A higher-order perspective on quantum signal processing

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1. Introduction and summary. Understanding the core structure underlying quantum algorithms has long been a central objective of the field of quantum computation. Such structural insights help clarify the origin of quantum advantage and often lead to the development of new and improved algorithms. In recent years, a unification encompassing most of the known quantum algorithms has emerged through the framework of *Quantum Signal Processing* (QSP) [LC17; GSLW19; MRTC21], which enables the manipulation of information encoded in quantum operations, thus initiating a functional programming scheme for quantum algorithms [RCC25].

A broad class of QSP-type algorithms follows a common blueprint: (1) encode a matrix A into a block of a larger unitary operator $U[A]$, and (2) interleave this block-encoding repeatedly with simple rotation unitaries known as processing operators. This procedure induces a polynomial transformation of A, which appears in a designated sub-block of the resulting unitary operator. This framework encompasses a variety of algorithms, differing in their choice of block-encodings and processing operators, yet with a remarkably unified underlying structure. For example, when A is non-unitary, the Quantum Singular Value Transformation (QSVT) [GSLW19] enables polynomial transformations of its singular values. When A is unitary, Generalised QSP (GQSP) [MW24] allows polynomial transformations of its eigenvalues.

At a structural level, these methods are understood through qubitization: a block-encoding can be decomposed into invariant two-dimensional subspaces. Applying QSP independently within each such subspace yields a polynomial transformation of the associated eigenvalues or singular values, thereby implementing a functional transformation of the original operator. Recent progress has been made in understanding the relationship between all the different QSP variants [Lan25] as well in finding a connection between GQSP and the nonlinear Fourier transform (NLFT) [AMT24; Lan25].

In parallel, a different approach to quantum functional programming has been built on *higher-order quantum transformations* [CDP08a; CDP08b; CDP09]. Such algorithms enable the transformations of unitary operations given as a black box, for example transforming a unitary U to its inverse, transpose, complex conjugate or controlled version [MSM19; QDS+19a; SBZ19; QE22; QDS+19b; YSM23; MZC+24; CML+25; DNSM19; CK19], as well as transformations of the underlying Hamiltonian H [OKSM24; OKTM25].

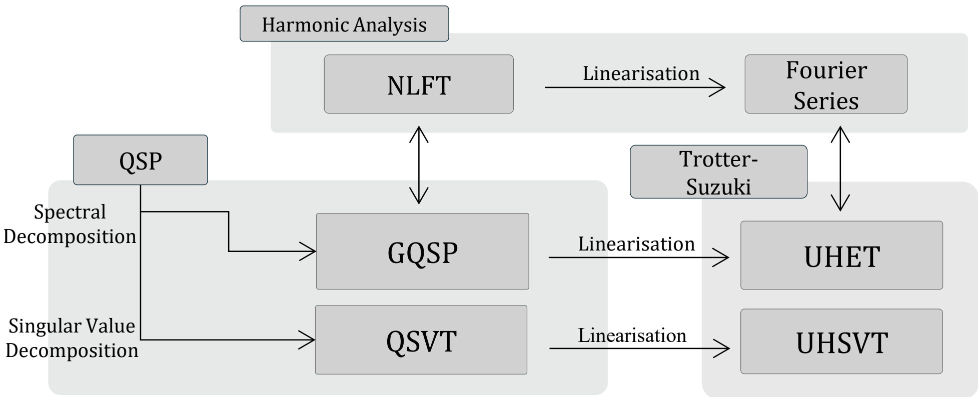

Figure 1: Conceptual map of the different quantum functional programming schemes: Generalised Quantum Singal Processing (GQSP) [MW24], Quantum Singular Value Transformation (QSVT) [GSLW19], Universal Hamiltonian Eigenvalue Transformation (UHET) [OKTM25] and our new Universal Hamiltonian Singular Value Transformation (UHSVT), as well as their relations to harmonic analysis including the non-linear Fourier transform (NLFT).

In particular, the Universal Hamiltonian Eigenvalue Transformation (UHET) algorithm [OKTM25] operates in a setting where one has access to free Hamiltonian dynamics through the evolution operator $e^{-iH\tau}$ for arbitrary time τ. UHET achieves a functional transformation of the eigenvalues of the original Hamiltonian dynamics, by any sufficiently differentiable function f, mapping the evolution operator $e^{-iH\tau}$ to $e^{-if(H)t}$.

