate. That is, we give a simple set of equalities between circuits of this fragment, and prove that any other true equation can be derived from these. This is the first such completeness result for a finitely-generated, universal fragment of quantum circuits, with no parameterized gates and no need for ancillas.

1 Introduction

Quantum circuits form the basic language which is essentially at the core of quantum computing. The role that they play in practice is sometimes compared to that of an assembly language in classical computing. But formally, they are in fact the quantum equivalent of *boolean circuits*. Like boolean circuits, they consist of a sequence of *gates*, which are applied to some quantum data which is most often in the form of *qubits*, the quantum equivalent of bits.

Quantum computing can *a priori* be realized on a variety of physical platforms with various characteristics, and for now it is essentially impossible to predict what physical implementation will turn out to be the best, neither is it clear whether one technology will completely supplant the others; therefore, various different classes of quantum circuits are *a priori* relevant to formalize and study. Even staying inside the most standard model of quantum computing, where the data is encoded in qubits, several *fragments* of the language of quantum circuits, corresponding to particular sets of elementary gates with which to build the circuits, are relevant to consider. Some of them have received particular attention, such as the well-known Clifford+T and Toffoli-Hadamard fragments. An essential characteristic for a gate set is to be *computationally universal*, that is, to make it possible in principle to implement any arbitrary quantum program, up to arbitrarily small error. One reason that makes Clifford+T and Toffoli-Hadamard interesting is that they are universal while being generated by a very small and simple set of gates – in addition to being possibly relevant for promising implementations such as superconducting circuits.

A central task in quantum computing is to manipulate quantum circuits, and in particular to transform circuits into equivalent ones that are better according to some criterion: this has various practical applications such as optimization, satisfaction of hardware constraints, error correction, or verification. A promising approach towards manipulation of quantum circuits is via graphical rewriting. In the recent years, several *graphical languages* for formal manipulation of quantum circuits (and of other aspects of quantum processes) have been introduced, such as the ZX-, ZW-, ZH-, and ZXW-calculi, among others [2, 16, 20, 22, 34]. A central question in the formalization of such languages is the completeness of their equational theory, that is, the fact that the set of rewriting rules with which they are equipped be powerful enough so that any semantically meaningful transformation can be done. A related important question is to make the complete equational theory as simple as possible, and if possible, to be finally able to prove that it is minimal in the sense that all redundancies have been simplified out. There are now various complete equational theories for graphical languages [23, 26–28, 42], some of which being minimal [18] or near-minimal [3, 4, 39–42].

The language of quantum circuits itself can be viewed as a graphical language, and finding complete equational theories for it has its own added value compared to relying on external auxiliary languages like the ZX-calculus. For instance, applying arbitrary ZX transformations to a quantum circuit can produce a ZX-diagram from which it may be hard to get back a quantum circuit [17, 20]. Although the question of finding a complete equational theory for quantum circuits had been formulated in the early 2000's [25, 29], the first complete equational theory was found only a few years ago [13]. It was then followed by some simplifications and a proof of minimality [11, 12]. Those results are only for the full language of quantum circuits in the qubit setting. In particular, the gate set is very simple but still contains a parameterized gate, and the minimal equational theory is relatively simple but involves non-trivial computations on the parameters.

In this paper, we introduce a complete equational theory for a finitely generated fragment of quantum circuits, without parameterized gates, namely the fragment generated by real Clifford gates – the Hadamard gate H, the Pauli Z gate, and the controlled-Z gate CZ – together with the controlled Hadamard gate CH. We call this fragment *Real-Clifford+CH*. This is a sub-fragment of Clifford+T, which contains Toffoli-Hadamard[1] and therefore is universal. To our knowledge, this is the first completeness result for a universal fragment of quantum circuits generated by a finite gate set, without parameterized gates.

Similarly to [13], our result is based on a back-and-forth translation into an auxiliary language for which a completeness result

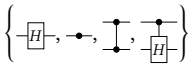
[1] Except in the case of 3-qubit circuits without ancillas.

is known, namely a presentation of the matrix group $U_n(\mathbb{Z}[\frac{1}{\sqrt{2}}])$ generated by one- and two-level matrices [21]. Those one- and two-level matrices have a similar structure to the optical circuits used in [13].

