Quantum theory over dual-complex numbers

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We take quantum theory and replace $\mathbb{C}$ by $\mathbb{C}[\varepsilon]$ where $\varepsilon^2 = 0$, i.e. we extend quantum theory to the ring of dual complex numbers. The aim is to develop a common language in which to treat continuous quantum physics and discrete quantum models in a unified manner, including their symmetries. Since quantum theory is linear, introducing ε is enough to model infinitesimals. A first objection to this programme is that $\mathbb{C}[\varepsilon]$ is not a field, since division by ε is undefined, while quantum mechanics typically relies on division. A second objection concerns whether unitarity still makes sense given $\varepsilon^2 = 0$. Hence, the core of this work is dedicated to proving that dual-complex quantum theory remains fully consistent. In particular, norm is preserved at all times, and renormalization never requires dividing by an infinitesimal. An equivalence with conventional quantum theory is demonstrated: the dual-complex extension of a parametrized quantum operation automatically provides a linear treatment of its first-order variations. As a first example application, we provide a unified description of both the Dirac equation in the continuum and the Dirac Quantum Walk in the discrete. We establish the discrete Lorentz covariance of the latter.

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1 Introduction

We replace the complex field by the dual-complex ring in the postulates of quantum mechanics. This modification allow us to linearly approximate the action of families of quantum operations $\mathscr{E}_z(\rho)$ parametrized by a complex number z. These numbers are of the form $z+t\varepsilon$ where $\varepsilon^2=0$ and z and t are complex numbers. Intuitively $t\varepsilon$ symbolizes a small variation of z. Dual-complex numbers have found wide applications in automatic differentiation, mechanics and geometry [9, 5, 33, 29]. Recent works have begun to extend our knowledge of dual-complex linear operators [26, 22]. An alternative synthetic treatment of infinities in quantum theory, based on non-standard analysis rather than nilpotent dual numbers, was developed in [18].

In this paper, we build upon these developments by formulating a self-consistent set of postulates that extend conventional quantum theory into the realm of dual-complex numbers. We introduce translation rules that automated mapping between the dual-complex and the conventional formalisms of quantum mechanics. Some of our results extend what was previously known about dual-complex unitary operators. These technical results may be of independent mathematical interest.

We argue that our extension is physical and offers a rigorous framework for a widely used but often informal technique in physicsthe neglect of higher-order terms in functional expansions. Since $\varepsilon^2=0$, the algebra of dual-complex numbers encodes first-order approximations, making it well-suited for modelling systems where higher-order effects should be neglected. We provide an example of application of our framework where it allows for an easy derivation of Dirac equation from a Discrete time Quantum Walk (QW).

Beyond their technical utility, dual-complex numbers also provide a compelling perspective in foundational debates, particularly the tension between discrete and continuous models of physics. While discrete models are computationally convenient and visually intuitive, continuous models have been more studied and are sometimes easier to work with, especially when symmetries are concerned for instance. This raises the question whether discrete models can fully represent Nature [21, 20, 31, 11, 10]. Through our example, we show that the dual-complex framework has the potential to unify these two perspectives by combining a continuous base field $\mathbb{C}$ with an infinitesimal variation which represents a discrete step. Such a hybrid structure may facilitate smooth translations between continuous and discrete frameworks, potentially offering new tools for quantum computation, simulation and foundations.

Dual-complex numbers constitute a new option in the foundational debate about which set of numbers is needed or useful to express quantum behaviours. Quantum theory based on extension of complex numbers such as quaternion numbers has been known for a long time ([16]) while the debate on the necessity of complex numbers over real numbers is still raging. [19]

In Sec.2, we recall all the required notions about dual-complex numbers for later sections. In Sec.3, we recall a few results about dual-complex linear operators and extend what is known about dual-complex unitary operators. We also define a notion of ordering required for our self-consistency check. In Sec.4, we use these results to present a set of postulates whose consistency is proved in Sec.5. Finally, in Sec.6, we present how dual-complex numbers are employed to automate the continuum limit calculations of the Dirac QW [2, 13], and make its discrete Lorentz covariance [1] exact.

2 Dual numbers

Extending real numbers with an imaginary unit i with $i^2=-1$ gives rise to the complex numbers. Complex numbers form a field. This field can be used to define a vector space. Quantum mechanics represents physical systems and their evolution within that vector space.

Notations. As usual the set of complex numbers $\{z=a+bi \mid a,b\in\mathbb{R} \text{ and } i^2=-1\}$ is denoted $\mathbb{C}$. As usual, we denote by real part (Re) and imaginary part (Im) the functions:

$$\mathrm{Re}:\mathbb{C}\to\mathbb{R}:a+bi\to a \quad (1) \qquad \mathrm{Im}:\mathbb{C}\to\mathbb{R}:a+bi\to b \quad (2) \qquad \forall z\in\mathbb{C}, z=\mathrm{Re}(z)+\mathrm{Im}(z)i \quad (3)$$

Dual numbers $\mathbb{D}$ were first introduced in [12] and follow a similar idea. They are generated by extending the real numbers with an *infinitesimal unit* ε such that $\varepsilon^2 = 0$. Multiplication and addition remain associative and commutative. Analogously, given a dual number $d \in \mathbb{D}$, we can take its *significant part* (Sig) and its *infinitesimal part* (Inf).

$$\mathrm{Sig}:\mathbb{D}\to\mathbb{R}:a+b\varepsilon\to a \quad (4) \qquad \mathrm{Inf}:\mathbb{D}\to\mathbb{R}:a+b\varepsilon\to b \quad (5) \qquad \forall d\in\mathbb{D}, d=\mathrm{Sig}(d)+\mathrm{Inf}(d)\varepsilon \quad (6)$$

Extending the real numbers with both i and ε gives rise to dual-complex numbers $\mathbb{DC}$. We can understand the complex, dual and dual-complex numbers as quotient rings:

$$\mathbb{C}\approx\mathbb{R}[i]/\langle i^2+1\rangle \quad (7) \qquad \mathbb{D}\approx\mathbb{R}[\varepsilon]/\langle\varepsilon^2\rangle \quad (8) \qquad \mathbb{DC}\approx\mathbb{C}[\varepsilon]/\langle\varepsilon^2\rangle. \quad (9)$$

Alternatively, dual-complex (resp. dual) numbers can be seen as the subalgebra of the matrices of the form

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, \tag{10}$$

where $a,b \in \mathbb{C}$ (resp. $a,b \in \mathbb{R}$). Dual and dual-complex numbers are widely used in automatic differentiation as they allow to "cut" the Taylor expansion and analytic functions from the second term on [5, 29].

$$f(z+\varepsilon) = f(z)+\varepsilon f'(z)+\varepsilon^2 f''(z)/2+\cdots = f(z)+\varepsilon f'(z). \tag{11}$$

Higher-order versions of dual-numbers were developed to deal with higher-order derivative, see [32, 6].

2.1 Ring operations and inversion on dual-complex numbers

Let $z_i, t_i \in \mathbb{C}, \forall i \in \mathbb{N}$ and $z_i + t_i\varepsilon \in \mathbb{DC}$. The definitions of addition and multiplication follow directly from associativity, commutativity, and $\varepsilon^2 = 0$:

$$(z_1+t_1\varepsilon)+(z_2+t_2\varepsilon) := (z_1+z_2)+(t_1+t_2)\varepsilon \tag{12}$$

$$(z_1+t_1\varepsilon)(z_2+t_2\varepsilon) := (z_1z_2)+(t_1z_2+z_1t_2)\varepsilon. \tag{13}$$

A dual-complex is called *infinitesimal* when its significant part is zero; the set of infinitesimal dual-complex numbers is denoted $\mathscr{O}_\varepsilon = \{0+t\varepsilon | t \in \mathbb{C}\} = \varepsilon\mathbb{C}$. A number that is not infinitesimal is called *appreciable*. When dividing $w_1 = z_1 + t_1\varepsilon$ by $w_2 = z_2 + t_2\varepsilon$ there are three cases:

1. When w_2 is appreciable, i.e. $z_2 \neq 0$. In this case, division is defined as

$$\frac{z_1+t_1\varepsilon}{z_2+t_2\varepsilon} := \frac{(z_1+t_1\varepsilon)(z_2-t_2\varepsilon)}{(z_2+t_2\varepsilon)(z_2-t_2\varepsilon)} = \frac{z_1z_2-z_1t_2\varepsilon+t_1z_2\varepsilon}{z_2^2} = \frac{z_1}{z_2}+\frac{z_2t_1-z_1t_2}{z_2^2}\varepsilon \tag{14}$$

   and w_2 has a unique inverse.

2. When w_1 and w_2 are infinitesimal, i.e. $t_1 = t_2 = 0$. The usual constraint that $w' = \frac{w_1}{w_2} \iff w'w_2 = w_1$ leaves us with infinitely many solutions. Indeed, take $w' = z' + t'\varepsilon \in \mathbb{DC}$ as a candidate solution:

$$z'+t'\varepsilon = \frac{t_1\varepsilon}{t_2\varepsilon} \iff z't_2\varepsilon+t't_2\varepsilon^2 = t_1\varepsilon \iff z' = t_1/t_2. \tag{15}$$

   We observe that t' is unconstrained. A natural choice would be $t' = 0$, implying $\frac{t_1'\varepsilon}{t_2'\varepsilon} = \frac{t_1'}{t_2'}$. This is only a convenience definition, as w_2 has no inverse.

3. When w_1 is appreciable and w_2 is infinitesimal. In this case we have no solution because $z' + t'\varepsilon = \frac{z_1}{t_2\varepsilon} \iff z' t_2 \varepsilon = z_1$ which is impossible.

Division between w_1 and w_2 is therefore defined for w_2 appreciable.

