Subsystems as subsets of quantum channels, and the strange case of blind agents

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The manuscript corresponding to this project can be found following this extended abstract (note that it is a full manuscript and not a draft). All our definitions and results in this extended abstract are given at a conceptual and non-rigorous level, to avoid clutter and the introduction of many definitions. In brackets we give the numbering to the corresponding technical statements in the manuscript.

1 Why subsystems matter

It is hard to state a theory of the world that would not involve making a distinction between subparts of it. Yet the way in which this splitting into subsystems is picked, and, even before that, the definition of what a subsystem may be, are often (we might say too often) viewed as unproblematic and going without saying. This is particularly true in the QPL-adjacent community, a logical byproduct of the starkly compositional flavour of category theory-inspired approaches: their tools are designed to combine 'small' systems to make up 'big' ones, but make it less natural to consider going in the other direction, i.e. starting with a big system and identifying a smaller one within it.

Yet in many situations that is indeed the direction of interest. For instance, recent developments in the field of quantum causality show that it is a critical matter, and a subject of controversy, to identify what counts as a suitable causal relatum. The best example is of course the (neverending) debate concerning the quantum switch and its purported implementations [1, 2], in which there is now a consensus among stakeholders that any causal account hinges on what are taken to be the causal relata under study – with these relata being variously called 'loci of intervention' [3], (possibly time-delocalised) 'subsystems' [4, 5, 6], 'events' [7, 8], 'labs' [9], etc.

Still on the subject of causality, the identification of causal relata with subalgebras of operators is also a central tenet of a recently proposed causality-based interpretation of quantum theory [10], and recent progress on the conjecture of causal decompositions [11] was made possible by the concurrent development of an (also algebra-based) theory of partitions into subsystems [12]. Finally, the notion of subsystems, and debates about it, have also been identified as central elements in other current areas of foundational investigation, such as quantum reference frames [13, 14], the appearance of locality in quantum cosmology [15, 16, 17], its definition in lattice-based theories [18, 19], or algebraic quantum field theory [20, 21].

In view of this, it might be surprising that there are still few works asking the question of what a subsystem is in general in the quantum world. In particular, a definition based on sub-C* algebras is often taken, as e.g. in Ref. [12] but it stands more as an axiom, and lacks a proper operational motivation. In a nutshell, in this project we explore operationally grounded ways to pin down subsystems in quantum theory, following the framework of Ref. [22] (see also Refs. [23, 24] for similar ideas). Because we work within the theory of unitary channels as opposed to unitary operators, this leads us away from the traditionally adopted C*-algebra subsystems, opening the possibility for new possible subsystems. We structurally characterise a subclass of those, and display some of their counter-intuitive properties, in particular as regards (the absence of) possible measurements on them.

2 How to pin down a subsystem

In this project, we explore the consequences of a recently proposed [22] operational definition of a quantum subsystem A of S, which identifies it with a subset $\mathrm{Act}(A, S)$ of the set $\mathrm{Transf}(S)$ of all available operations on S. One can think of these as the operations that an agent Alice, with only access to A, can implement. In that context, an important notion is Ref. [22]'s definition of complementary subsystems using commutants in $\mathrm{Transf}(S)$.

Definition 1 (Definition 2.2)**.** *For a subsystem A of S we define $\mathrm{Act}(A', S)$ to be the commutant of $\mathrm{Act}(A, S)$, i.e. the set of elements of $\mathrm{Transf}(S)$ that commute with all elements of $\mathrm{Act}(A, S)$. A'', the commutant of A', is A's bicommutant.*

We show that 'bicom-closed subsystems' (those that are equal to their bicommutant) enjoy nice properties: mathematically, they are closed under many algebraic and topological structures (Lemma 2.1); and conceptually, they admit a satisfactory notion of a complement A', the 'rest of S' compared to A. This yields our first tenet.

Tenet 1. *A 'good' subsystem is a bicom-closed $\mathrm{Act}(A, S)$.*

This identifies dual sets of operations that are acausally related, e.g. due to spacelike separation, in the sense that the relative ordering of two operations $U \in \mathrm{Act}(A, S), V \in \mathrm{Act}(A', S)$ is operationally irrelevant.

