Shadows and subsystems of generalised probabilistic theories: when tomographic incompleteness is not a loophole for contextuality proofs

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It is commonly believed that failures of tomographic completeness undermine assessments of nonclassicality in noncontextuality experiments. In this work, we study how such failures can indeed lead to mistaken assessments of nonclassicality. We then show that proofs of the failure of noncontextuality are robust to a very broad class of failures of tomographic completeness, including the kinds of failures that are likely to occur in real experiments. We do so by showing that such proofs actually rely on a much weaker assumption that we term *relative tomographic completeness*: namely, that one's experimental procedures are tomographic *for each other*. Thus, the failure of noncontextuality can be established even with coarse-grained, effective, emergent, or virtual degrees of freedom. This also implies that the existence of a deeper theory of nature (beyond that being probed in one's experiment) does not in and of itself pose any challenge to proofs of nonclassicality. To prove these results, we first introduce a number of useful new concepts within the framework of generalised probabilistic theories (GPTs). Most notably, we introduce the notion of a *GPT subsystem*, generalising a range of preexisting notions of subsystems (including those arising from tensor products, direct sums, decoherence processes, virtual encodings, and more). We also introduce the notion of a *shadow* of a GPT fragment, which captures the information lost when one's states and effects are unknowingly not tomographic for one another.

This is an extended abstract for Quantum 9, 1880 (2025). All proofs and technicalities can be found there.

Preliminaries

A gold standard method for demonstrating that a given theory resists classical explanation is assessing generalized contextuality [1–6]. Studies characterizing exactly which phenomena constitute proofs of generalized contextuality have been carried out in several contexts [7–17, 19–29, 31–44]. It is often claimed that assessments of noncontextuality rest on the assumption that one's laboratory procedures are *tomographically complete* [1, 45, 46]: that is, that they allow one to fully characterize the properties of the physical system in question (the causal mediary between one's preparations and measurements). However, we show that assessments of noncontextuality rest on a weaker assumption, namely that one's preparations and measurements are sufficient to fully characterise *each other*. Consequently, we prove that assessments of nonclassicality will be valid even if one's laboratory operations are far from tomographically complete for any given systems in one's experiment or theory, provided that the preparations and measurements used in one's proof of nonclassicality probe *the same degree of freedom*.

Methods. We formalise the notion of degree of freedom within a framework for possible theories known as the framework of generalised probabilistic theories (GPTs) [47–49]. While the notion of a system within a GPT is completely standard, we introduce the notion of a GPT *subsystem*, and demonstrate how this captures the relevant notion of a 'degree of freedom' for the study of noncontextuality. We provide a more flexible representation of GPTs and demonstrate that it subsumes standard representations in the literature.

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Summary of Results

We hereby summarise the most important definitions.

Definition (GPT systems). A GPT system G comprises of a state space Ω_G, an effect space $\mathscr{E}_G$ and a tomographic probability rule $p_G : \Omega_G \times \mathscr{E}_G \to [0,1]$, where Ω_G and $\mathscr{E}_G$ are convex subspaces of a linear space V_G such that all properties of GPT states (effects) are satisfied by any vectors living in Ω_G ($\mathscr{E}_G$). p_G is tomographic if $p_G(\Omega_G, \cdot)$ separates $\mathscr{E}_G$, as well as $p_G(\cdot, \mathscr{E}_G)$ separates Ω_G.
Definition (GPT fragments). A GPT fragment f is constituted by convex subspaces $\Omega_f \subseteq \Omega_G$ and $\mathscr{E}_f \subseteq \mathscr{E}_G$ such that p_f is defined from p_G by limiting its domain to Ω_f and $\mathscr{E}_f$. Naturally, p_f is often non-tomographic.
Definition (GPT embeddings and subsystems). A GPT system/fragment is said to embed into another if there are linear maps $\iota : \Omega_G \to \Omega_H$ and $\kappa : \mathscr{E}_G \to \mathscr{E}_H$ such that $p_G(\cdot,\cdot) = p_H(\iota(\cdot), \kappa(\cdot))$. A GPT G is a **subsystem** of H if it embeds into H. Two GPT systems/fragments are *equivalent* if they embed into each other.
A specially interesting case of the above definition is the one in which the GPT G embeds into a simplicial GPT $\triangle$, since this is equivalent to the existence of a noncontextual ontological model for G [6].
Definition (GPT fragment shadows). A GPT fragment shadow $\mathscr{S}(f)$ is constructed from the fragment f, for instance, by quotienting Ω_f and $\mathscr{E}_f$ with respect to each other through the (possibly non-tomographic) probability rule p_f. Notice that any shadow $(\Omega_f/\sim, \mathscr{E}_f/\sim, \tilde{p}_f)$ is a valid GPT system (i.e., with a tomographic probability rule).