Exponentiation and rooting of dual-complex are given by

Proposition 1 (dual-complex exponentiation)**.** $(z+t\varepsilon)^n = z^n + nz^{n-1}t\varepsilon$

Proof in App. A.

Corollary 2 (dual-complex roots)**.** *Let $z_k^{1/n}$ be the k-branch of the n-th roots of $z \neq 0$. Then the k-th branch of the n-th roots of $z+t\varepsilon$ is given by*

$$(z+t\varepsilon)_k^{1/n} = z_k^{1/n} + \frac{t}{n(z_k^{1/n})^{n-1}}\varepsilon. \tag{16}$$

Proof in App. A.

2.2 Automatic differentiation

Dual numbers are useful for automatic differentiation. Given a complex function $f(z)$ which is holomorphic on the open set D, then for any $z, z_0 \in D$, we can express $f(z)$ as a sum of $f^{(n)}(z_0) := \frac{d^n f(z_0)}{dz^n}$ by means of the Taylor expansion $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)(z-z_0)^n}{n!}$. Following [24], a dual-complex function $\hat{f}(z+t\varepsilon)$ is holomorphic on $D + \mathscr{O}_\varepsilon$ if and only if for any $z+t\varepsilon, z_0 \in D + \mathscr{O}_\varepsilon$,

$$\hat{f}(z+t\varepsilon) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)(z+t\varepsilon-z_0)^n}{n!} \tag{17}$$

Proposition 3 (Automatic differentiation)**.** *[24] Given a complex function $f(z)$ which is holomorphic on the open set D, there exists a unique dual-complex function $\hat{f}(z+t\varepsilon)$ which is holomorphic on $D+\mathscr{O}_\varepsilon$ and agrees with $f(z)$ for every $z \in D$. For any $z \in D$, $z+t\varepsilon \in D+\mathscr{O}_\varepsilon$, the function is given by*

$$\hat{f}(z+t\varepsilon) = f(z) + t\varepsilon f^{(1)}(z). \tag{18}$$

Proof in App. A.

Corollary 4. *Dual-complex holomorphic extensions of exponential, logarithm, sine and cosine are given by*

$$\exp(z+t\varepsilon) = \exp(z)(1+t\varepsilon) \tag{19}$$

$$\log(z+t\varepsilon) = \log(z) + \frac{t}{z}\varepsilon \tag{20}$$

$$\sin(z+t\varepsilon) = \sin(z) + \cos(z)t\varepsilon \tag{21}$$

$$\cos(z+t\varepsilon) = \cos(z) - \sin(z)t\varepsilon \tag{22}$$

3 Linear algebra over dual-complex numbers

This section provides those properties of $\mathbb{D}$ and $\mathbb{DC}$ that are needed in order to formulate postulates of quantum theory over dual-complex numbers and prove their consistency. We introduce an order on dual numbers that will allow us to define valid probabilities. We motivate the choice of conjugation and norm on $\mathbb{DC}$, and establish some properties about them—before recalling results about diagonalization of dual-complex Hermitian operators. Finally, we prove that dual-complex unitary operators are also diagonalizable, and establish their relation to dual-complex anti-Hermitian operators, as exponentials and logarithms of each other.

3.1 Ordering dual and dual-complex numbers

In the usual definition of a 'total ring order' [17], one needs to have that (i) for all c, for all $a \leq b$, $a+c \leq b+c$ and (ii) for all $a \geq 0, b \geq 0$, $ab \geq 0$. We know that while $\mathbb{R}$ is a totally ordered ring, this is not the case for $\mathbb{C}$. Indeed, if $\mathbb{C}$ could be totally ordered we would have either $i \geq 0$ or $-i \geq 0$ and then $(\pm i)^2 = -1 \geq 0$. It turns out that there is a very natural lexicographic total ring order for $\mathbb{D}$:

Proposition 5 (dual-complex order). *There are exactly two extensions of the natural order on $\mathbb{R}$ that make $\mathbb{D}$ a totally ordered ring, determined by the choice $\varepsilon > 0$ or $\varepsilon < 0$.*

Proof in App. A.

3.2 Linear algebra

In [24] the authors suggests four different possible ways of conjugating dual complex numbers:

$$w^* = z^* + t^*\varepsilon \quad (23) \qquad \widetilde{w} = z - t\varepsilon \quad (24) \qquad \overline{\widetilde{w}} = z^* - t^*\varepsilon \quad (25) \qquad \overline{w} = z^*(1 - t\varepsilon/z) \quad (26)$$

The $\overline{w}$ option is the only one which satisfies both $\overline{w}w \in \mathbb{R}$ and $\mathrm{Re}(\mathrm{Sig}(w)) = \mathrm{Re}(\mathrm{Sig}(\overline{w}))$, but it is non-linear. Since our purpose is to extend quantum mechanics, which is fundamentally linear, we must give up this option and favour the first property over the second. Indeed, the two options $\widetilde{w}$ and $\overline{\widetilde{w}}$ lead to norms having an imaginary part, which is difficult to interpret in quantum mechanics. We are left with the w^* options, which leads to norms having an infinitesimal part. Those can be understood as representing infinitesimal variations of the norm, we opt for this definition as done in [28].

Definition 6. *Let $n \in \mathbb{N}$. We define $\mathbb{DC}^n$ to be the module of n-dimensional vectors with numbers in $\mathbb{DC}$, equipped with the inner-product:*

$$\langle\psi|\phi\rangle = \sum_k \psi_k^* \phi_k \tag{27}$$

The norm $|w|$ of $w = z + t\varepsilon$ is defined as $|w| = \sqrt{w^*w}$. We can observe the following properties

$$(w_1 + w_2)^* = w_1^* + w_2^* \quad (28) \qquad (w_1 w_2)^* = w_1^* w_2^* \quad (31) \qquad (w^*)^* = w \quad (34)$$

$$w + w^* = 2\mathrm{Re}(z) + 2\mathrm{Re}(t)\varepsilon \quad (29) \qquad w^*w = |z|^2 + 2\mathrm{Re}(z^*t)\varepsilon \in \mathbb{D} \quad (32) \qquad |w| = \sqrt{w^*w} = |z| + \frac{Re(z^*t)}{|z|}\varepsilon \quad (35)$$

$$|w| = 0 \iff z = 0 \quad (30) \qquad |w| \geq 0 \quad (33)$$

In particular equations (33) and (35) entail the following propositions.

Proposition 7 (Dual-complex norm). *For every $|\psi_\varepsilon\rangle = |\psi\rangle + \varepsilon\,|\phi\rangle \in \mathbb{DC}^n$, $\langle\psi_\varepsilon|\psi_\varepsilon\rangle = \langle\psi|\psi\rangle + 2\mathrm{Re}(\langle\psi|\phi\rangle)\varepsilon$ and therefore*

$$||\,|\psi_\varepsilon\rangle\,|| = \sqrt{\langle\psi_\varepsilon|\psi_\varepsilon\rangle} = ||\,|\psi\rangle\,|| + \frac{\mathrm{Re}(\langle\psi|\phi\rangle)}{||\,|\psi\rangle\,||}\varepsilon. \tag{36}$$

We say that $|\psi_\varepsilon\rangle$ is unit *norm iff $||\,|\psi_\varepsilon\rangle\,|| = 1$.*

Corollary 8 (Dual-complex norm non-negativity). *For any $|\psi\rangle \in \mathbb{DC}^n$, $||\,|\psi\rangle\,|| = \sum_{k=1}^n \psi_k^* \psi_k \geq_{\mathbb{D}} 0$.*

Notice that the norm of a vector is infinitesimal if and only if all entries are infinitesimal. In this case the norm of the vector is zero, and the vector itself is said to be infinitesimal. Conversely, the norm of a vector is appreciable if and only if at least one entry is appreciable. In this case the vector itself is said to be appreciable. Two vectors are called orthogonal when their inner product is zero when their inner product is infinitesimal.

We define the adjoint as the transpose conjugate, i.e. $A^\dagger = (A^T)^*$. As usual an operator H is said to be Hermitian if $H^\dagger = H$ and anti-Hermitian if $H^\dagger = -H$, for both its significant part and infinitesimal part. An operator U is unitary when $U^\dagger U = I$. It follows that:

Proposition 9 (Dual-complex unitarity). *The following propositions are equivalent for U_ε a square linear operator:*

(a) U_ε *is dual-complex unitary, i.e.* $U_\varepsilon^\dagger U_\varepsilon = I$*.*

(b) U_ε *preserves the inner product, i.e.* $(U_\varepsilon |\psi\rangle)^\dagger (U_\varepsilon |\phi\rangle) = \langle\psi|\phi\rangle$ *and therefore the norm* $||U_\varepsilon |\psi\rangle|| = || |\psi\rangle||$*.*

(c) Rows of U_ε *form an orthonormal basis.*

(d) Columns of U_ε *form an orthonormal basis.*

(e) $U_\varepsilon = (I + i\varepsilon H)U$ *where* U *is a complex unitary and* H *is a complex Hermitian.*

Proof in App. B.

In the rest of this section, we prove that the exponential of dual-complex anti-Hermitian is unitary and that the logarithm of a dual-complex unitary is anti-Hermitian. We first introduce the exponential of a dual-complex operator and recall a known result about dual-complex Hermitian diagonalization. Next, we give two new results about dual-complex unitary operators.

Proposition 10 (Dual-complex matrix exponentiation)**.** *The dual-complex matrix exponential always exists.*

Proof in App. B.

It is interesting to note that a proof identical to that of [22] for the case of dual numbers can be used to show that the exponential of dual-complex matrix $A + B\varepsilon$ is $\exp(A) + \varepsilon L_{\exp}(A, B)$ where $L_{\exp}(A, B)$ is the Fréchet derivative of the exponential function of A in the direction of B.

