Noncontextuality inequalities for prepare-transform-measure scenarios

David Schmid[1] Roberto D. Baldijão[1,2] John H. Selby[2,3] Ana Belén Sainz[2,3,4] Robert W. Spekkens[1]
[1] Perimeter Institute for Theoretical Physics, 31 Caroline Street North, Waterloo, Ontario Canada
[2] ICTQT, Uniwersytet Gdański, Gdańsk, Poland
[3] Theoretical Sciences Visiting Program, OIST, Graduate University, Onna, Japan
[4] Basic Research Community for Physics e.V., Germany

Introduction

Generalized noncontextuality provides a universally applicable notion of classical explainability for operational theories [1]. In the Spekkens framework, a theory or a fragment of it is said to be noncontextual if there exists an ontological model that represents operationally equivalent procedures identically and reproduces the observed statistics. This notion applies to arbitrary experimental scenarios and unifies many manifestations of nonclassicality across quantum information and foundations.

One of the compelling features of this program is that it alows to derive *noise-robust, computable* tests of noncontextuality. In prepare-and-measure (PM) scenarios, this goal has been achieved: the set of correlations admitting a noncontextual model can be characterized via linear programming techniques [2, 3], some of which yield noncontextuality inequalities that are necessary and sufficient relative to a fixed set of linear operational identities. However, these methods cannot account for intermediate transformations.

Scenarios involving transformations (e.g., in prepare-transform-measure (PTM) fragments) are of fundamental importance in any theory and might have many applications, but have so far lacked any computable characterization. Transformations introduce dynamical structure and compositional constraints that obstruct a direct extension of PM techniques. In particular, the standard convex-decomposition strategy used to reduce the ontological problem to a finite linear program fails in the presence of transformations.

In this work, we provide the first linear feasibility program giving necessary conditions for the existence of a noncontextual ontological model in arbitrary finite PTM fragments (relative to linear operational identities). The program is constructed by combining flag-convexification, Bayesian inversion, and vertex enumeration of suitable assignment polytopes, thereby reducing the *a priori* infinite-dimensional ontological problem to a finite linear one. Its dual formulation yields explicit noncontextuality inequalities certifying nonclassicality of experimental data. As an illustration, we recover a noise-robust inequality for transformations violated by the single-qubit stabilizer fragment which admits of a noncontextual explanation in the prepare-and-measure case.

Preliminaries

We consider a finite prepare–transform–measure (PTM) fragment of a finite-dimensional generalized probabilistic theory (GPT). Such a fragment consists of finite sets of GPT states $\{\omega_s\}_s$, GPT transformations $\{T_t\}_t$, and GPT effects $\{e_k\}_k$, together with the associated statistics $p(k|s,t) = e_k \circ T_t \circ \omega_s$. We assume that all transformations in $\{T_t\}_t$ have the same input system type and the same output system type, although these two system types may differ. The fragment includes all linear operational identities among these GPT elements, namely all relations of the form

$$\sum_s \alpha_s \omega_s = 0, \quad \sum_k \beta_k e_k = 0, \quad \sum_t \gamma_t T_t = 0, \tag{1}$$

which hold in the GPT representation. Since the processes live in finite-dimensional real vector spaces, the set of such linear operational identities is finitely generated.

An ontological model (OM) for the GPT prepare-transform-measure fragment consists of measurable spaces Λ and Λ' of ontic states, associated respectively with the input and output systems, together with a linear representation mapping states ω_s to probability distributions $\mu_s(\lambda)$ over Λ, transformations T_t to stochastic maps $\Gamma_t(\lambda'|\lambda)$, and effects e_k to response functions $\xi_k(\lambda')$ on Λ', such that

$$p(k|s,t) = \int_\Lambda d\lambda \int_{\Lambda'} d\lambda' \, \xi_k(\lambda')\Gamma_t(\lambda'|\lambda)\mu_s(\lambda). \tag{2}$$

Linearity of the representation ensures that all linear operational identities among GPT elements are automatically respected at the ontological level.

In the traditional formulation of generalized contextuality, one instead begins with a fragment of an operational theory whose elements are laboratory procedures satisfying certain operational equivalences. Noncontextuality is defined as the existence of an ontological representation that assigns identical ontic representations to operationally equivalent procedures. Quotienting the operational description by these equivalences yields precisely the GPT fragment described above. The existence of an ontological model for the GPT fragment is equivalent to the existence of a generalized-noncontextual ontological model for the original operational fragment [4]. Hence, testing generalized noncontextuality of an operational PTM fragment reduces to testing existence of an OM for its associated GPT PTM fragment.

Results

In finite PM fragments, the noncontextual set admits a finite linear description relative to fixed operational identities [2]. Extending this reduction to PTM fragments is nontrivial due to the coupling between preparations and transformations in the ontological representation. In Ref. [5] we overcome this obstruction.

Theorem (Main Result). Let a finite PTM GPT fragment be specified together with all its linear operational identities. Then there exist a matrix $\mathbf{M}$ and vector $\mathbf{b}$, computable from the fragment, such that feasibility of the linear program

$$\exists\, x \quad \text{s.t.} \quad \mathbf{M}x = \mathbf{b}, \qquad x \geq 0, \tag{3}$$

is a necessary condition for the fragment to admit an ontological model.

