Tomographically Non-Local Entanglement

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Entanglement, defined as non-separability, is a central phenomenon of quantum theory. However, many foundational and information-theoretic claims about what entanglement can or cannot achieve rely, often implicitly, on *tomographic locality* (TL). Applied to states, TL asserts that a bipartite state is fully characterized by the statistics it generates for product measurements. While TL holds in unrestricted complex quantum theory, it can fail in perfectly consistent physical theories. It may also fail once superselection rules or other restrictions are imposed, so that quantum theory itself becomes tomographically nonlocal in real [1] or fermionic [2] variants. We refer to theories in which TL fails as *tomographically nonlocal* (TNL) theories. In such theories, composite systems may possess genuinely holistic degrees of freedom that are invisible to all local probes.

When TL fails, the conceptual structure of entanglement requires refinement. In Ref. [3], we introduced a distinction capturing the fact that entanglement may reside either in the subspace accessible to local tomography (tomographically local entanglement) or in purely holistic components that evade local reconstruction (tomographically-nonlocal entanglement). The distinction sheds light on several puzzles arising in TNL theories and clarifies which operational features depend on locally accessible correlations and which stem from genuinely holistic structure.

Preliminaries

We work in the framework of generalized probabilistic theories (GPTs). A system $\mathcal{A}$ is associated to a finite-dimensional real vector space A, a compact convex set of normalized states $\Omega_{\mathcal{A}} \subset A$, and a convex effect space $E_{\mathcal{A}} \subset A^*$. We denote by $\mathcal{S}_{\mathcal{A}} := \mathrm{ConvHull}(0, \Omega_{\mathcal{A}})$ the set of possibly subnormalized states.

Given systems $\mathcal{A}$ and $\mathcal{B}$, a composite $\mathcal{AB}$ is specified by a vector space AB together with bilinear maps $\boxtimes$ defining product states $\omega^{\mathcal{A}} \boxtimes \nu^{\mathcal{B}}$ and product effects $e^{\mathcal{A}} \boxtimes f^{\mathcal{B}}$, such that product statistics factorize $(e^{\mathcal{A}} \boxtimes f^{\mathcal{B}})[\omega^{\mathcal{A}} \boxtimes \nu^{\mathcal{B}}] = e^{\mathcal{A}}(\omega^{\mathcal{A}}) f^{\mathcal{B}}(\nu^{\mathcal{B}}))$ and $u^{\mathcal{AB}} = u^{\mathcal{A}} \boxtimes u^{\mathcal{B}}$. We also assume closure under conditioning (steering). Details can be seen in Ref. [3].

A composite GPT system is *tomographically local* if whenever $e \boxtimes f[\omega^{\mathcal{AB}}] = e \boxtimes f[\nu^{\mathcal{AB}}]$ for all product effects implies $\omega^{\mathcal{AB}} = \nu^{\mathcal{AB}}$, and a similar condition holding for effects. A bipartite state $\omega^{\mathcal{AB}}$ is *separable* if it can be written as a convex mixture of product states,

$$\omega^{\mathcal{AB}} = \sum_i p_i \, \omega_i^{\mathcal{A}} \boxtimes \nu_i^{\mathcal{B}}, \tag{1}$$

and *entangled* otherwise. Again, a similar condition defines separable and entangled effects.

Tools for exploring the failure of TL

When tomographic locality fails, product effects do not span the full dual space of the composite. To isolate the degrees of freedom invisible to local tomography, we define the Holistic subspaces $H_S \subset AB$ and $H_E \subset AB^*$. Let $AB_{\otimes} \subset AB$ denote the linear span of product states $\omega^{\mathcal{A}} \boxtimes \nu^{\mathcal{B}}$ and let $AB^*_{\otimes} \subset (AB)^*$ denote the span of product effects $e^{\mathcal{A}} \boxtimes f^{\mathcal{B}}$.

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Definition. The *holistic state subspace* is the annihilator of the product-effect subspace, $AB^*_\otimes$,

$$H_S := \{\, h \in AB \;\big|\; (e^{\mathcal{A}} \boxtimes f^{\mathcal{B}})(h) = 0 \text{ for all } e^{\mathcal{A}}, f^{\mathcal{B}} \,\}, \tag{2}$$

dually, the *holistic effect subspace* $H_E \subset AB^*$ is the annihilator of $AB_\otimes$.

Holistic degrees of freedom are precisely those undetectable by product elements of the dual space. We therefore do not define H_S as the orthogonal complement of $AB_\otimes$, as is sometimes informally suggested in the literature. Orthogonality would require choosing an inner product, which is neither canonical in a general GPT nor necessarily aligned with the operational structure of holistic subspace. The annihilator definition, by contrast, is intrinsic: it captures exactly the defining feature of holistic degrees of freedom—their invisibility to all product dual objects.

Lemma. In finite dimensional composite GPT systems $\mathcal{AB}$, the following direct-sum decompositions hold:

$$AB = AB_\otimes \oplus H_S, \quad (AB)^* = AB^*_\otimes \oplus H_E. \tag{3}$$

The proof can be found in Ref. [3]. This Lemma implies that every vector $v \in AB$, in particular states $\omega^{\mathcal{AB}} \in \mathcal{S}_{\mathcal{A}}$, can be uniquely written as

$$v = v_{\mathrm{TL}} + \tilde{h}, \text{with } v_{\mathrm{TL}} \in AB_\otimes \text{ and } \tilde{h} \in H_S \tag{4}$$

and a similar condition holds for vectors in AB^*(particularly effects $e^{\mathcal{AB}} \in E_{\mathcal{AB}}$).