Next is the problem of applying this to quantum theory. A point of critical importance here is that there are (at least) three theories (i.e. three possible definitions of $\mathrm{Transf}(S)$) that we could mean by this: unitary operators $\mathrm{UnitOp}(\mathcal{H}_S)$, unitary channels $\mathrm{UnitChan}(\mathcal{H}_S)$ (corresponding to conjugations by unitary operators), and general quantum channels $\mathrm{Chan}(\mathcal{H}_S)$. In the theory of unitary operators, we show (Proposition 2.1) that any bicom-closed subsystem corresponds to (i.e. is the set of the unitary elements of) a sub-C* algebra of the algebra of operators on $\mathcal{H}_S$, thus recovering the traditional, and widely adopted, definition of subsystems using sub-C* algebras.

However, we remark on how – even if we restrict to a quantum theory of closed systems – it is preferable to work not in $\mathrm{UnitOp}(\mathcal{H}_S)$, but in $\mathrm{UnitChan}(\mathcal{H}_S)$, because the former keeps track of global phase.

Tenet 2. *Global phase is unphysical; thus we should work with unitary channels, not unitary operators.*

For a seminal example of the difference this entails: on a qubit, the operator Z does not commute with X, but the channel $\hat{Z}$ (conjugation by Z) commutes with $\hat{X}$; so if we work in unitary channels, the two can be seen as acting on two complementary subsystems. The goal of this submission is to explore the consequences of these two tenets.

3 A characterisation of the (bi-)commutant of a unitary channel

Our first major result is a full characterisation of the commutant and bicommutant (in $\mathrm{UnitChan}(\mathcal{H}_S)$) of a given unitary channel $\mathcal{U}$. This brings us closer to the characterisation of all bicom-closed subsystems of $\mathrm{UnitChan}(\mathcal{H}_S)$, and substantially clarifies their structure.

First, $\mathcal{U}(\cdot) = U \cdot U^\dagger$ commutes with $\mathcal{V}(\cdot) = V \cdot V^\dagger$ if and only if $UV = e^{i\phi} VU$ for some ϕ (Proposition 3.1), so that our task turns into a characterisation of '(bi-)commutants up to a global phase' in $\mathrm{UnitOp}(\mathcal{H}_S)$. We show that commutation up to a phase is a very constrained phenomenon, controlled by a quantity we call the spectral period.

Lemma 1 (Lemma 3.1)**.** *Let $U \in \mathrm{UnitOp}(\mathcal{H}_S)$. Defining its spectral period as the maximal b such that $\mathrm{Spec}(U) = e^{\frac{2\pi i}{b}} \cdot \mathrm{Spec}(U)$, U can commute with other unitaries up to phases $2\pi \frac{m}{b}$, $m = 0, \ldots, b-1$.*

We introduce the shift and clock matrices in dimension b, $\Sigma_1^{(b)} := \sum_k |k\rangle\langle k-1 \mod b|$ and $\Sigma_3^{(b)} := \sum_i e^{2\pi i \frac{k}{b}} |k\rangle\langle k|$ (Definition 3.3). These precisely commute up to $e^{2\pi i \frac{1}{b}}$. More than that, they are 'essentially the only case': our following results can be broadly understood as showing that commutation up to a phase can always be attributed to the appearance (up to a change of basis) of a $\Sigma_3^{(b)}$ in one of the operators and of a $\Sigma_1^{(b)}$ in the other.

Proposition 1 (Proposition 3.3)**.** *A U with spectral period b admits a canonical form in a certain basis, in which it can be seen as being built from a $\Sigma_3^{(b)}$.*

Theorem 1 (Theorem 3.1)**.** *Given a U with spectral period b, and working in the basis adapted to its canonical form, we have a characterisation of its commutant and bicommutant up to a phase. Essentially, a given element of its commutant is built from a $(\Sigma_1^{(b)})^l$ for a certain $l = 0, \dots, b-1$, and a given element of its bicommutant is built from a $(\Sigma_3^{(b)})^k$ for a certain $k = 0, \dots, b-1$.*

This Theorem yields a full characterisation of the commutant and bicommutant of any unitary channel. Besides its foundational relevance to us to characterise subsystems in $\mathrm{UnitChan}(\mathcal{H}_S)$, it might be of a broader appeal as a mathematical result.