The structure of UHET appears immediately familiar to those accustomed to QSP-like frameworks: controlled evolution operators are interleaved with $SU(2)$ rotations on an ancillary qubit. However, while QSP carefully selects phase angles to coherently engineer a target polynomial within a sub-block of the resulting unitary, UHET relies instead on a conceptually simpler, yet powerful, randomised Trotter-Suzuki-based construction (also known as qDRIFT [Cam18]).

Due to this randomised Trotter-Suzuki-based nature, the gate complexity of UHET typically scales as $\mathcal{O}(1/\epsilon)$ with the target error ϵ, whereas standard QSP approaches achieve logarithmic scaling in $1/\epsilon$ for smooth functions and $\frac{1}{\epsilon^{1/(J-1)}}$ for J-smooth functions. Nevertheless UHET can outperform QSP in settings where only the free Hamiltonian evolution is available, such that the controlled evolution itself must also be implemented via a randomized Trotter-Suzuki method [DNSM19]. This advantage arises from a *compilation procedure* that leverages correlated randomness to suppress the overall error of concatenated randomised subroutines [OKTM25].

An important question that naturally arises when considering UHET is how it fits into the broader "map" of quantum functional algorithms. What is its precise relationship to QSP and QSVT? Understanding the structural connections between these different approaches is crucial to developing a complete theory of quantum functional programming that can lead to the realisation of new algorithmic primitives.

In this work, we address this issue in two parts. First, we show that UHET can be viewed as a linearisation of Generalised QSP (GQSP). Second, using this understanding, we lift the intuition to construct a new quantum functional programming algorithm called Universal Hamiltonian Singular Value Transformation (UHSVT), which can be conceptualised as a linearisation of QSVT. These linearisation mappings are analogous to that between the Fourier series and the non-linear Fourier transform (NLFT), which in turn has been shown to be equivalent to GQSP [Lan25], all summarised in Figure 1.

2. Universal Hamiltonian Eigenvalue Transformation as a linearisation of Generalised QSP. We place UHET on the "map" (Figure 1) of quantum functional algorithms by establishing its structural connection to GQSP. We construct a variant of GQSP whose architecture closely mirrors that of UHET. This construction allows us to interpret UHET as a linearisation of GQSP, as summarised in Figure 2.

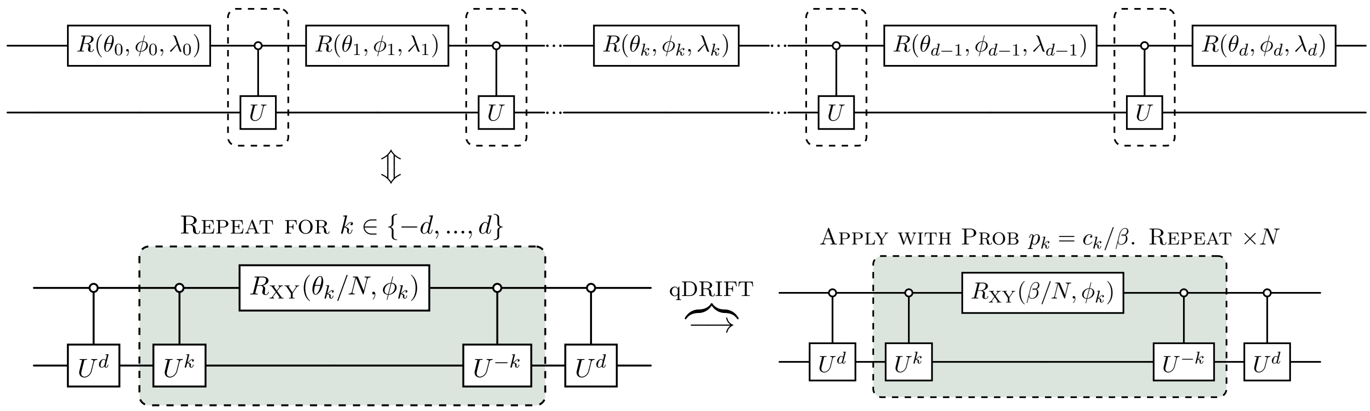

Figure 2: Transformation from a GQSP (with arbitrary SU(2) processing operators $R(\theta, \phi, \lambda)$) (top), to UHET (with an XY-plane rotation) (bottom right) using the small angle approximation ($\theta_k \to \theta_k/N$) (equivalent to Trotterisation) which keeps only linear terms, and then adding randomised compiling (qDRIFT) to the Trotterisation procedure.