Related works. In addition to the graphical languages mentioned, other languages for quantum computing, more or less close to quantum circuits, are equipped with complete equational theories [8, 19, 21, 24]. The main difference is that those languages are equipped with a control constructor (or an equivalent construction) allowing one to express arbitrary multi-controlled gates, which means that they are not finitely generated from a quantum circuit point of view. Additionally, before the results of [11–13], some completeness results were known for non-universal fragments of quantum circuits [1, 6, 7, 14, 15, 25, 33, 35, 38] (see also [9, 30]).

2 Real-Clifford+CH Quantum Circuits

We consider the *real-Clifford+CH* fragment of quantum circuits, without ancillas, generated by the following gate set:


(A)

Following [11–13], we give a rigorous formalization to real-Clifford+CH quantum circuits by using the structure of PROP [10, 32, 43].

Remark 1. *The minimal structure required to rigorously formalize the circuits is a structure of PRO. Using instead the slightly richer structure of PROP means that the swap gate is implicitly added to the set of generators and satisfies basic topological properties. Note however that the swap gate can be expressed from the generators of* (A) *(see Equation* (8)*), and that its topological properties can be expressed as a finite number (namely 4) of simple equations.*

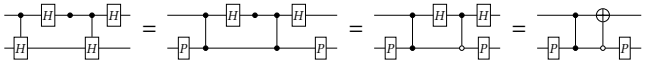
Additionally, note that the other standard real Clifford gates, such as CNOT, can also be recovered from those of (A).

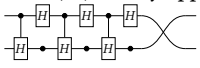
The circuits are given the standard semantics based on unitary matrices. The semantics of a circuit C is denoted by $[\![C]\!]$. Note that the matrices represented by real-Clifford+CH circuits belong to the group $U_{2^n}(\mathbb{Z}[\frac{1}{\sqrt{2}}])$.

3 Equational Theory

The set QC of equations that we prove to be complete is given in Figure 1. We write $\mathrm{QC} \vdash C_1 = C_2$ when C_1 and C_2 can be transformed into one another by means of these equations.

Equations (1) to (10) and (12) to (15) are relatively standard. Equations (16) and (17) also express a well-known property, namely that a positively-controlled circuit commutes with a negatively-controlled one. Equation (19) partly witnesses the fact that the shortcut behaves as if it did not touch the bottom qubit. Finally, Equations (11) and (18) can be more easily understood by using the change-of-basis matrix $P \coloneqq \begin{pmatrix} \cos(\frac{\pi}{8}) & \sin(\frac{\pi}{8}) \\ \sin(\frac{\pi}{8}) & -\cos(\frac{\pi}{8}) \end{pmatrix}$ which diagonalizes H into Z (one can check that $P^2 = I$ and $PZP = H$): then, reasoning from a purely semantic point of view, one could *informally* write



which makes the commutation of Equation (11) clearly appear; and Equation (18) says that the top part of  behaves like a P gate (indeed, one can check that the semantics of that circuit is equal to $P \otimes P$).

Proposition 3.1 (Soundness). *For any circuits C_1, C_2,*

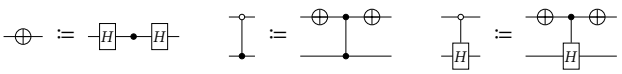
$$\textit{if } \mathrm{QC} \vdash C_1 = C_2 \textit{ then } [\![C_1]\!] = [\![C_2]\!].$$

Theorem 3.2 (Completeness). *For any circuits C_1, C_2,*

$$\textit{if } [\![C_1]\!] = [\![C_2]\!] \textit{ then } \mathrm{QC} \vdash C_1 = C_2.$$

4 Shortcuts

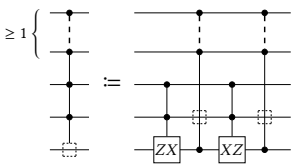
The complete equational theory of Figure 1 is written using a few basic shortcuts, in order to lighten the notations:

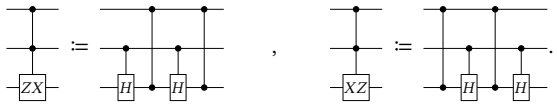


It also uses a multi-controlled $Z \otimes I$ gate, which is defined as a shortcut as well, by induction on the number of qubits. Note that this needs to be a multi-controlled $Z \otimes I$ gate, and not just an n-qubit multi-controlled Z gate, since for $n \geq 3$ the latter cannot be represented as a real-Clifford+CH circuit (without ancillas).[2] We depict the borrowed qubit by using a small dashed box.[3]