Proposition 11 ([27])**.** *Suppose* H *is a Hermitian operator. Then there exists an orthonormal dual-complex basis* $\{|j_\varepsilon\rangle\}$ *and* n *dual eigenvalues* $\lambda_j + \varepsilon\mu_j$ *of* H *such that* $H = \sum_j (\lambda_j + \varepsilon\mu_j) |j_\varepsilon\rangle\langle j_\varepsilon|$*.*

We now establish that dual-complex unitary operators, just like dual-complex Hermitian operators, are diagonalizable and deduce that their logarithms are anti-Hermitian.

Proposition 12 (Spectral theorem for dual-complex unitary)**.** *A unitary operator is unitarily diagonalizable, i.e. for every dual-complex unitary operator* U_ε*, there is a dual-complex orthonormal eigenbasis* $\{|j_{U_\varepsilon}\rangle\}$ *with associated dual-complex eigenvalues* $e^{i\theta_j}(1 + \varepsilon i\mu_j)$ *such that* $U_\varepsilon = \sum_j e^{i\theta_j}(1 + \varepsilon i\mu_j) |j_{U_\varepsilon}\rangle\langle j_{U_\varepsilon}|$*, where* θ_j *and* μ_j *are real numbers.*

Proof in App. B.

Thanks to the diagonalization of unitary operators, we can establish the following result.

Proposition 13 (Dual-complex unitary as exponential of anti-Hermitian)**.** *The exponential of anti-Hermitian linear operator is a unitary. Moreover, every unitary* U_ε *is of the form* $\exp(iH_\varepsilon)$ *where* H_ε *is dual-complex Hermitian.* H_ε *is unique modulo* 2π*.*

Proof in App. B.

Another result is required to extend the measurement postulate. We say the operator E is appreciably semipositive if $\langle\psi|E|\psi\rangle$ is positive and appreciable or exactly zero for every vector $|\psi\rangle$.

Proposition 14 (Dual-complex semipositivity)**.** *For every dual-complex linear operator* M_ε*,* $M_\varepsilon^\dagger M_\varepsilon$ *is appreciably semipositive.*

Proof in App. B.

The Stinespring dilation theorem also continues to hold with dual-complex numbers:

Proposition 15 (Dual-complex Stinespring dilation)**.** *For every family of dual-complex operators* $\{M_m^\varepsilon\}$*, let* $V_\varepsilon := \sum_m |m\rangle \langle 0| \otimes M_m^\varepsilon$*. We have that*

$$M_m^\varepsilon |\psi\rangle = (\langle m| \otimes I)V_\varepsilon(|0\rangle \otimes |\psi\rangle)$$
$$\sum_m {M_m^\varepsilon}^\dagger M_m^\varepsilon = I \Leftrightarrow V_\varepsilon^\dagger V_\varepsilon = I$$

If the completeness relation is satisfied, V_ε *is a dual-complex isometry and can be extended to a dual-complex unitary operator* U_ε*, called a* Stinespring dilation*.*

Proof. The proof is the same as for the case of complex numbers. □

4 Postulates of quantum theory over dual-complex numbers

In this section we take the postulates of conventional quantum theory, and replace the complex scalar field, by the dual-complex ring. This defines a quantum theory over dual-complex numbers, which we refer to as dual-complex quantum theory for short.

Postulate 1 (Dual-complex states)**.** *At all times, a dual-complex quantum system can be described by a dual-complex state $|\psi_\varepsilon\rangle$, defined to be a unit dual-complex vector.*

Every dual-complex state $|\psi_\varepsilon\rangle$ decomposes as into a significant part $|\psi\rangle$ and an infinitesimal part $|\phi\rangle$, i.e. $|\psi_\varepsilon\rangle = |\psi\rangle + \varepsilon\,|\phi\rangle$. By Eq. (36):

$$\begin{aligned} ||\,|\psi\rangle + \varepsilon\,|\phi\rangle\,|| = 1 &\Leftrightarrow ||\,|\psi\rangle\,|| + \varepsilon\,\mathrm{Re}(\langle\psi|\phi\rangle) = 1 \\ &\Leftrightarrow (i) \quad ||\,|\psi\rangle\,|| = 1 \quad \text{and} \quad (ii) \quad \mathrm{Re}(\langle\psi|\phi\rangle) = 0 \end{aligned}$$

Interestingly, if we choose to interpret the infinitesimal part as some kind of perturbation of the significant part, i.e. $|\phi\rangle = A\,|\psi\rangle$, then this perturbation A is necessarily anti-Hermitian. Indeed, by (ii):

$$\mathrm{Re}(\langle\psi|\phi\rangle) = \mathrm{Re}(\langle\psi|A|\psi\rangle) = 0 \Leftrightarrow \langle\psi|(iA)|\psi\rangle \in \mathbb{R} \Leftrightarrow A \text{ is anti-Hermitian.}$$

Postulate 2 (Dual-complex evolutions)**.** *Over any finite period of time, the evolution of a dual-complex quantum system is described by the application of a dual-complex evolution U_ε, defined to be a unitary dual-complex operator.*

Every dual-complex evolution U_ε decomposes as into a significant part U and an infinitesimal part V, i.e. $U_\varepsilon = U + \varepsilon V$. Recall from our Prop. 9:

$$\begin{aligned} U_\varepsilon^\dagger U_\varepsilon = 1 &\Leftrightarrow (i) \quad U^\dagger U = I \quad \text{and} \quad (ii) \quad U^\dagger V \text{ anti-Hermitian} \\ &\Leftrightarrow (i) \quad U^\dagger U = I \quad \text{and} \quad (ii) \quad V = iHU \text{ with } H \text{ hermitian} \end{aligned}$$

This fits the interpretation that the infinitesimal part of a dual-complex state is indeed produced by an anti-Hermitian perturbation of the significant part of the unitary evolution. As a first example consider:

$$U_\varepsilon = \begin{pmatrix} -im\varepsilon & 1 \\ 1 & -im\varepsilon \end{pmatrix} \quad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad V = \begin{pmatrix} -im & 0 \\ 0 & -im \end{pmatrix} \quad H = \begin{pmatrix} 0 & -m \\ -m & 0 \end{pmatrix}$$

Notice although U_ε perfectly unitary in the sense of Prop. 9, it feels dangerously remote from a unitary in the conventional sense, i.e. if we replace ε by $h \in \mathbb{R}$ the obtained U_h is not a unitary. Still we will prove in Sec. 5, Prop. 18 that U_h is at most $O(h^2)$ away from a unitary, here

$$\widetilde{U}_h = \begin{pmatrix} -i\sin(mh) & \cos(mh) \\ \cos(mh) & -i\sin(mh) \end{pmatrix}.$$

Another way to put this is that U_ε could equally well have been presented as $\widetilde{U}_\varepsilon$, because over dual-complex numbers we just have $U_\varepsilon = \widetilde{U}_\varepsilon$. The apparent relaxation of unitarity is therefore innocuous.
In conventional quantum mechanics, the evolution postulate is equivalently expressed in continuous time through the Schrödinger equation. From our Prop. 13, the same equivalence holds here.

Postulate 3. *(continuous-time evolutions) The continuous time evolution of a closed system is described by the Schrödinger equation,*

$$i\hbar \frac{\mathrm{d}\,|\psi_\varepsilon\rangle}{\mathrm{d}t} = H_\varepsilon\,|\psi_\varepsilon\rangle \tag{37}$$

where H_ε is a dual-complex Hermitian linear operator.

Postulate 4 (Dual-complex measurements)**.** *A family of dual-complex operators $\{M_m^\varepsilon\}$ describes a dual-complex measurement iff it is bound by the completeness relation $\sum_m {M_m^\varepsilon}^\dagger M_m^\varepsilon = I$. The outcome m appears with the dual probability $p(m) = \langle\psi| {M_m^\varepsilon}^\dagger M_m^\varepsilon |\psi\rangle$. The resulting dual-complex state is*

$$|\psi'_\varepsilon\rangle = \frac{M_m^\varepsilon |\psi\rangle}{\sqrt{p(m)}}. \tag{38}$$

Every dual-complex measurement operator M_m^ε decomposes into a significant part M_m and an infinitesimal part N_m, i.e. $M_m^\varepsilon = M_m + \varepsilon N_m$. Expanding the completeness relation:

$$\begin{aligned}\sum_m (M_m+\varepsilon N_m)^\dagger (M_m+\varepsilon N_m) = I &\Leftrightarrow \sum_m M_m^\dagger M_m + \varepsilon \sum_m (M_m^\dagger N_m + N_m^\dagger M_m) = I \\ &\Leftrightarrow (i) \quad \sum_m M_m^\dagger M_m = I \quad \text{and} \quad (ii) \quad M_m^\dagger N_m \text{ anti-Hermitian.}\end{aligned}$$

To ensure our dual-complex quantum theory does not cause unwanted behaviors, we must check that probabilities are positive and sum to one. We must also check that the resulting dual-complex state is well-defined, i.e. that it does not get renormalized by an infinitesimal probability. Fortunately, none of these bad behaviors arise, as we will show in Sec. 5, Prop. 16.

As in conventional quantum theory, we can see that when we have exactly one measurement operator, then it must be unitary, which means that Postulate 2 can be deduced from Postulate 4. On the other hand, we will show in Sec. 5 that, through Stinespring dilation, dual-complex measurement operators can also be seen as arising from a dual-complex unitary evolution of a larger closed system. Moreover, Postulate 3 implies that the infinitesimal part of the dual-complex unitary evolution arises from the infinitesimal part of the energy observable, which, as a consequence of Postulate 4, can have dual-complex eigenvalues and eigenvectors. In Sec. 5 it becomes apparent that ε can be interpreted as an infinitesimal variation of a parameter of the experiment being carried.