4 Measurements and the strange case of 'blind agents'

Our second major result concerns the measurements that can be performed on our subsystems. Since we set our considerations within the theory of unitary channels, there is no direct definition of the measurements that can be performed on a given subsystem A of S.

Our strategy is to consider purifications of these measurements, i.e. to see them as unitary interactions engineered by the agent Alice between her subsystem A and an ancillary quantum system E into her entire control. In other words, Alice performs measurements on A by implementing a unitary channel on AE (with E initialised however she wants), then measuring E. This leaves us with the task of inferring the set $\mathrm{Act}(AE, SE)$ of unitary channels that Alice can perform on the subsystem AE of SE, from the data of $\mathrm{Act}(A, S)$. Our strategy is here again to leverage commutants.

Definition 2 (Definition 4.1)**.** *Given $\mathrm{Act}(A, S) \subseteq \mathrm{Transf}(S)$ and an ancillary system E, A's <u>E-extension</u> is the subsystem AE of SE defined by*[1]

$$\mathrm{Act}(AE, SE) := \Big(\mathrm{Act}(A, S)'_{\mathrm{Transf}(S)} \otimes \mathbb{1}_E\Big)'_{\mathrm{Transf}(SE)} . \tag{1}$$

Under this definition, we can characterise the set of extended operations available to an agent in possession of a subsystem covered by Theorem 1.

Theorem 2 (Theorem 4.1)**.** *Given a U with spectral period b and working in the basis adapted to its canonical form, we have a characterisation of the E-extension of the subsystem $\{\hat{U}\}'' \subseteq \mathrm{UnitChan}(\mathcal{H}_S)$.*

We then note that $\mathrm{Act}(AE, SE)$ is in fact greatly constrained. In particular, there exist instances of non-trivial subsystems A for which $\mathrm{Act}(AE, SE)$ only includes factorised channels, of the form $\mathcal{U}_A \otimes \mathcal{V}_E$ (Example 3). Thus, an agent in the possession of such an A cannot engineer any interaction between A and an ancilla in her possession, and in particular, cannot measure anything about A, even though she can perform non-trivial operations on A: she is a *blind agent*.

We discuss the status of subsystems admitting blind agents; their strange features put us in front of a choice. A first possible attitude is to consider that such subsystems should not be taken seriously. Although perfectly admissible, we remark that such a contention would sit awkwardly with the fact that most researchers would agree with our two tenets above, which naturally lead to a theory of subsystems incorporating these instances. We argue that choosing this response compels one to find sound operational arguments restricting the range of 'valid' subsystems down to more standard ones – without e.g. invoking global phase.

The other response is to hold that, even though odd, these subsystems are structurally sound as they satisfy our two tenets, and should be taken seriously. This calls, however, for the display of more concrete instances of physical situations in which it makes sense to talk about such subsystems, something that we have not been able to find. Maybe this requires to abandon some unjustified blinkers for how we think about subsystems; but in any case the challenge is now clear.

We remain agnostic with respect to these two positions, preferring to present the conundrum and let everyone draw their own conclusions from our work. QPL would be the perfect venue to start that discussion.

[1] We specify as a subscript the ambient set in which each commutant is taken.

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Subsystems as subsets of quantum channels, and the strange case of blind agents

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Defining subsystems is a critical task in any theory of the world, and especially in causal modelling, where these subsystems are the causal relata. We investigate the operational definition of subsystems in quantum theory, within a framework recently proposed by Chiribella, in which those are identified with subsets of transformations. We argue that it is reasonable to restrict to what we call 'bicom-closed' subsystems, admitting a notion of complementation. We make the point that, due to the unphysicalness of global phase, the transformations at hand for a closed quantum system are not unitary operators, but unitary channels; this has major consequences for subsystems, opening the way to new instances beyond the traditionally adopted C*-algebraic ones. We provide a full mathematical classification of the channel-based subsystems that arise as the commutant or bicommutant of one unitary channel. We investigate the admissible measurements that can be performed on such subsystems, and point out that some of them admit no measurement at all, despite being non-trivial. We discuss the two possible attitudes regarding these results – dismissing such subsystems, or taking them seriously – and the challenges that each entails.