3. Universal Hamiltonian Singular Value Transformation. We apply the same type of linearisation procedure to QSVT, resulting in the formulation of a new algorithm which we call Universal Hamiltonian Singular Value Transformation (UHSVT) (Figure 3), adding one more node into the map of quantum functional algorithms (Figure 1). We consider a setting where we want to transform the singular values of an arbitrary matrix A, given black-box access to the free evolution $e^{-iH[A]\tau}$ for arbitrary time τ of an unknown Hamiltonian $H[A]$, which is a *Hamiltonian block-encoding* [LKA+; KS25] of A. That is,

$$H(A) = \begin{bmatrix} 0 & A^\dagger \\ A & 0 \end{bmatrix} \overset{SVD}{\underbrace{\longrightarrow}} \begin{bmatrix} V & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} 0 & \Sigma \\ \Sigma & 0 \end{bmatrix} \begin{bmatrix} V^\dagger & 0 \\ 0 & U^\dagger \end{bmatrix}$$

where A has a singular value decomposition $A = U\Sigma V^\dagger$. This query model has potential applications in Hamiltonian learning when one seeks to learn the properties of a given block within an unknown Hamiltonian.

We then employ a randomised Trotter-Suzuki protocol to obtain a Hamiltonian block-encoding of the target matrix $f_{\mathrm{SV}}(A)$, corresponding to a desired function f_{SV} acting on the singular values of A. The general idea is to construct f with a truncated sine Fourier series (for odd functions):

$$f_{\mathrm{SV}}(A) = U f(\Sigma) V^\dagger \approx U \left(\sum_{k=1}^{K} b_k \sin(\pi \Sigma k/\Lambda) \right) V^\dagger \tag{1}$$

and build this series by combining each frequency term using randomised Trotterisation (Figure 3).

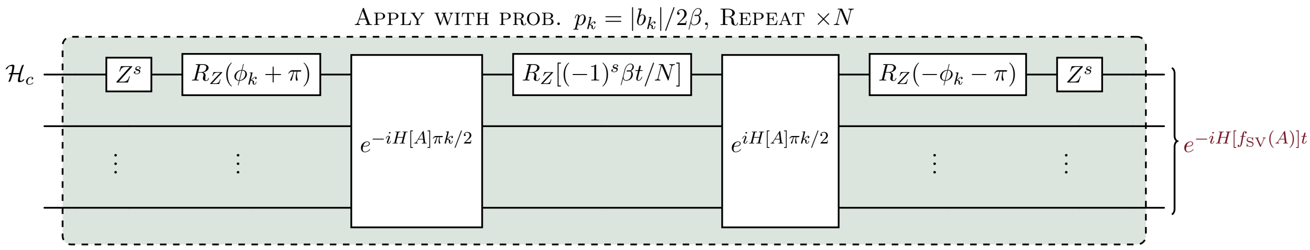

Figure 3: Sine Fourier series simulation for the singular values of A block-encoded into a Hamiltonian $H[A]$, whose dynamics is accessible as a black box. By choosing the frequency $k \in \{1, 2, ..., K\}$ and bit $s \in \{0, 1\}$ with probability $p_{k,s} = p_k = |b_k|/2\beta$, where b_k is the the k-th coefficients of the sine Fourier series of $f(x)$, one can achieve a functional transformation $A \to f_{\mathrm{SV}}(A)$. Here $b_k = |b_k|e^{-i\phi_k}$ and $\beta = \sum_{k=1}^{K} |b_k|$.