Postulate 5 (Composite dual-complex quantum systems)**.** *Given two dual-complex states $|\psi_\varepsilon\rangle^A$ and $|\psi_\varepsilon\rangle^B$ describing two dual-complex quantum systems A and B that have not yet interacted, the composite dual-complex quantum system AB is described by the dual-complex state $|\psi_\varepsilon\rangle^A \otimes |\psi_\varepsilon\rangle^B$ with $\otimes$ the usual tensor aka Kronecker product.*

Notice that this postulate preserves the unit norm. Indeed, if we have two dual-complex states $|\psi_\varepsilon\rangle^A$ and $|\psi_\varepsilon\rangle^B$, the norm of $|\psi_\varepsilon\rangle^A \otimes |\psi_\varepsilon\rangle^B$ is still $\sqrt{\langle\psi_\varepsilon|\psi_\varepsilon\rangle \langle\phi_\varepsilon|\phi_\varepsilon\rangle} = 1$.

5 Consistency of dual-complex quantum theory

Self-consistency

We first prove that dual-complex quantum theory is internally consistent, in the sense that dual-complex evolutions and measurements of a system preserves the fact of being a dual-complex state, and that measurement outcome probabilities remain well-behaved.

Proposition 16. *A system that starts in a dual-complex state and is evolved according to dual-complex evolutions and dual-complex measurements, remains described by a dual-complex state at all times. Moreover, probabilities from any dual-complex measurement are either zero or appreciably positive, and sum to one.*

Proof. By Prop. 9, a dual-complex unitary preserves the norm. For measurements, from Eq. (38) norm preservation is given by

$$\left\| \frac{M_m^\varepsilon |\psi\rangle}{\sqrt{p_\varepsilon(m)}} \right\| = \left\| \frac{M_m^\varepsilon |\psi\rangle}{\sqrt{\langle\psi| {M_m^\varepsilon}^\dagger M_m^\varepsilon |\psi\rangle}} \right\| = \frac{||M_m^\varepsilon |\psi\rangle||}{||M_m^\varepsilon |\psi\rangle||} = 1. \tag{39}$$

where $p_\varepsilon(m)$ is the probability of outcome m.

Probabilities sum to one since $\sum_m p_\varepsilon(m) = \sum_m \langle\psi| {M_m^\varepsilon}^\dagger M_m^\varepsilon |\psi\rangle = \langle\psi| (\sum_m {M_m^\varepsilon}^\dagger M_m^\varepsilon) |\psi\rangle = \langle\psi| I |\psi\rangle = 1$ and are either zero or appreciably positive by Prop. 14. □

If we decompose into significant and infinitesimal parts as $p_\varepsilon(m) = p_m + \varepsilon \dot{p}_m$, $M_m^\varepsilon = M_m + \varepsilon \dot{M}_m$ and $|\phi_\varepsilon\rangle = |\phi\rangle + \varepsilon |\dot{\phi}\rangle = M_m^\varepsilon |\psi\rangle$, we can easily check that $|\phi\rangle = M_m \operatorname{Sig}(|\psi\rangle)$, which only depends on the significant part of M_ε and $|\psi\rangle$. Interestingly, if we think of the infinitesimal parts as representing the derivatives with respect to a variable z of the significant parts in each of these decompositions, then by Eq. 14 and the quotient rule,

$$\frac{M_m^\varepsilon |\psi\rangle}{\sqrt{p_\varepsilon(m)}} = \frac{|\phi\rangle}{p_m} + \varepsilon \frac{|\phi\rangle \dot{p}_m - |\dot{\phi}\rangle p_m}{p_m^2} = \frac{|\phi\rangle}{p_m} + \varepsilon \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{|\phi\rangle}{p_m}\right) \tag{40}$$

Consistency with conventional quantum theory

In Sec 2.2 we showed how holomorphic functions can be extended from complex numbers to dual-complex . In a similar vein, we now show how conventional quantum theory operators parametrized by h, can be extended to become dual-complex quantum theory operators.

We subsequently answer to concerns that the dual-complex quantum theory operators obtained that way, or otherwise, may only be valid only in dual-complex quantum theory. E.g. dual-complex unitary operators and dual-complex complete measurements are not in general unitary and complete in the conventional sense—when ε is replaced by a real number h. We give a general method to add an $O(h^2)$ correction which "unitarizes/completes" them, in the conventional sense.

Definition 17 (Dual-complex extension and complex correction for unitaries)**.** *Let U_h be a conventional unitary which is analytic in its real parameter h. Let $U := U_0$ and recall that $\frac{\mathrm{d}U_h}{\mathrm{d}h}\big|_{h=0} = iHU$ for some Hermitian H. The* dual-complex extension *of U_r is defined by:*

$$\hat{U}_\varepsilon := U + \varepsilon iHU.$$

Let U_ε be a dual-complex unitary. Recall that $U_\varepsilon = U + \varepsilon iHU$. Its complex correction *is the conventional unitary:*

$$\widetilde{U}_h := U_h + h^2 \sum_{n=0}^{\infty} \frac{1}{(n+2)!}(iH)^{n+2} h^n U = \exp(ihH)U \tag{41}$$

Proposition 18 (Dual-complex extension and complex correction for unitaries)**.** *Consider U_h a conventional unitary analytic in its real parameter h. We have $U_h = \hat{U}_h + O(h^2)$.*
Consider U_ε a dual-complex unitary. We have $U_\varepsilon\big|_{\varepsilon=h} + O(h^2) = \widetilde{U}_h$.

Proof. First part, the Taylor expansion of U_h gives

$$U_h = U + h\,iHU + O(h^2) = \hat{U}_h + O(h^2). \tag{42}$$

Second part,

$$\begin{aligned} \widetilde{U}_h &= \exp(ihH)U = \left(I + ihH + O(h^2)\right)U \quad \text{by def. of matrix exponentials} \\ &= U + ihHU + O(h^2) = U_\varepsilon\big|_{\varepsilon=h} + O(h^2). \end{aligned}$$

□

Definition 19 (Dual-complex extension and complex correction for measurements)**.** *Let $\{M_m^h\}$ be a conventional complete measurement analytic in its real parameter h. Let $M_m := M_m^0$ and $N_m := \frac{\mathrm{d}M_m^h}{\mathrm{d}h}\big|_{h=0}$. The* dual-complex extension *of $\{M_m^h\}$ is $\{\hat{M}_m^\varepsilon\}$ with*

$$\hat{M}_m^\varepsilon := M_m + \varepsilon N_m.$$

Let $\{M_m^\varepsilon\}$ be a dual-complex complete measurement with Stinespring dilation $U_\varepsilon = U + \varepsilon iHU$ (Prop. 15). The corresponding complex correction *is the conventional measurement $\{\widetilde{M}_m^h\}$ with:*

$$\widetilde{M}_m^h := (\langle m| \otimes I)\,\widetilde{U}_h\,(|0\rangle \otimes I). \tag{43}$$

Proof. $\{\hat{M}_m^\varepsilon\}$ is complete as dual-complex measurement since

$$\sum_m (\hat{M}_m^\varepsilon)^\dagger \hat{M}_m^\varepsilon = \sum_m (M_m^\dagger + \varepsilon N_m^\dagger)(M_m + \varepsilon N_m) = \sum_m M_m^\dagger M_m + \varepsilon \sum_m (M_m^\dagger N_m + N_m^\dagger M_m)$$
$$= \sum_m M_m^\dagger M_m + \varepsilon \frac{\mathrm{d}}{\mathrm{d}h}\Big|_{h=0} \sum_m (M_m^h)^\dagger M_m^h = I.$$

$\{\widetilde{M}_m^h\}$ is a valid conventional complete measurement since $\widetilde{U}_h$ is a conventional unitary. □

Proposition 20 (Dual-complex extension and complex correction for measurements)**.** *Consider $\{M_m^h\}$ a conventional measurement analytic in its real parameter h. We have $M_m^h = \hat{M}_m^h + O(h^2)$ and $p^h(m) = \hat{p}^h(m) + O(h^2)$.*
Consider $\{M_m^\varepsilon\}$ a dual-complex measurement with Stinespring dilation U_ε. We have $M_m^h\big|_{\varepsilon=h} + O(h^2) = \widetilde{M}_m^h$ and $p^\varepsilon\big|_{\varepsilon=h}(m) + O(h^2) = \widetilde{p}^h(m)$.

Proof. First part, the Taylor expansion of M_m^h gives

$$M_m^h = M_m + hN_m + O(h^2) = \hat{M}_m^h + O(h^2).$$

Hence, $(M_m^h)^\dagger M_m^h = (\hat{M}_m^h)^\dagger \hat{M}_m^h + O(h^2)$, so $p^h(m) = \hat{p}^h(m) + O(h^2)$.
Second part, by Prop. 18, $\widetilde{U}_h = U_\varepsilon\big|_{\varepsilon=h} + O(h^2)$ with $U_\varepsilon\big|_{\varepsilon=h} = U + hiHU$. Thus

$$\widetilde{M}_m^h = (\langle m| \otimes I)\, \widetilde{U}_h\, (|0\rangle \otimes I) = (\langle m| \otimes I)(U_\varepsilon\big|_{\varepsilon=h} + O(h^2))(|0\rangle \otimes I)$$
$$= (\langle m| \otimes I) U_\varepsilon\big|_{\varepsilon=h} (|0\rangle \otimes I) + O(h^2) = M_m^h + O(h^2).$$

So $(\widetilde{M}_m^h)^\dagger \widetilde{M}_m^h = (M_m^h)^\dagger M_m^h + O(h^2)$, hence $\widetilde{p}^h(m) = p^\varepsilon\big|_{\varepsilon=h}(m) + O(h^2)$. □

Example 21 illustrates the use of this proposition.