1 Introduction

1.1 Background

An idea that too often goes without saying in physics is that the world is split into different parts in an obvious manner. This 'natural decomposition' is typically, and most often unconsciously, made with respect to point-like particles, or to local field modes – the incompatibility between these two views, which readily becomes apparent with a bit of reflection, should in fact be sufficient to cast doubt on their supposed transparency.

Against this, it has been argued many times already, and coming from a wide variety of directions, that the way in which the world is split is not a given, but may depend on circumstances – e.g. on time, on the specific state of the world, on the dynamics, and/or on the perspective that one decides to adopt. The latter dependency on perspectives, for instance, is a (we would even say *the*) core conceptual component of the idea of quantum reference frames, currently a subject of much investigation.

Even before we reach the question of how the world is split into parts, however, we have to answer an even more basic one:

What is a part?

i.e., if we take a system (say, a quantum one, described by some Hilbert space), what might we mean when we talk about a part (or subsystem) of it? Maybe a substructure that also corresponds to a Hilbert space? Although this response sounds sensible, perhaps we should, here again, be suspicious of its apparent obviousness: just because the Hilbert-space way of pinning down a system is clearly (in finite dimension) the one that encompasses all possible quantum structures, this does not mean that subsystems should themselves correspond to Hilbert spaces – only that they appear somewhere within a Hilbert space description. There may, in fact, be some prejudice in a dogmatic assertion that a subsystem is necessarily 'something that looks like a Hilbert space'.

In fact, a first loosening of that assertion has already been achieved, at least in some circles, through the recourse to generic sub-C* algebras[1] to describe quantum subsystems. *Factor* sub-C* algebras – the ones isomorphic to spaces of all operators over a certain Hilbert space – formally pin down what we loosely characterised above as 'subsystems that look like a Hilbert space'. But there also exist non-factor sub-C* algebras, and indeed these actually fit naturally into many physical situations, typically to describe subsystems that are subject to symmetry constraints, or equivalently to conservation laws [1].

Still, even this broadened definition might raise suspicion. Why, after all, should a subsystem necessarily be described by a sub-C* algebra? This has too many times been taken as a mere axiom, and an obscure one at that. To finally escape circular definitions, we need to *derive* a definition of subsystems, from some requirements with a clear meaning. The gold standard is for these requirements to be *operational* statements, i.e. to refer as directly as possible to lab operations, as opposed to metaphysical or mathematical statements with ambiguous or obscure physical meaning.

Following the blossoming of rigorous quantum foundations in the wake of the quantum information revolution, significant progress in this direction has recently been made [2, 3, 4]. These references proposed to define subsystems as corresponding to subsets of operations on the larger systems, a definition that we will enthusiastically adopt. However, just accepting any subset now sounds too broad: we would need further requirements to characterise 'reasonable' subsystems. Further than that, we would need to figure out where these requirements lead us, i.e. get a proper structural classification of the subsystems thus identified, and an investigation of their concrete properties. This is what this paper achieves in the case of quantum theory.

1.2 Results

First, in Section 2, using the framework and results of Ref. [3], we motivate and propose a further restriction for a subsystem A of a system S to be considered 'good', or structurally sound: that A should be 'bicom-closed' – we can broadly describe this as requiring the existence of a good 'complementary system' A', the rest of S relative to A.

We then discuss how this may be applied to quantum theory. Restricting to quantum theory for closed systems, we note that it in fact exists in two versions: a theory of unitary operators, and a theory of unitary channels. We prove that bicom-closed subsystems in the theory of unitary operators precisely recover the traditional C*-algebraic definition. However, we note that the theory of unitary channels should be preferred, as it gets rid of the unphysical global phase; so we should investigate the more general notion of subsystems that it yields.

[1] or Von Neumann algebras; here we restrict to finite dimension, so we can overlook the distinction between the two notions.

This is what we achieve in Section 3: we structurally characterise a significant subclass of unitary-channel-subsystems, namely, those arising as the commutant and bicommutant of a given unitary channel. We show that these admit a tight structure, given by the appearance of 'clock' and 'shift' matrices interplaying in a specific way. Beyond its foundational significance, our result is also valuable as a technical statement, as it identifies the structural underpinnings of the fact that two unitary operators commute up to a phase.