For each frequency k in the series, we construct two different circuits for $H^{(k,s)}$ depending on a boolean variable $s \in \{0, 1\}$. This boolean variable is introduced to annihilate the on-diagonal terms such that the overall Hamiltonian block-encoding structure is conserved. The final Hamiltonian after randomised Trotterisation is therefore:

$$\begin{aligned} H[A] &\to \sum_{k=1}^{K} \frac{|b_k|}{2} \overbrace{\begin{bmatrix} \cos(\pi A k) & e^{i\phi_k}\sin(\pi A k) \\ e^{-i\phi_k}\sin(\pi A k) & \cos(\pi A k) \end{bmatrix}}^{H^{(k,0)}} + \frac{|b_k|}{2} \overbrace{\begin{bmatrix} -\cos(\pi A k) & e^{i\phi_k}\sin(\pi A k) \\ e^{-i\phi_k}\sin(\pi A k) & -\cos(\pi A k) \end{bmatrix}}^{H^{(k,1)}} \\ &= \begin{bmatrix} 0 & \sum_{k=1}^{K} b_k^* \sin(\pi A k) \\ \sum_{k=1}^{K} b_k \sin(\pi A k) & 0 \end{bmatrix} = \begin{bmatrix} 0 & f_{\mathrm{SV}}^*(A^\dagger) \\ f_{\mathrm{SV}}(A) & 0 \end{bmatrix} = H[f_{\mathrm{SV}}(A)] \, . \end{aligned} \tag{2}$$

For sufficiently (J)-smooth functions, we show that our algorithm achieves a time complexity scaling of $\mathcal{O}(1/\epsilon)$, inherited from the underlying randomised Trotter-Suzuki approximation.

We compare this approach with the QSVT-based method of [KS25], which, despite achieving a better scaling of $\mathcal{O}(\log(1/\epsilon)/\epsilon^{1/J-1})$ requires the ability to perform an X gate on the qubitized subspace, a requirement that goes beyond standard QSVT and is not obviously compatible with a fully black-box setting. Moreover, inspired by [LKA+], we developed another QSVT-based algorithm that only requires a lower bound $\|\delta\|$ on the singular values of A. These comparisons are summarized in Tab 1. Since our algorithm does not require such assumptions, it is more versatile in a the context of black-box evolution. Finally, UHSVT is the only algorithm so far that can implement a complex-valued function.

Method	C^2 and odd	C^J and odd $(J \geq 3)$	C^∞ and odd	Assumptions
UHSVT (3)	$\mathcal{O}\left(\frac{1}{\epsilon}\log\left(\frac{1}{\epsilon}\right)\right)$	$\mathcal{O}\left(\frac{1}{\epsilon}\right)$	$\mathcal{O}\left(\frac{1}{\epsilon}\right)$	–
H-QSVT	$\mathcal{O}\left(\frac{1}{\delta^2\epsilon}\log^2\left(\frac{1}{\delta\epsilon}\right)\right)$	$\mathcal{O}\left(\frac{1}{\delta^2\epsilon^{1/J-1}}\log^2\left(\frac{1}{\delta\epsilon}\right)\right)$	$\mathcal{O}\left(\frac{1}{\delta^2}\log^3\left(\frac{1}{\delta\epsilon}\right)\right)$	lower bound $\|\|A\|\| \geq \delta$
[KS25]	$\mathcal{O}\left(\frac{1}{\epsilon}\log\left(\frac{1}{\epsilon}\right)\right)$	$\mathcal{O}\left(\frac{1}{\epsilon^{1/J-1}}\log\left(\frac{1}{\epsilon}\right)\right)$	$\mathcal{O}\left(\log^2\left(\frac{1}{\epsilon}\right)\right)$	X-gates on qubitised subspace

Table 1: Time complexity as a function of error ϵ between our and two QSVT-based algorithms for implementing a functional transformation of the singular values of a matrix A block-encoded into a Hamiltonian.

As a direct application of our algorithm, we consider the problem of inverse block-encoding introduced in [LKA+], which consists of transforming a Hamiltonian block-encoding $e^{-iH[A]t}$ into a unitary block-encoding $U[A]$, which is a unitary containing A as a block. Applying Algorithm 3 with $f(x) = \arcsin(x)$ yields the desired transformation with a complexity of $\mathcal{O}(\epsilon^{-1})$, which, unlike the approach of [LKA+], does not depend on a lower bound δ on the singular values of A.

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