6 Application: the Dirac Quantum Walk

In this section, we apply dual-complex quantum theory in order to provide a unified treatment of both 1/ the Dirac QW [4]—which is a fundamentally discrete space discrete time quantum algorithm simulating the propagation of a fermion and 2/ the Dirac equation—which is a fundamentally continuous space continuous time PDE describing the propagation of a fermion. Essentially, all we need to do formulate the Dirac QW, i.e. take seriously the fact that $\varepsilon^2 = 0$.

The Dirac QW is a discrete model a particle moving in discrete steps $\Delta_x = \Delta_t = \varepsilon$. It moves either left or right, accordingly to its spin degree of freedom, but sometimes changes direction is a way that is proportional to εm. Let $|x+\rangle$ be the basic state that represents a right-mover at x, and similarly let $|x-\rangle$ represent the left-mover. At any given time, the dual-complex states we employ are of the form $|\psi(t)\rangle = \sum_x \psi^+(x,t)\,|x+\rangle + \psi^-(x,t)\,|x-\rangle$. The relation between the amplitudes at step t and at step $t+\varepsilon$ is given by

$$\begin{pmatrix} \psi^-(x-\varepsilon, t+\varepsilon) \\ \psi^+(x+\varepsilon, t+\varepsilon) \end{pmatrix} = U_\varepsilon \begin{pmatrix} \psi^+(x,t) \\ \psi^-(x,t) \end{pmatrix} \tag{44}$$

where U_ε is the dual-complex evolution

$$U_\varepsilon = \begin{pmatrix} -im\varepsilon & 1 \\ 1 & -im\varepsilon \end{pmatrix}. \tag{45}$$

In the literature, one usually finds

$$\widetilde{U}_\varepsilon = \begin{pmatrix} -i\sin(m\varepsilon) & \cos(m\varepsilon) \\ \cos(m\varepsilon) & -i\sin(m\varepsilon) \end{pmatrix} \tag{46}$$

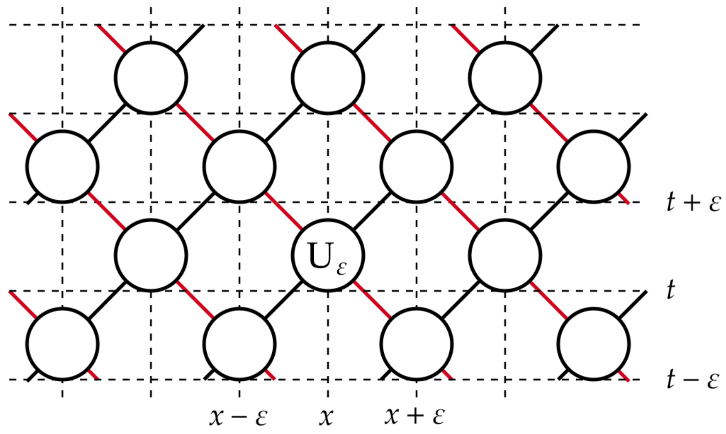


Figure 1: The Dirac Quantum Walk

as it feels more unitary. But, over the dual-complex numbers, $U_\varepsilon = \widetilde{U_\varepsilon}$, and in dual-complex quantum theory U_ε is already unitary. These gates are arranged as in Fig. 1. Applying the automatic differentiation Prop. 3, to the second row of Eq. (6), we get

$$\begin{aligned} & \psi^+(x+\varepsilon, t+\varepsilon) = \psi^+(x,t) - im\varepsilon\psi^-(x,t) \\ \Leftrightarrow \quad & \psi^+(x,t) + \varepsilon\frac{\partial}{\partial x}\psi^+(x,t) + \varepsilon\frac{\partial}{\partial t}\psi^+(x,t) = \psi^+(x,t) - im\varepsilon\psi^-(x,t) \\ \Leftrightarrow \quad & \frac{\partial}{\partial t}\psi^+(x,t) = -\frac{\partial}{\partial x}\psi^+(x,t) - im\psi^-(x,t) \end{aligned}$$

Similarly, with the first row we get

$$\frac{\partial}{\partial t}\psi^-(x,t) = \frac{\partial}{\partial x}\psi^-(x,t) - im\psi^+(x,t).$$

These two PDEs can be stacked in a vector form, this is the Dirac equation:

$$\partial_t\psi = -\sigma_3\partial_x\psi - im\sigma_1\psi \quad \text{with} \quad \psi(x,t) = \begin{pmatrix}\psi^+(x,t)\\ \psi^-(x,t)\end{pmatrix}.$$

In other words, in the dual-complex quantum theory formulation of the Dirac QW, the process of taking the continuum limit is fully automated. In fact, the very same object describes both the QW and the PDE, they are equated.

A major byproduct of this equality is that the continuous symmetries of the PDE are no longer broken by the QW formulation.

Indeed, the fundamental symmetry upon which the Dirac equation is built is Lorentz covariance, capturing the fact that Physics is unchanged under Lorentz transformations, which correspond to rescalings along the lightlike directions. For QWs, a precise notion of discrete Lorentz covariance exists and has been studied in detail in [1]. In that setting, a discrete Lorentz transform with integer parameters α, β is defined by replacing each spacetime point with a lightlike $\alpha \times \beta$ rectangular patch. Each such patch is obtained by applying isometric encodings E_α and E_β on the left-moving and right-moving wires, then evolving under the same QW, although with a rescaled mass, throughout the patch. Performing such a discrete Lorentz transform and then taking the continuum limit, is the same as taking the continuum limit first and then performing the continuous Lorentz transform. Discrete Lorentz covariance is the requirement that these patches join correctly, which amounts to the quantum circuit equality shown in Fig. 2. I.e. there must exist a family of isometric encodings E_α, E_β such that for all α, β, the QW evolution and the encoding commute in the sense of this figure.

Unfortunately, the conventional quantum theory formulation of the Dirac QW is not exactly discrete Lorentz covariant. As shown by Fig. 3a, some terms in $O(\varepsilon^2)$ mess up the required circuit equality. Discrete Lorentz covariance holds only to first order, as illustrated in Fig. 3b.

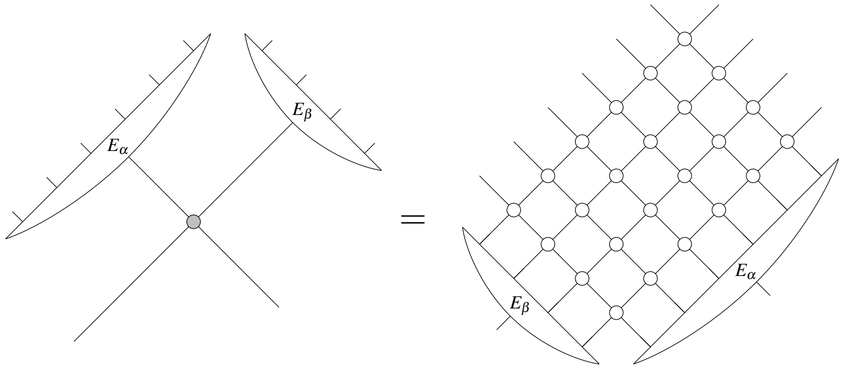
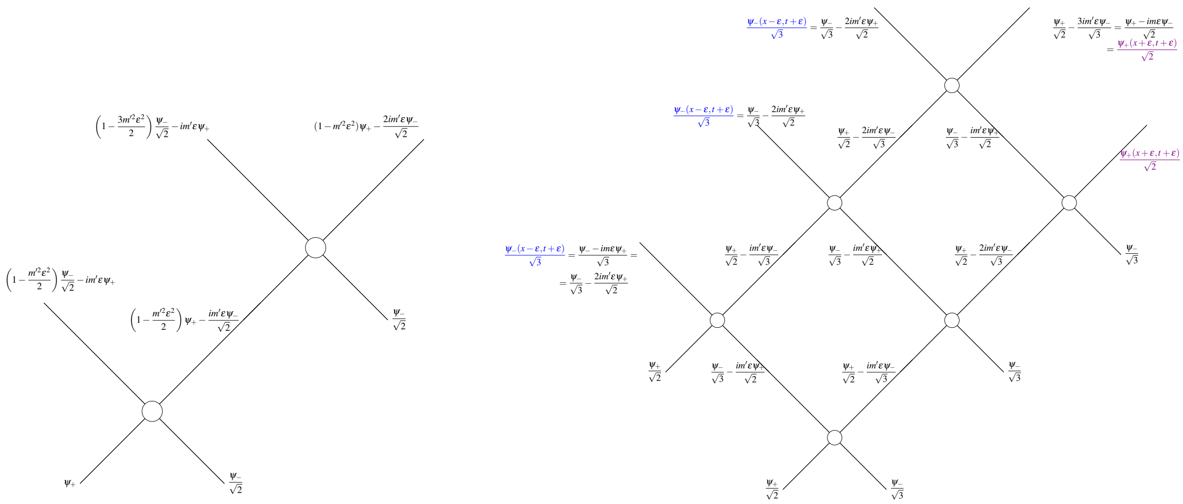


Figure 2: Discrete Lorentz covariance as a quantum circuit equality.



(a) Failure of covariance at second order. (b) First-order covariance, hence exact dual-complex covariance.

Figure 3: *Discrete Lorentz covariance for the Dirac Quantum Walk.* Here the chosen isometries E_α and E_β spread the incoming amplitude uniformly, and $m' = m/\sqrt{\alpha\beta}$.

In the dual-complex quantum theory formulation of the Dirac QW, those second-order terms therefore vanish, thereby restoring exact discrete Lorentz covariance. This is a contribution, bearing in mind the many works dealing with discreteness induced Lorentz-symmetry breaking e.g. [30, 23, 15, 7, 8, 14].

Finally, notice that by applying the complex correction of Def. 17, we recover the conventional quantum theory Dirac QW $\widetilde{U}_h$. Interestingly, this automates the exact path Meyer took, by hand, to design the first ever QW [25]. Prop. 18 ensures that $\widetilde{U}_h = U_h + O(h^2)$.