Finally, in Section 4, we investigate these new subsystems by asking which measurements can be performed by agents in their possession. We define these measurements through the introduction of ancillary systems with which the subsystems are made to interact, and pin down the set of possible such interactions for the subsystems that wwe characterised in the previous section. We then remark on how some instances of channel-subsystems admit no measurements at all: we describe them as corresponding to 'blind agents', who can act but not see. We discuss the choice that this puts us in front of: either discarding such subsystems, but at the cost of introducing further operational requirements, yet to be made explicit; or taking them seriously, which would require understanding better how they may make sense in concrete physical situations.

1.3 Discussion of previous literature

There is already an extensive literature on what one might call quantum mereology – the study of the relationship between the parts and the whole in quantum theory – from a top-down perspective, i.e. starting from a larger system and identifying subsystems within it.

As we mentioned, most of this literature has been starting from the premise that these subsystems correspond to subalgebras of the algebra of operators on the global system (in infinite dimension, those could be C* algebras or Von Neumann algebras). A prototypical case is that of algebraic quantum field theory [5, 6]. The same premise was adopted in more informational approaches as well, for instance in early work on quantum error correction, which identified error-free subsystems in the algebra commuting with noise [7, 8, 9], and in recent works discussing the appearance of a preferred notion of locality [10, 11, 12, 13, 14, 15, 16, 17, 18]. Approaches based on this premise might differ depending on whether they restrict themselves to the consideration of factor subalgebras, or incorporate the non-factor ones. In Ref. [1, 19], one of us and colleagues started from this same premise (including the non-factor case) to define partitions into an arbitrary number of subsystems.

C* algebraic approaches have traditionally insisted on interpreting the subalgebras through their observables (i.e. Hermitian operators). It generally remains unclear, however, why one should restrict the focus to subsets of observables that correspond to a certain sub-C* algebra. Due to this and to other considerations (for instance, the will to define locality in physical theories on a dynamical lattice), some papers have traded sub-C* algebras for other, laxer structures, among which we can cite Lie algebras [20, 21, 22], convex cones [23], groupoids [24], or subsets based on a certain basis choice [25, 26].

Another recent approach has been to define subsystems from an operational perspective, by linking them to subsets of potential operations on a lab [2, 3, 4]. This is the approach that we will mainly adopt, with one slight change of focus: while all of these references dedicated some work to the discussion of the set of states of a subsystem (with Ref. [2] incorporating that set into their definition of a subsystem, and Refs. [3, 4] deducing it from their definition), here we will closely stick with discussions of transformations only. Some reasons are 1/ that we will mainly care about unitary quantum theory, in which states are not a straightforward notion (as a category, for instance, this theory has strictly speaking no states at all); 2/ that, given the problematic nature of measurements for agents in possession of some of our subsystems (discussed in Section 4), the operational meaning of these states would be highly debatable anyway; and 3/ that we believe (in line with Ref. [27]) that states may be given undue importance in discussions of physical theories.

With that slight divergence in mind, our goal here is to carry forward the project of Refs. [2, 3, 4], which mostly kept to a presentation of general frameworks for thinking of subsystems in arbitrary physical theories (with some quantum examples), by moving towards a systematic classification of all cases of subsystems that they yield in the specific case of quantum theory.[2]

2 Defining subsystems through subsets of operations

In this section, we quickly present the framework of Ref. [3], within which we place ourselves, and we motivate and discuss an additional restriction we make in that context: restricting to the consideration of what we call bicom-closed subsystems.

2.1 An operational definition of subsystems

The tenet of Ref. [3] is that *subsystems should be operationally defined as corresponding to subsets of operations*. This eschews more metaphysical definitions (based, for instance, on the state space of S) by focusing on the directly meaningful data of available operations, for instance to an experimenter in a lab.