The base cases of the definition are as follows:

and the recursive step is as follows:



where 

5 Proof Overview

This paper relies essentially on a result of [21] which gives a presentation (that is, a set of generators together with a complete equational theory) of the group $U_N(\mathbb{Z}[\frac{1}{\sqrt{2}}])$ of N-dimensional unitary matrices with entries in $\mathbb{Z}[\frac{1}{\sqrt{2}}]$. Similarly to [13], we turn that complete equational theory into one for quantum circuits by means of a back-and-forth encoding. However, contrary to the situation in [13], for $n \geq 3$, the matrices represented by n-qubit real-Clifford+CH circuits do not span the whole group $U_{2^n}(\mathbb{Z}[\frac{1}{\sqrt{2}}])$, but only a subgroup of it of index 4.

There is a standard technique for obtaining a complete equational theory for a subgroup when one is known for the whole group, based on the *Reidemeister-Schreier theorem* [36, 37]. This theorem has been adapted to monoids in [6] and used to obtain complete equational theories for 2-qubit Clifford+T [6] and 3-qubit

[2] This is due to the fact that it has determinant -1 and that we are only working with real matrices.

[3] One can interpret the dashed box as representing a 1-qubit $-I$ gate, and our drawing as representing a multi-controlled $-I$ gate.