Proof sketch. A naive extension of the PM scenario techniques fails. We circumvent this by passing to a flag-convexified scenario with a single source and performing a Bayesian inversion, which is always possible if an OM for the original PTM scenario exists. In this reformulation, epistemic states $\mu_s(\lambda)$ are replaced by *source response functions* $\widetilde{\Psi}_\lambda$ describing retrodictive probabilities conditioned on ontic states (ie, which state s was prepared, given the ontic state λ describing the system). These assignments satisfy linear constraints inherited from the identities among states and, together with positivity and normalization, define a polytope. Measurement response functions similarly form a polytope determined by linear identities among effects (though no inversion is needed here). After vertex enumeration of both polytopes, any ontological model can be rewritten using finitely many ontic states labeled by such vertices, denoted κ or κ'. Transformations are then encoded by joint conditional probabilities $p(\kappa, \kappa'|t)$ over these extremal assignments, and all ontological constraints become linear in these variables, yielding the matrix $\mathbf{M}$ and vector $\mathbf{b}$ above. See diagram below.

$$\begin{tikzcd} \underset{s}{\boxed{\mu_s(\lambda)}} \arrow[r, "\lambda"] & \underset{t}{\boxed{\Gamma_t(\lambda'|\lambda)}} \arrow[r, "\lambda'"] & \underset{k}{\xi_k(\lambda')} \arrow[r, phantom, "\Longrightarrow"] & \widetilde{\Psi}_\kappa(s) \quad \widetilde{\Phi}_{\kappa'}(k) \\ & & & \underset{t}{\boxed{p(\kappa, \kappa'|t)}} \arrow[u, "\kappa", "\kappa'"'] \end{tikzcd} \tag{4}$$

Thus feasibility of the above linear program is necessary for the existence of an ontological model of the PTM fragment. A detailed proof appears in Ref. [5].

By linear programming duality (Farkas' lemma), we obtain the following.

Corollary (Noncontextuality inequalities). The linear program in Eq. (3) is feasible if and only if the dual optimization problem

$$\min_{y}\ y \cdot \mathbf{b} \quad \text{such that} \quad \mathbf{1} \geq y\mathbf{M} \geq 0 \tag{5}$$

has nonnegative minimum value. If there exists a vector y satisfying $\mathbf{1} \geq y\mathbf{M} \geq 0$ and $y \cdot \mathbf{b} < 0$, then the primal program is infeasible. In this case, $y \cdot \mathbf{b} \geq 0$ is a noncontextuality inequality violated by the data.

Example: Single-qubit stabilizer fragment We apply our method to the single-qubit stabilizer fragment, consisting of the six Pauli eigenstates and projectors together with the four transformations $T = \{I, Z, S, S^{-1}\}$, where S is the phase gate. These satisfy the linear operational identity $\frac{1}{2}(I(\cdot) + Z(\cdot)) = \frac{1}{2}(S(\cdot) + S^{-1}(\cdot))$. While this fragment admits a noncontextual model in the prepare–measure case (e.g. Spekkens' toy model), it is contextual in the full PTM scenario.

Running our LP yields infeasibility. The dual program therefore returns the noncontextuality inequality (with $p_{skt} := e_k \circ T_t \circ \omega_s$, states and effects are ordered as $\{|+\rangle, |-\rangle, |+y\rangle, |-y\rangle, |0\rangle, |1\rangle\}$, and transformations as $\{I, Z, S, S^{-1}\}$):

$$\begin{aligned}\beta := {} & 3[p_{121} - p_{212} + p_{644} + p_{234} + p_{335} - p_{445} - p_{146} + p_{326} + p_{416}] \\ & + 2[-p_{311} + p_{322} - p_{333} + p_{344} - p_{115} - p_{125} - p_{235}] \\ & + p_{431} + p_{521} - p_{622} + p_{413} + p_{513} + p_{643} - p_{524} \\ & - p_{634} + p_{245} + p_{635} - p_{516} - p_{526} + p_{546} + 5[p_{442} - p_{324}] \geq -6.\end{aligned} \tag{6}$$

For the stabilizer fragment, quantum predictions yield $\beta = -12$, violating the noncontextual bound. Hence our program witnesses nonclassicality of the single-qubit stabilizer PTM fragment.

Closing remarks

Our LP provides necessary conditions for the existence of a noncontextual ontological model in PTM fragments. In general, it is not sufficient: while the program enforces linear operational identities, it does not incorporate nonlinear ones (such as compositional constraints), so the LP might not signal nonclassicality sometimes. An important direction for future work is therefore finding necessary and sufficient conditions for nonclassicality in scenarios with transformations.

Nevertheless, the ability to derive necessary conditions efficiently already has significant implications. Nonclassicality of transformations has remained comparatively unexplored, both theoretically and experimentally, in contrast to prepare–measure scenarios. On the theoretical side, no protocols are currently known that explicitly exploit noncontextuality (or contextuality) of transformations as a resource; on the experimental side, dynamical tests of contextuality have not been implemented. A central obstacle in both directions has been the lack of systematic and tractable tools for detecting transformation contextuality efficiently. Our LP addresses this gap: it provides a systematic and noise-robust method for deriving noncontextuality inequalities in dynamical settings. By rendering transformation nonclassicality efficiently detectable, our framework enables experimental investigations and provides concrete motivation for identifying and characterizing operational advantages rooted in transformation contextuality.

References

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