Definition 2.1. *Suppose we have a 'global' system S, whose set of physical transformations we denote as* $\mathrm{Transf}(S)$*, and suppose we want to define a subsystem A of it. The way to do so is to specify a subset*

$$\mathrm{Act}(A, S) \subseteq \mathrm{Transf}(S)\,. \tag{1}$$

The interpretation of $\mathrm{Act}(A, S)$ *is that it is the set of transformations of S that only affect A.*

As a convenient conceptual tool, to a subsystem A we can also associate an agent, defined as the agent that only has the ability to perform the operations $\mathrm{Act}(A, S)$. This should be seen only as an intuitive way of speaking, and does not commit us to a fundamentally agentialist view of physics.

Let us discuss examples. While the framework of Ref. [3] can be applied to any physical theory, our point here is to study quantum theory.[3] Even the latter term, however, might in fact refer to a few subtly different versions. It can mean:[4]

[2]In that regard, Ref. [4] claims, in their Section 5, to have a characterisation of self-bicommutant subgroups (the analogue of what we shall call bicom-closed subsystems) in projective quantum theory, i.e. with respect to the projective unitary group $U(n)/U(1)$. However, this is incorrect: their result is a characterisation of self-bicommutant subgroups of the unitary group $U(n)$. In other words, with respect to our terminology below, they have the characterisation of bicom-closed subsystems in $\mathrm{Transf}(S) = \mathrm{UnitOp}(\mathcal{H}_S)$ in terms of sub-C* algebras (analogue to our Proposition 2.1), not in $\mathrm{Transf}(S) = \mathrm{UnitChan}(\mathcal{H}_S)$ as claimed.

[3]In this paper, we restrict to finite dimension.

[4]Yet other versions could be mentioned, for instance those including non-deterministic transformations.

1. a theory of *unitary operators*: to S we associate a Hilbert space $\mathcal{H}_S$, and we define

$$\mathrm{Transf}(S) := \mathrm{UnitOp}(\mathcal{H}_S)\,, \tag{2}$$

where $\mathrm{UnitOp}(\mathcal{H}_S)$ is the set of the unitary operators on $\mathcal{H}_S$, i.e. of the linear maps $U : \mathcal{H}_S \to \mathcal{H}_S$ satisfying $UU^\dagger = U^\dagger U = \mathbb{1}$;

2. a theory of *unitary channels*: to S we associate a Hilbert space S, but we now define

$$\mathrm{Transf}(S) := \mathrm{UnitChan}(\mathcal{H}_S)\,, \tag{3}$$

where $\mathrm{UnitChan}(\mathcal{H}_S)$ is the set of the unitary channels on the space $\mathrm{Lin}(\mathcal{H}_S)$ of operators on $\mathcal{H}_S$, i.e. of the linear maps $\mathcal{U} : \mathrm{Lin}(\mathcal{H}_S) \to \mathrm{Lin}(\mathcal{H}_S)$ for which there exists a unitary map $U : \mathcal{H}_S \to \mathcal{H}_S$ such that $\mathcal{U}(\rho) = U\rho U^\dagger$ for any $\rho \in \mathrm{Lin}(\mathcal{H}_S)$;[5]

3. a theory of *channels*: to S we associate a Hilbert space S, and we define

$$\mathrm{Transf}(S) := \mathrm{Chan}(\mathcal{H}_S)\,, \tag{4}$$

where $\mathrm{Chan}(\mathcal{H}_S)$ is the set of the channels on the space $\mathrm{Lin}(\mathcal{H}_S)$ of operators on $\mathcal{H}_S$, i.e. of the completely positive linear maps $\mathcal{C} : \mathrm{Lin}(\mathcal{H}_S) \to \mathrm{Lin}(\mathcal{H}_S)$;

In this paper, our main focus will be on the second case. The relationship with more standard approaches, based on the first case (or equivalently, on observables), will be discussed in a moment.