7 Conclusion

In this paper, we demonstrated that quantum theory can safely be extended from complex numbers to dual-complex numbers, and that this offers a common language in order to deal with both continuous and discrete physical models.

First, we advanced our knowledge of dual-complex linear algebra by demonstrating that dual-complex unitary are 1) diagonalizable and 2) related to dual-complex Hermitian *via* exponentiation and logarithm maps. We also introduced a notion of ordering on dual-complex numbers.

Second, we presented a set of postulates extending quantum theory to $\mathbb{C}[\varepsilon]$ with $\varepsilon^2 = 0$. We showed that the equivalence between the discrete-time unitary formulation and the continuous-time Schrödinger equation continues to hold.

Third, we established the self-consistency of dual-complex quantum theory by showing norm and probability preservation, and introducing a translation between parametrized conventional quantum theory and dual-complex quantum theory. One can always be viewed as approximating the other up to $O(h^2)$.

Fourth, we applied this formalism to a concrete example, namely we were able to describe both the Dirac equation and its corresponding Discrete Time Quantum Walk, as a single object.

These results show the possibility and usefulness of applying dual-complex numbers to quantum theory. They are also of general mathematical interest as they contribute to the ongoing research on dual-complex linear operators.

We aim to pursue these results in order to reach a mathematical framework in which we can unify discrete and continuous Physics symmetries beyond Lorentz boosts.

Additionally, the foundational debate on the set of numbers to use for quantum mechanics could also benefit of generalization of our extension to higher-order dual-complex numbers or general quaternions $H(\alpha, 0)$.

Another direction would be to explore connections with Grassman numbers, which also obeys nilpotency conditions and may offer new ways of describing fermionic systems in a generalized dual-complex formalism of quantum mechanics.

Ultimately, we wish perfect the formalism in order to reconcile the following apparent paradox. On the one hand, quantum cellular automata seem to require $\varepsilon^2 = 0$ in order to be Lorentz covariant [1]. But on the other hand, they seem to require $\varepsilon^2 \neq 0$ in order the model particular phenomena, such as the electric term of QED [3].

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A Proofs for Sec. 2

Proposition 1 (dual-complex exponentiation)**.** $(z+t\varepsilon)^n = z^n + nz^{n-1}t\varepsilon$

Proof. $\mathbb{DC}$ is a commutative ring and therefore the binomial theorem holds.

$$(z+t\varepsilon)^n = \sum_{k=0}^{n} \binom{n}{k} z^{n-k}(t\varepsilon)^k = z^n + nz^{n-1}t\varepsilon + (t\varepsilon)^2 \sum_{k=2}^{n} \binom{n}{k} z^{n-k}(t\varepsilon)^{k-2} = z^n + nz^{n-1}t\varepsilon \tag{47}$$

□

Corollary 2 (dual-complex roots)**.** *Let $z_k^{1/n}$ be the k-branch of the n-th roots of $z \neq 0$. Then the k-th branch of the n-th roots of $z+t\varepsilon$ is given by*

$$(z+t\varepsilon)_k^{1/n} = z_k^{1/n} + \frac{t}{n(z_k^{1/n})^{n-1}}\varepsilon. \tag{16}$$

Proof. Suppose $(z'+t'\varepsilon)^n = z+t\varepsilon$ for some $z', t' \in \mathbb{C}$. By Prop. 1, we have

$$(z'+t'\varepsilon)^n = z'^n + nz'^{n-1}t'\varepsilon = z+t'\varepsilon. \tag{48}$$

Equating the significant and infinitesimal parts, we obtain $z'^n = z$ and $nz'^{n-1}t' = t$. Since $z \neq 0$, $z' = z_k^{1/n}$. From the second equation, we get $t' = \frac{t}{nz'^{n-1}} = \frac{t}{n(z_k^{1/n})^{n-1}}$. □

Proposition 3 (Automatic differentiation)**.** *[24] Given a complex function $f(z)$ which is holomorphic on the open set D, there exists a unique dual-complex function $\hat{f}(z+t\varepsilon)$ which is holomorphic on $D+\mathscr{O}_\varepsilon$ and agrees with $f(z)$ for every $z \in D$. For any $z \in D$, $z+t\varepsilon \in D+\mathscr{O}_\varepsilon$, the function is given by*

$$\hat{f}(z+t\varepsilon) = f(z) + t\varepsilon f^{(1)}(z). \tag{18}$$

Proof. The following proof being simplified and only meant for intuition, see [24] for a full formal proof. Given $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)(z-z_0)^n}{n!}$ and $\hat{f}(z+t\varepsilon)$ agreeing with f for complex values and holomorphic on $D+\mathscr{O}_\varepsilon$, we have:

$$\hat{f}(z+t\varepsilon) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)(z-z_0+t\varepsilon)^n}{n!} = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)(z-z_0)^n + n(z-z_0)^{n-1}t\varepsilon}{n!} \quad \text{(by Prop. 1)} \tag{49}$$

this can be rewritten as

$$\begin{aligned} \hat{f}(z+t\varepsilon) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)(z-z_0)^n}{n!} + \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)n(z-z_0)^{n-1}t\varepsilon}{n!} \\ &= f(z) + t\varepsilon \frac{d}{dz}\Big(\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)(z-z_0)^n}{n!}\Big) = f(z) + t\varepsilon f^{(1)}(z). \end{aligned} \tag{50}$$

□

Proposition 5 (dual-complex order)**.** *There are exactly two extensions of the natural order on $\mathbb{R}$ that make $\mathbb{D}$ a totally ordered ring, determined by the choice $\varepsilon > 0$ or $\varepsilon < 0$.*

Proof. Suppose $\leq_{\mathbb{D}}$ is a total ring order on $\mathbb{D}$ such that for any $a,b \in \mathbb{R}$, $a \leq b \iff a \leq_{\mathbb{D}} b$. Without loss of generality, assume $\varepsilon \geq_{\mathbb{D}} 0$ (since $\mathbb{R}[\varepsilon] \cong \mathbb{R}[-\varepsilon]$).

Suppose for the sake of a contradiction that there exists $a > 0$ such that $b\varepsilon \geq_{\mathbb{D}} a$. Then by (i) $b\varepsilon - a \geq_{\mathbb{D}} 0$ and $b\varepsilon + a \geq_{\mathbb{D}} 2a > 0$. By (ii) again $(b\varepsilon - a)(b\varepsilon + a) \geq_{\mathbb{D}} 0$, yielding $b^2\varepsilon^2 \geq_{\mathbb{D}} a^2$. Since $\varepsilon^2 = 0$ we get $a^2 \leq 0$, contradicting $a^2 > 0$. Therefore, for all $a > 0$, $b\varepsilon <_{\mathbb{D}} a$.

We now show that $\leq_{\mathbb{D}}$ must be lexicographic, i.e. $a_1 + b_1\varepsilon \leq_{\mathbb{D}} a_2 + b_2\varepsilon$ if and only if $a_1 < a_2$ or ($a_1 = a_2$ and $b_1 \leq b_2$).
Case $a_1 < a_2$. Then $a_2 - a_1 > 0$. It follows that $(b_1 - b_2)\varepsilon <_{\mathbb{D}} (a_2 - a_1)$ and hence $a_1 + b_1\varepsilon <_{\mathbb{D}} a_2 + b_2\varepsilon$.
Case $a_1 > a_2$ is symmetric, yielding $a_1 + b_1\varepsilon >_{\mathbb{D}} a_2 + b_2\varepsilon$.
Case $a_1 = a_2$ and $b_1 \leq b_2$. Then $b_2 - b_1 \geq 0$ and since $\varepsilon >_{\mathbb{D}} 0$, we have $(b_2 - b_1)\varepsilon \geq_{\mathbb{D}} 0$, as a consequence $b_1\varepsilon \leq_{\mathbb{D}} b_2\varepsilon$ and hence $a_1 + b_1\varepsilon \leq_{\mathbb{D}} a_2 + b_2\varepsilon$.
Case $a_1 = a_2$ and $b_1 > b_2$ is similar with $b_2 - b_1 < 0$, yielding $a_1 + b_1\varepsilon >_{\mathbb{D}} a_2 + b_2\varepsilon$.
Having verified that $\leq_{\mathbb{D}}$ is the only option, let us check that it is indeed a total ring order.
(i) Consider $w_1 = a_1 + b_1\varepsilon \leq_{\mathbb{D}} a_2 + b_2\varepsilon = w_2$ and some $w_3 = a_3 + b_3\varepsilon$. By definition of $\leq_{\mathbb{D}}$ we have that

$$a_1 < a_2 \Rightarrow (a_1 + a_3) < (a_2 + a_3) \Rightarrow w_1 + w_3 <_{\mathbb{D}} w_2 + w_3.$$
$$a_1 = a_2 \text{ and } b_1 \leq b_2 \Rightarrow (a_1 + a_3) = (a_2 + a_3) \text{ and } (b_1 + b_3) \leq (b_2 + b_3) \Rightarrow w_1 + w_3 \leq_{\mathbb{D}} w_2 + w_3$$

(ii) Consider $w_1 = a_1 + b_1\varepsilon \geq_{\mathbb{D}} 0$ and $w_2 = a_2 + b_2\varepsilon \geq_{\mathbb{D}} 0$, since the order is lexicographic, $a_1 \geq 0$ and $a_2 \geq 0$, and so $a_1a_2 \geq 0$. Now, notice that

$$w_1w_2 \geq_{\mathbb{D}} 0 \Leftrightarrow a_1a_2 + (a_1b_2 + a_2b_1)\varepsilon \geq_{\mathbb{D}} 0. \tag{51}$$

We distinguish three cases. If $a_1a_2 > 0$, the significant part is positive, so $w_1w_2 \geq_{\mathbb{D}} 0$. If $a_1 = 0$ and $a_2 \geq 0$ (or symmetrically $a_2 = 0$ and $a_1 \geq 0$), then $w_1 = b_1\varepsilon$ with $b_1 \geq 0$, and $w_1w_2 = a_2b_1\,\varepsilon \geq_{\mathbb{D}} 0$. □

B Proofs for Sec. 3

Proposition 9 (Dual-complex unitarity)**.** *The following propositions are equivalent for U_ε a square linear operator:*

(a) U_ε is dual-complex unitary, i.e. $U_\varepsilon^\dagger U_\varepsilon = I$.