This splitting induces canonical linear projections onto the two sectors. We denote by Π_{TL} the projection onto $AB_\otimes$, which acts as $\Pi_{\mathrm{TL}}(v) = v_{\mathrm{TL}}$ and by Π_{TNL} the projection onto H_S, which acts as $\Pi_{\mathrm{TNL}}(v) = \tilde{h}$. They satisfy $\Pi_{\mathrm{TL}} + \Pi_{\mathrm{TNL}} = \mathrm{id}$ and Π_{TL} preserves the tomographically local sectors (in particular leaving the deterministic effect invariant, and thus preserves normalization).

Main definition. A composite state $\omega^{\mathcal{AB}}$ has *tomographically-local entanglement* iff $\omega_{\mathrm{TL}} = \Pi_{\mathrm{TL}}(\omega^{\mathcal{AB}}) \notin \mathsf{Sep}[\Omega_{\mathcal{AB}}]$ (in some cases, ω_{TL} might not be a state ($\omega_{\mathrm{TL}} \notin \mathcal{S}_{\mathcal{A}}$), in which case $\omega^{\mathcal{AB}}$ has TL-entanglement). State $\omega^{\mathcal{AB}}$ has *tomographically-nonlocal entanglement* iff it has non-null holistic component, $\Pi_{\mathrm{TNL}}(\omega^{\mathcal{AB}}) \neq 0$.

A state may exhibit both forms simultaneously or just one of them; e.g., a state is *solely TNL entangled* if ω_{TL} is separable while $\Pi_{\mathrm{TNL}}(\omega^{\mathcal{AB}}) \neq 0$. Bare quantum theory has trivial holistic subspaces, so all entanglement is TL. In contrast, in real and fermionic quantum theories both forms of entanglement are possible. In theories such as bilocal classical theory, all entanglement is of the TNL form.

Application of the new definitions

This perspective clarifies counterintuitive features of entanglement in bipartite real quantum theory. In the standard representation of two-rebits, holistic degrees of freedom correspond to the $\sigma_y \otimes \sigma_y$ component. We characterize two-rebit states lacking TL-entanglement and prove the following.

Result. An entangled two-rebit state is locally broadcastable only if it lacks TL-entanglement.

As a concrete illustration, consider the two-rebit state $\rho^{\text{rebits}} = \frac{1}{4}\big(\mathbb{I} \otimes \mathbb{I} + \sigma_y \otimes \sigma_y\big)$. This state was shown to have maximal entanglement of formation (concurrence), yet it is infinitely shareable [4] and locally broadcastable [5]—features typically regarded as incompatible with entanglement.

From the present viewpoint, the tension disappears. The incompatibility between entanglement and local broadcasting stems from TL-entanglement. The state above, however, carries only TNL-entanglement: its TL component is separable, so its entanglement resides entirely in the holistic sector. Thus its broadcastability and infinite shareability are possible due to the absence of TL-entanglement.

Operational consequences

We show that TNL entanglement alone is useless for some tasks for which entanglement is known to be required.

Result. A bipartite state that lacks TL-entanglement cannot violate a Bell inequality, demonstrate steering, or enable teleportation.

The proof follows directly from the general decomposition of states $\omega^{\mathcal{AB}} = \omega_{\mathrm{TL}} + \tilde{h}$ together with two other facts: first, that holistic components are invisible to product measurements; second, that Bell tests, steering scenarios, and teleportation protocols ultimately rely on statistics generated by local operations (and classical communication for some of them). Formal statements and proofs appear in Propositions 9, 10, and 11 of Ref. [3]. This result helps explain, for example, why some states with maximal concurrence in TNL theories do not violate any Bell inequalities.

We now turn to tasks for which TNL-entanglement is sufficient. A first example is dense coding in tomographically nonlocal theories.

Result. In many TNL theories, TNL-entanglement alone suffices for dense-conding.

This shows that bilocal classical theories, shown to allow for dense coding, are just particular examples of TNL theories allowing for dense-coding.

More strikingly, TNL-entanglement enables *perfectly secure data hiding* [6]: the encoding of classical information into bipartite states that are globally distinguishable but completely indistinguishable by all local measurements, even with unlimited classical communication. Ie, it requires a pair of states $\{\omega_x^{\mathcal{AB}}\}_{x=0}^1$ that is perfectly distinguishable so it perfectly encodes the bit x, but also $e^{\mathcal{A}} \boxtimes f^{\mathcal{B}}(\omega_x)$ is independent of x.

Such perfectly secure data hiding is impossible in unrestricted quantum theory, where TL holds. In contrast, TNL admit states whose distinguishability resides entirely in the holistic sector.

Result. Perfectly secure data hiding encodings require TNL-entanglement. A class of TNL GPTs allows for such protocols.

This is established in Proposition 12 of Ref. [3]. Intuitively, perfect hiding demands the existence of nontrivial holistic components: if all entanglement were TL, local measurements would signal some difference between the states $\{\omega_x\}_x$. We further identify a set of structural features of GPT composites that suffice for perfect data hiding to be possible in a TNL theory.

These results suggest several broader directions. First, many claims commonly made about "entanglement" simpliciter—particularly no-go theorems—should be revisited to determine whether they concern entanglement in general or only its tomographically-local component. Second, the analysis reinforces the foundational question of whether symmetries and superselection rules are fundamental. If they are, then tasks regarded as impossible in unrestricted quantum theory may become achievable in physically relevant restricted theories. Finally, the decomposition naturally points toward resource theories of TL- and TNL-entanglement, as well as toward understanding their interplay and possible interconversion.

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