Completeness for flow-preserving rewrite rules

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Introduction

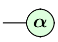
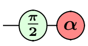
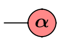
The ZX-calculus is a powerful diagrammatic language that enables fully graphical reasoning about quantum computations and has been an active area of research for almost two decades. Equipped with complete graphical rewriting rules, the ZX-calculus finds applications in both the foundational study of quantum mechanics and various quantum computing tasks, such as circuit optimization and the development of error-correcting codes. A key property of the ZX-calculus, compared to traditional quantum circuits, is its flexibility, exemplified by the 'only connectivity matters' and bialgebra rules, as well as the ability to merge or split spiders at will. This feature greatly facilitates graphical reasoning but makes the extraction of a quantum circuit #P-hard in the general case [5].

In this context, it is desirable to identify classes of ZX-diagrams for which circuit extraction is efficient, while still retaining the expressive flexibility of the ZX-rewriting rules. Natural candidates are the classes of ZX-diagrams that have a *flow*. Introduced in the setting of measurement-based quantum computation [11], the graphical property of flow provides a rewriting strategy that enables the efficient extraction of quantum circuits from ZX-diagrams possessing this property [2, 6, 12].

There exist several kinds of flow, from the restrictive causal flow [4] to the more general Pauli flow [3], forming a hierarchy of ZX-diagram classes. However, these classes are not closed under standard ZX-rewriting rules. It is therefore essential to develop alternative rewriting rules that preserve the existence of flows.

Contributions

While a few rewriting rules, such as local complementation and vertex deletion, were already known to be flow preserving [1, 2, 6–10] and a complete set of flow-preserving rewrite rules was known for the Clifford fragment [9], we extend this toolkit by identifying a set of rewriting rules that preserve various kinds of flow, and we prove the completeness of this rule set.

Consequently, we equip the most general known fragment of ZX-diagrams that admit efficient circuit extractions, with a complete set of diagrammatic rewriting rules. In the first instance, this includes all diagrams for which it is meaningful to define flow, in particular graph-like ZX-diagrams [6] and MBQC-form ZX-diagrams [2]. The former are diagrams that consist only of Z-spiders with Hadamard edges between them, without self-loops or multiple edges, and where every input or output is connected to a Z-spider. The latter consist of an open graph state diagram – basically a graph-like diagram without phases – where each spider is connected to either an output wire or a measurement effect from the set  ,  and  .

Theorem 1 (Informal)**.** The flow-preserving rewrite rules given in fig. 1 are complete.

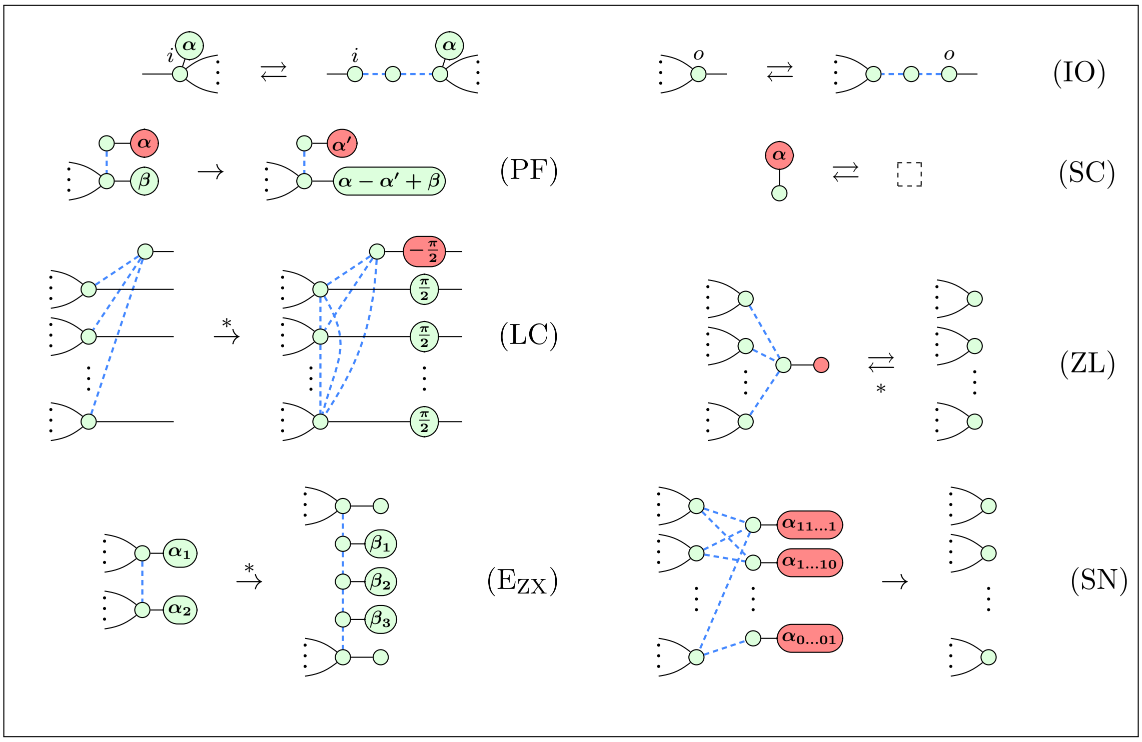


Figure 1: The complete flow-preserving rule set. The vertices marked with i in (IO) are inputs and the vertices marked with o are outputs. In (LC), none of the vertices may be outputs and the top vertex cannot be an input. While (LC) always preserves gflow and Pauli flow, it preserves extended causal flow only in special cases. The right-to-left direction of rule (ZL), can be used if either the new vertex is assigned a measurement label Z or if it is assigned YZ and there exists a flow on the original diagram satisfying certain conditions. Similarly, rule (E_{ZX}) requires a specific kind of flow on the original diagram to be flow-preserving. For (E_{ZX}) and (SN) to preserve the interpretation, the respective phase angles need to satisfy certain conditions given in the full paper. The remaining rule directions apply unconditionally for any type of flow.

The complete rewriting rules are: (IO) rules are input/output expansion/contraction, they are counterparts of the standard ZX spider fusion rule, their application is however restricted to input and ouputs spiders and the angles cannot be split. Addition of angles can however be performed through the (PF) rule. The (SC) rule is a standard scalar rule, which expresses that the rewriting rules preserves the semantics only up to a global phase. The (LC) rule corresponds to the local complementation operation that can be implemented by Clifford $\pi/2$ rotations on graph states. The (ZL) rule can be interpreted as the red copy, each degree-1 red spider being color changed to green by the Hadamard edges and then merged with the neighbouring green spiders. The (E_{ZX}) rule is a particular instance of Euler angle decomposition with only two parameters on the LHS, finally the (SN) rule is the spider-nest identity where the angles are appropriate dyadic multiples of π.

The proof of completeness proceeds in essentially two main stages. First, we show that any ZX-diagram with a Pauli flow can be transformed into a circuit-like ZX-diagram equipped with a causal flow. As ZX-diagrams with causal flow can easily be interpreted as quantum circuits, this first stage is called circuit extraction. We use as intermediate steps particular ZX-diagrams with gflow. Note that circuit extraction already appears in circuit optimization procedures [2, 6, 12], that we refine here by proving that it can be performed by means of flow preserving rules. The first stage guarantees that any ZX-diagram with

Pauli flow can be transformed, in a flow preserving way, into a circuit-like ZX-diagram. To complete the proof of completeness, the second stage relies on a complete set of circuit rewriting rules that we introduce for quantum circuits with initialisation. To the best of our knowledge, this represents the first application of quantum circuit completeness results within the ZX-calculus.

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