(b) U_ε preserves the inner product, i.e. $(U_\varepsilon|\psi\rangle)^\dagger(U_\varepsilon|\phi\rangle) = \langle\psi|\phi\rangle$ and therefore the norm $||U_\varepsilon|\psi\rangle|| = |||\psi\rangle||$.

(c) Rows of U_ε form an orthonormal basis.

(d) Columns of U_ε form an orthonormal basis.

(e) $U_\varepsilon = (I + i\varepsilon H)U$ where U is a complex unitary and H is a complex Hermitian.

Proof. For point equivalence between point (e) and (a), suppose $U_\varepsilon = U + i\varepsilon HU$.

$$\begin{aligned} U_\varepsilon^\dagger U_\varepsilon = I &\iff U^\dagger U + \varepsilon U^\dagger(i\varepsilon HU) + \varepsilon(i\varepsilon HU)^\dagger U = I \\ &\iff \begin{cases} U^\dagger U = I \\ i\varepsilon U^\dagger HU - i\varepsilon U^\dagger H^\dagger U = 0 \end{cases} \\ &\iff \begin{cases} U \text{ is a unitary.} \\ H \text{ is a Hermitian.} \end{cases} \end{aligned} \tag{52}$$

Points from (a) to (e) are easily seen to be equivalent by following a reasoning analogous to the case of complex numbers. □

Proposition 10 (Dual-complex matrix exponentiation)**.** *The dual-complex matrix exponential always exists.*

Proof. For scalar exponentiation in Eq. (19), we used the dual-complex extension from Eq. (3) which works thanks to binomial theorem and, therefore, commutativity. Commutativity is notoriously not a property of linear operators in general. Without commutativity, we get

$$\exp(A+B\varepsilon) = \sum_{n=0}^{\infty} \frac{1}{n!}(A+B\varepsilon)^n = \sum_{n=0}^{\infty} \frac{A^n}{n!} + \varepsilon \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} A^k B A^{n-k} \tag{53}$$

The significant part is the exponential of the complex matrix A which is known to always converge. Using the usual operator norm, we see that the infinitesimal part also converges as

$$\begin{aligned} \|\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} A^k B A^{n-k}\| &\leq \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \|A^k\| \, \|B\| \, \|A^{n-k}\| \\ &\leq \|B\| \sum_{n=0}^{\infty} \frac{2^n \|A\|^n}{n!} = \|B\| \exp(2\|A\|) \end{aligned} \tag{54}$$

□

Proposition 12 (Spectral theorem for dual-complex unitary)**.** *A unitary operator is unitarily diagonalizable, i.e. for every dual-complex unitary operator U_ε, there is a dual-complex orthonormal eigenbasis $\{|j_{U_\varepsilon}\rangle\}$ with associated dual-complex eigenvalues $e^{i\theta_j}(1+\varepsilon i\mu_j)$ such that $U_\varepsilon = \sum_j e^{i\theta_j}(1+\varepsilon i\mu_j)\,|j_{U_\varepsilon}\rangle\langle j_{U_\varepsilon}|$, where θ_j and μ_j are real numbers.*

Proof. Let $U_\varepsilon = U + i\varepsilon JU$ be a dual-complex unitary where $U = \exp(iH)$ is a complex unitary operator and H, J are Hermitian operators. We are going to build an eigenbasis $|j_{U_\varepsilon}\rangle$ for U_ε from an eigenbasis of $H_\varepsilon = H + \varepsilon J$.

$[H_\varepsilon]$. Indeed, by Prop. 11, we have that H_ε is diagonalizable. Let $|j_\varepsilon\rangle = |j_0\rangle + \varepsilon\,|j_1\rangle$ be an orthonormal eigenbasis with the eigenvalues $\theta_j + \varepsilon\mu_j$ where $\theta_j, \mu_j \in \mathbb{R}$, in which H_ε is diagonal. Then, we have the following eigenvalue equation,

$$\begin{aligned} (H+\varepsilon J)(|j_0\rangle + \varepsilon\,|j_1\rangle) &= (\theta_j + \varepsilon\mu_j)(|j_0\rangle + \varepsilon\,|j_1\rangle) \\ \iff H\,|j_0\rangle + \varepsilon(J\,|j_0\rangle + H\,|j_1\rangle) &= \theta_j\,|j_0\rangle + \varepsilon(\theta_j\,|j_1\rangle + \mu_j\,|j_0\rangle) \end{aligned} \tag{55}$$

From the significant part of Eq. (55), we find that $|j_0\rangle$ forms an eigenbasis for H. Then we can uniquely express J in the $\{|j_0\rangle\}$ eigenbasis, i.e. $J = \sum_{jk} h_{kj}\,|k_1\rangle\langle j_1|$ where $h^*_{kj} = h_{jk}$. For the infinitesimal part of Eq. (55), we see that

$$H\,|j_1\rangle + J\,|j_0\rangle = \theta_j\,|j_1\rangle + \mu_j\,|j_0\rangle \iff (H - \theta_j I)\,|j_1\rangle = (\mu_j I - J)\,|j_0\rangle \tag{56}$$

A prior remark is that this implies that for every $j \neq k$,

$$\langle k_0|\,(H - \theta_j I)\,|j_1\rangle = \langle k_0|\,(\mu_j I - J)\,|j_0\rangle \iff (\theta_k - \theta_j)\,\langle k_0|j_1\rangle = -h_{kj}, \tag{57}$$

which implies that for each $j \neq k$ such that $\theta_j = \theta_k$, $h_{jk} = 0$.

$[U_\varepsilon]$. With this prior remark in mind, we go back to constructing an orthonormal eigenbasis for U_ε. First, $U = \exp(iH)$ can be again be diagonalized in the same orthonormal basis $\{|j_0\rangle\}$, with corresponding eigenvalues $\lambda_j = e^{i\theta_j}$. Hence, what is left is to find is $\{|\tilde{j}_1\rangle\}$ and $\{\sigma_j\}$ satisfying the eigenvalue equation:

$$U_\varepsilon(|j_0\rangle + \varepsilon\,|\tilde{j}_1\rangle) = (\lambda_j + \varepsilon\sigma_j)(|j_0\rangle + \varepsilon\,|\tilde{j}_1\rangle). \tag{58}$$

and such that the $|j_{U_\varepsilon}\rangle = |j_0\rangle + \varepsilon\,|\tilde{j}_1\rangle$ are orthonormal.
Second we analyze the infinitesimal part of the equation above. We obtain

$$(U - \lambda_j I)\,|\tilde{j}_1\rangle = (\sigma_j I - iJU)\,|j_0\rangle \tag{59}$$

Since $U = \sum_k \lambda_k\,|k_0\rangle\langle k_0|$, the LHS must necessarily be orthogonal to $|j_0\rangle$ which means that

$$\langle j_0|\,(\sigma_j I - iJU)\,|j_0\rangle = 0 \iff \sigma_j = i\lambda_j h_{jj} \tag{60}$$

Expressed in terms of $\alpha_{jk} := \langle k_0|\tilde{j}_1\rangle$, Eq. (59) be restated as

$$\langle k_0|(U-\lambda_j I)|\tilde{j}_1\rangle = \langle k_0|(\sigma_j I - iJU)|j_0\rangle \iff (\lambda_k-\lambda_j)\alpha_{jk} = \delta_{kj}\sigma_j - i\lambda_j h_{kj} \tag{61}$$

In the case $\lambda_j \neq \lambda_k$, this means $\alpha_{jk} = \frac{i\lambda_j h_{kj}}{\lambda_j-\lambda_k}$. In the case $j \neq k$ but $\lambda_j = \lambda_k$, our prior remark showed that $h_{jk} = 0$, hence any $\alpha_{jk} \in \mathbb{C}$ is a solution. The same goes for every α_{jj}. Choosing $\alpha_{jk} = 0$ when $\lambda_j = \lambda_k$ therefore defines a valid eigenbasis $|j_{U_\varepsilon}\rangle = |j_0\rangle + \varepsilon|\tilde{j}_1\rangle$ for U_ε with eigenvalues $e^{i\theta_j}(1+i\varepsilon h_{jj})$.