The above notion generalises the concept of when one is mistaken about the tomographic completeness of the description of a scenario or theory [50–52]. There are many other ways of constructing shadows from a fragment f. Any map taking a fragment f into its shadow is called a **shadow map**.

We now list our main results:

Transitivity of subsystems. The notion of a GPT subsystem is transitive, i.e., if G is a subsystem of H and H is a subsystem of K, then G is a subsystem of K. This is particularly relevant for contextuality assessments, since in this framework G being noncontextual is equivalent to G being a subsystem of a classical GPT system.
Equivalence of GPT shadows. Any two shadows of a fragment f are equivalent to each other (no matter how they are constructed). Any GPT system that is equivalent to a shadow is itself a shadow. Shadow maps act trivially on GPT systems and subsystems.
Theorem (Holevo projection). Every polytopic GPT system is the shadow of a fragment of a finite-dimensional, strictly simplicial GPT system.

The above result was first exemplified by Holevo [50], and subsumes the core interest of formalising shadows for the purposes of nonclassicality assessments. However, our next result provides a sufficient condition for trusting assessments of noncontextuality:
Simplex embedding assessments are always trustworthy. Consider a fragment f of a GPT system G. If the shadow $\mathscr{S}(f)$ is simplex-embeddable, than f is simplex-embeddable.
The failure of simplex embedding is trustworthy for subsystems. Consider a fragment f of a GPT system G. If f is a subsystem of G, then the shadow $\mathscr{S}(f)$ is simplex-embeddable if and only if f is simplex-embeddable.

These are the core results of this paper. Interesting additional results are the following:
Hyperdecohered theories are subsystems. Any GPT system G that undergoes a process of hyperdecoherence [53] results in a GPT system $H(G)$ that is a subsystem of G. It follows that any theory that hyperdecoheres into quantum theory cannot be simplex-embeddable, a result first shown in Ref. [52] that we recast into this new formalisation.
Theory-agnostic tomography output. The GPT system obtained by performing theory-agnostic tomography [45, 54] over a set of data is, in fact, a GPT shadow.

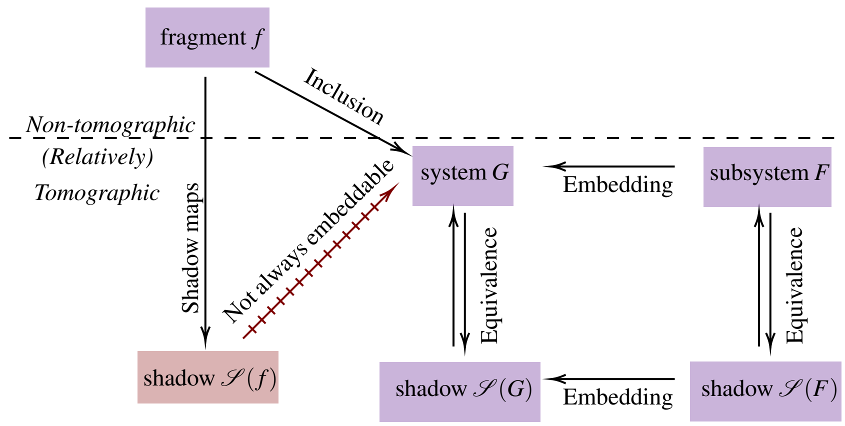

Figure 1: Summary of concepts introduced and how they relate to each other.

For derivations and additional results, please refer to the full publication and Figure 1.

Final remarks

This work proposes a formal way of investigating mistakenly constructed GPTs from scenarios that are not, in fact, tomographically complete, which we call *GPT fragment shadows*. We provide many results concerning the validity of contextuality assessments over these shadows, and show that the so-called tomography loophole for contextuality proofs is in fact much less stringent than it has been thought: as soon as one can ensure that the system under investigation is a *GPT subsystem*, assessments of contextuality will be trustworthy.

It would be interesting to see whether GPT subsystems subsume still other notions of systems that have been considered. In forthcoming work, we generalise the idea of a GPT subsystem to the broader notion of a *GPT subtheory*, capturing the idea of what it means for a *theory* to live inside another, adding a compositional structure on top of the convex structure that must also be preserved. Finally, it remains an open question whether one can find other sufficient conditions for the preservation of simplex-embeddability, which of these conditions might be easiest to test experimentally, and whether there are any necessary conditions for such a preservation.

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