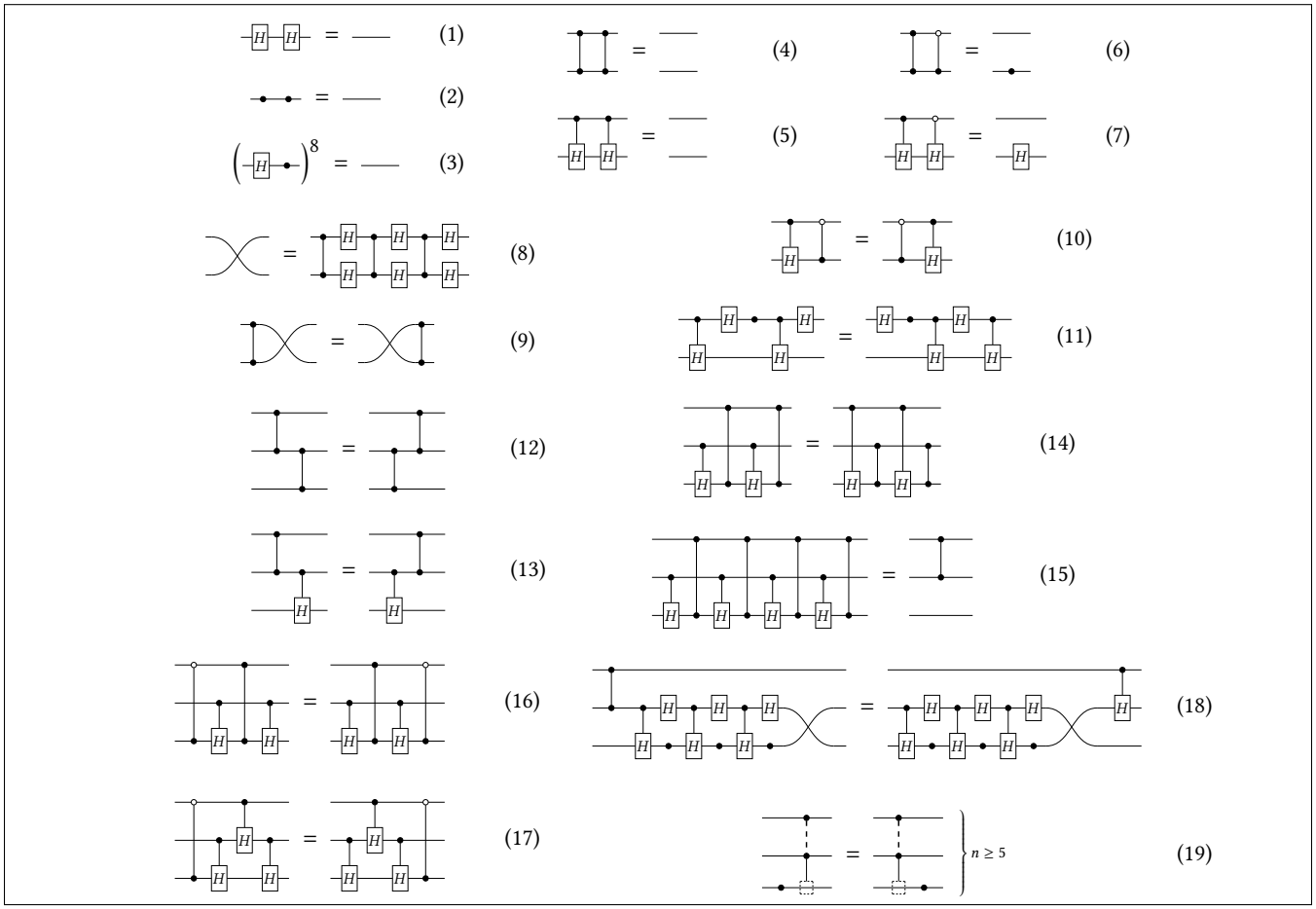


Figure 1: Complete equational theory for real-Clifford+CH quantum circuits. Equation (19) is an equation schema, consisting of an equation on n qubits for each $n \geq 5$. Equations (6), (7), (10), (16), (17), and (19) use shortcuts, defined in Section 4.

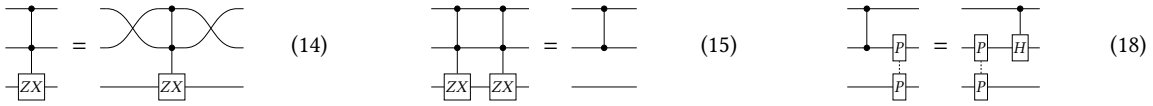


Figure 2: Some equations of Figure 1, written with more shortcuts.

Clifford+CS [7] quantum circuits. One drawback of this technique is that it often leads to a combinatorial explosion that makes it computationally costly, even in relatively simple cases. This is the reason that the completeness result of [7] is limited to 3 qubit, whereas the presentation of the underlying matrix group is *a priori* valid in arbitrary dimension.

Rather than trying to obtain a complete equational theory for real-Clifford+CH quantum circuits directly by applying the Reidemeister-Schreier theorem to the equational theory of [21], we adopt an alternative approach in two steps:

- first, we use the Reidemeister-Schreier theorem to obtain a presentation of the appropriate subgroup of $U_N(\mathbb{Z}[\frac{1}{\sqrt{2}}])$, using the same kind of generators as [21] (namely, using one- and two-level matrices);
- then we use a back-and-forth encoding similar to that of [13] in order to turn this presentation into a complete equational theory for real-Clifford+CH circuits.

Additionally, we perform some simplifications of the equational theories before and after each step. In particular, we start by slightly simplifying the equational theory of [21]. More importantly, after the first step, before proceeding to the back-and-forth encoding, we

greatly simplify the equational theory given by the Reidemeister-Schreier theorem. Finally, similarly to [13], the equational theory obtained by means of the back-and-forth encoding consists of several complex, unbounded equation schemas involving multi-controlled gates; therefore, a large part of our work is to simplify that equational theory into the much simpler one shown in Figure 1.

6 Conclusion

We have introduced a complete equational theory for unitary quantum circuits over the gate set $\left\{ -\boxed{H}-, -\bullet-, \begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}, \begin{smallmatrix}\bullet\\|\\\boxed{H}\end{smallmatrix} \right\}$. To our knowledge, this is the second universal fragment of unitary quantum circuits to be given a complete axiomatization, and the first which is finitely generated. One can note the relative similarity between our complete equational theory and that of [11], which is proved to be minimal: both equational theories are composed of a few simple equations on at most 3 qubits, plus an equation schema consisting of one equation for each number of qubits; in both cases, the equation schema has a trivial semantic interpretation, and its meaning comes from the fact that the multi-controlled gate is expressed as a macro built from 1- and 2- qubit gates. A difference between our equational theory and that of [11] is that since we are working in a discrete setting, ours does not require working with continuous parameters.

Moreover, the proof scheme followed in this paper could a priori be reused essentially as is in order to obtain complete equational theories for other fragments of quantum circuits. In particular for Toffoli-Hadamard and Clifford+CS, by relying on the corresponding matrix group presentations given respectively in [31] and [5].

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