[Orthonormality]. We are left to show that $|j_{U_\varepsilon}\rangle$ forms an orthonormal basis. With our definition, when $\lambda_j = \lambda_k$, $\alpha^*_{jk} = 0 = -\alpha_{kj}$. When $\lambda_j \neq \lambda_k$,

$$\alpha^*_{jk} = \frac{-i\lambda^*_j h^*_{kj}}{\lambda^*_j - \lambda^*_k} = \frac{-i\lambda_j^{-1} h_{jk}}{\lambda_j^{-1}-\lambda_k^{-1}} = \frac{-i\lambda_k h_{jk}}{\lambda_k-\lambda_j} = -\alpha_{kj}$$
$$\implies \langle j_{U_\varepsilon}|k_{U_\varepsilon}\rangle = \langle j_0|k_0\rangle + \varepsilon\langle\tilde{j}_1|k_0\rangle + \varepsilon\langle j_0|\tilde{k}_1\rangle = \delta_{jk} + \varepsilon(\alpha^*_{kj}+\alpha_{jk}) = \delta_{jk}$$

[Diagonalization recap.]. By construction we have $\theta_j, h_{jj} \in \mathbb{R}$ and

$$\begin{aligned}
\sum_j \lambda_j(|j_0\rangle\langle\tilde{j}_1| + |\tilde{j}_1\rangle\langle j_0|) &= \sum_{jk}\lambda_j(\alpha^*_{jk}|j_0\rangle\langle k_0| + \alpha_{jk}|k_0\rangle\langle j_0|) \\
&= -\sum_{\substack{jk \\ \lambda_j\neq\lambda_k}} ih^*_{jk}\frac{\lambda_k\lambda_j}{\lambda_k-\lambda_j}|j_0\rangle\langle k_0| + \sum_{\substack{jk \\ \lambda_j\neq\lambda_k}} ih_{kj}\frac{\lambda_j^2}{\lambda_j-\lambda_k}|k_0\rangle\langle j_0| \\
&= -\sum_{\substack{jk \\ \lambda_j\neq\lambda_k}} ih_{kj}\frac{\lambda_j\lambda_k}{\lambda_j-\lambda_k}|k_0\rangle\langle j_0| + \sum_{\substack{jk \\ \lambda_j\neq\lambda_k}} ih_{kj}\frac{\lambda_j^2}{\lambda_j-\lambda_k}|k_0\rangle\langle j_0| \\
&= \sum_{\substack{jk \\ \lambda_j\neq\lambda_k}} ih_{kj}\lambda_j\frac{\lambda_j-\lambda_k}{\lambda_j-\lambda_k}|k_0\rangle\langle j_0| = \sum_{\substack{jk \\ \lambda_j\neq\lambda_k}} ih_{kj}\lambda_j|k_0\rangle\langle j_0| \\
&= \sum_{\substack{jk \\ j\neq k}} ih_{kj}\lambda_j|k_0\rangle\langle j_0| \quad \text{by the prior remark.}
\end{aligned} \tag{62}$$

This gives us

$$\begin{aligned}
\sum_j e^{\theta_j}(1+\varepsilon ih_{jj})|j_{U_\varepsilon}\rangle\langle j_{U_\varepsilon}| &= \sum_j \lambda_j|j_0\rangle\langle j_0| + \varepsilon i\lambda_j h_{jj}|j_0\rangle\langle j_0| + \varepsilon\lambda_j(|j_0\rangle\langle\tilde{j}_1| + |\tilde{j}_1\rangle\langle j_0|) \\
&= U + \varepsilon i\sum_j h_{jj}\lambda_j|j_0\rangle\langle j_0| + \varepsilon\sum_{\substack{jk \\ j\neq k}} ih_{kj}\lambda_j|k_0\rangle\langle j_0| \\
&= U + i\varepsilon\sum_{jk} h_{kj}\lambda_j|k_0\rangle\langle j_0| \\
&= U + i\varepsilon(\sum_{jk} h_{kj}|k_0\rangle\langle j_0|)(\sum_j \lambda_j|j_0\rangle\langle j_0|) \\
&= U + i\varepsilon JU
\end{aligned} \tag{63}$$

□

Proposition 13 (Dual-complex unitary as exponential of anti-Hermitian)**.** *The exponential of anti-Hermitian linear operator is a unitary. Moreover, every unitary U_ε is of the form* $\exp(iH_\varepsilon)$ *where H_ε is dual-complex Hermitian. H_ε is unique modulo 2π.*

Proof. To prove the first statement, let H_ε be a Hermitian operator. Then we have

$$\exp(iH_\varepsilon)^\dagger = (\sum_{n=0}^{\infty}\frac{(iH_\varepsilon)^n}{n!})^\dagger = \sum_{n=0}^{\infty}\frac{(-iH_\varepsilon^\dagger)^n}{n!} = \exp(-iH_\varepsilon) \tag{64}$$

This shows us that the adjoint of the exponential of iH is its inverse:

$$\exp(iH_\varepsilon)^\dagger \exp(iH_\varepsilon) = \exp(-iH_\varepsilon)\exp(iH_\varepsilon) = \exp(-iH_\varepsilon + iH_\varepsilon) = \exp(0) = I \tag{65}$$

For the second statement, let $U_\varepsilon = \sum_j (\lambda_j + \varepsilon\sigma_j)\,|j_{U_\varepsilon}\rangle\langle j_{U_\varepsilon}|$ be a dual-complex unitary operator diagonalized as in Prop. 12. We can write $\lambda_j + \varepsilon\sigma_j = e^{i\theta_j}(1+i\varepsilon\mu_j)$ where $\theta_j, \mu_j \in \mathbb{R}$. Therefore, $\log(U_\varepsilon) = \sum_j \log(\lambda_j + \varepsilon\sigma_j)\,|j_{U_\varepsilon}\rangle\langle j_{U_\varepsilon}| = \sum_j (i\theta_j + \varepsilon i\mu_j)\,|j_{U_\varepsilon}\rangle\langle j_{U_\varepsilon}|$ (by Eq. (20)) which is indeed anti-Hermitian.

Uniqueness modulo 2π can be seen from the fact that the other solutions $z + t\varepsilon$ of $\log(e^{i\theta_j}(1+i\varepsilon\mu_j))$, i.e. $z,t \in \mathbb{C}$ such that $\exp(z+t\varepsilon) = e^{i\theta_j}(1+i\varepsilon\mu_j)$ are those such that $e^{i\theta_j} = e^z$ and $t = i\mu_j$ by using Eq. (19).

□

Proposition 14 (Dual-complex semipositivity)**.** *For every dual-complex linear operator M_ε, $M_\varepsilon^\dagger M_\varepsilon$ is appreciably semipositive.*

Proof. Let $|\psi\rangle$ be an arbitrary vector and $|\phi\rangle = \sum_j \phi_j |j\rangle = M_\varepsilon |\psi\rangle$ for an orthonormal basis $|j\rangle$. Then $\langle\psi|M_\varepsilon^\dagger M_\varepsilon|\psi\rangle = \langle\phi|\phi\rangle = \sum_j \phi_j^* \phi_j$. From Eqs (30), (32) and (33) combined, we have that for all j, $\phi_j^*\phi_j$ is either appreciable or exactly zero and always greater or equal to zero. It follows that $M_\varepsilon^\dagger M_\varepsilon$ is appreciably semipositive. □

C Example for Sec. 5

Example 21 (Complex correction of a dual-complex measurement)**.** *Consider the dual-complex measurement $\{M_m^\varepsilon\}$ with $M_m^\varepsilon = M_m + \varepsilon N_m$ given by:*

$$M_0 = |0\rangle\langle 0| \tag{66}$$

$$M_1 = |0\rangle\langle 1| \tag{67}$$

$$N_0 = i\pi\,|0\rangle\langle 0| \tag{68}$$

$$N_1 = i\pi\left(\frac{1}{\sqrt{3}}\,|1\rangle\langle 0| + |0\rangle\langle 1| - \frac{2}{\sqrt{6}}\,|1\rangle\langle 1|\right) \tag{69}$$

By Prop. 15 there exists $U_\varepsilon = U + \varepsilon iHU$ is a Stinespring dilation of $\{M_m^\varepsilon\}$. Let us construct such an U_ε. We need a conventional unitary U and Hermitian H such that $(\langle m|\otimes I)U(|0\rangle\otimes I) = M_m$ and $(\langle m|\otimes I)\,iHU\,(|0\rangle\otimes I) = N_m$.

The constraint $(\langle m|\otimes I)U(|0\rangle\otimes I) = M_m$ is satisfied by the SWAP gate on the ancilla. In the chosen basis:

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{70}$$

The first two columns of $U^\dagger(iHU) = iU^\dagger HU$ are fixed by N_m; completeness of the dilation requires $-iU^\dagger(iHU) = U^\dagger HU$ to be Hermitian. One can complete the matrix accordingly; one solution is

$$H = \pi \begin{pmatrix} 1 & 0 & 0 & \frac{1}{\sqrt{3}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} & 1 \end{pmatrix} \tag{71}$$

We get $(\langle m|\otimes I)U_\varepsilon(|0\rangle\otimes I) = M_m^\varepsilon$.

By Def. 19 the complex correction $\{M_m^\varepsilon\}$ is $\widetilde{M}_m^h = (\langle m|\otimes I)\,\widetilde{U}_h\,(|0\rangle\otimes I)$ with $\widetilde{U}_h = \exp(ihH)\,U$.

$$\widetilde{U}_h = \begin{pmatrix} \frac{z(\cos(\pi h)+2)}{3} & \frac{\sqrt{2}z(1-\cos(\pi h))}{3} & 0 & \frac{\sqrt{3}(z^2-1)}{6} \\ 0 & 0 & z & 0 \\ \frac{\sqrt{2}z(1-\cos(\pi h))}{3} & \frac{z(2\cos(\pi h)+1)}{3} & 0 & \frac{\sqrt{6}(1-z^2)}{6} \\ \frac{\sqrt{3}(z^2-1)}{6} & \frac{\sqrt{6}(1-z^2)}{6} & 0 & \frac{z^2+1}{2} \end{pmatrix} \tag{72}$$

where $z = e^{i\pi h}$. We get

$$\widetilde{M}_0^h = \frac{1}{3}\begin{pmatrix} z(\cos(\pi h)+2) & \sqrt{2}z(1-\cos(\pi h)) \\ 0 & 0 \end{pmatrix} \tag{73a}$$

$$\widetilde{M}_1^h = \frac{1}{3}\begin{pmatrix} \sqrt{2}z(1-\cos(\pi h)) & z(2\cos(\pi h)+1) \\ i\sqrt{3}z\sin(\pi h) & -i\sqrt{6}z\sin(\pi h) \end{pmatrix} \tag{73b}$$

Then $\{\widetilde{M}_m^h\}$ is a valid conventional complete measurement. By Prop. 20, its effects and outcome probabilities agree up to $O(h^2)$ with those of $\{M_m^\varepsilon\}$.