Example 1 (Subsystems corresponding to subfactors)**.** *In the quantum context, a first, natural instance of a subsystem is of course given by a subfactor. This corresponds to the case in which the Hilbert space decomposes as a tensor product, $\mathcal{H}_S = \mathcal{H}_A \otimes \mathcal{H}_B$, so that* $\mathrm{Act}(A, S)$ *can be defined as the set of transformations 'only acting on $\mathcal{H}_A$', namely:*

1. if $\mathrm{Transf}(S) = \mathrm{UnitOp}(\mathcal{H}_S)$*, we take* $\mathrm{Act}(A, S) := \{U_{\mathcal{H}_A} \otimes \mathbb{1}_{\mathcal{H}_B} \,|\, U \in \mathrm{UnitOp}(\mathcal{H}_A)\}$*;*

2. if $\mathrm{Transf}(S) = \mathrm{UnitChan}(\mathcal{H}_S)$*, we take* $\mathrm{Act}(A, S) := \{\mathcal{U}_{\mathrm{Lin}(\mathcal{H}_A)} \otimes \mathbb{1}_{\mathrm{Lin}(\mathcal{H}_B)} \,|\, \mathcal{U} \in \mathrm{UnitChan}(\mathcal{H}_A)\}$*;*

3. if $\mathrm{Transf}(S) = \mathrm{Chan}(\mathcal{H}_S)$*, we take* $\mathrm{Act}(A, S) := \{\mathcal{C}_{\mathrm{Lin}(\mathcal{H}_A)} \otimes \mathbb{1}_{\mathrm{Lin}(\mathcal{H}_B)} \,|\, \mathcal{C} \in \mathrm{Chan}(\mathcal{H}_A)\}$.

Subfactor subsystems may thus be framed in any of our three versions of quantum theory. A central point of this paper is that this will not be the case for more exotic types of subsystems.

2.2 Bicom-closed subsystems

Another important notion introduced in Ref. [3] is that of what we will call *complementary subsystems* (named 'maximal adversaries' in Ref. [3]). The idea is, given a subsystem A, to obtain a natural definition of 'the part of S complementary to A'. We suppose that $\mathrm{Transf}(S)$ is equipped with an associative composition $\circ$ admitting a unit element $\mathbb{1}$, making $\mathrm{Transf}(S)$ a monoid – the obvious interpretation is that $\circ$ means 'doing one transformation after another', and $\mathbb{1}$ is the trivial transformation (doing nothing to S).

[5] $\mathrm{UnitChan}(\mathcal{H}_S)$ could equivalently be defined to be the projective unitary group $\mathrm{PU}(\mathcal{H}_S)$, to which it is isomorphic as a group.

Definition 2.2 (Complementary subsystem)**.** *Let A be a subsystem of S, specified by* $\mathrm{Act}(A,S) \subseteq \mathrm{Transf}(S)$*. Its complementary subsystem is the subsystem A' of S specified by the commutant* $\mathrm{Act}(A,S)'$ *of* $\mathrm{Act}(A,S)$*, i.e. by*

$$\mathrm{Act}(A',S) = \mathrm{Act}(A,S)' := \{\mathcal{T} \in \mathrm{Transf}(S) \,|\, \forall \mathcal{A} \in \mathrm{Act}(A,S), \mathcal{T} \circ \mathcal{A} = \mathcal{A} \circ \mathcal{T}\}\,. \quad (5)$$

The operational motivation for the use of commutants in this definition is particularly neat: A', the 'rest of S' relative to A, is defined by the fact that operations on A' never 'see' operations on A, i.e. commute past those. This makes it operationally natural to assert that A' is indeed a subsystem independent of A. Furthermore, A' is maximal in that respect, i.e. *all* of the operations that commmute with all operations on A are taken to be operations on A'; this makes it natural to assert that A' includes any part of S that is independent of A.

Another very valuable property of complementary subsystems is that they are naturally closed under many algebraic structures of $\mathrm{Transf}(S)$.

Lemma 2.1. *Let* $\mathrm{Act}(A,S) \subseteq \mathrm{Transf}(S)$*. Then:*

1. $\mathrm{Act}(A',S)$ *is closed under* $\circ$*, even if* $\mathrm{Act}(A,S)$ *is not;*
2. *if* $\mathrm{Transf}(S)$ *admits some notion of sum distributing over* $\circ$ *(denoting, for instance, probabilistic mixtures of transformations), then* $\mathrm{Act}(A',S)$ *is closed under it, even if* $\mathrm{Act}(A,S)$ *is not;*
3. *if* $\mathrm{Transf}(S)$ *is a Hausdorff topological space and* $\circ$ *is continuous, then* $\mathrm{Act}(A',S)$ *is a topologically closed set, even if* $\mathrm{Act}(A,S)$ *is not.*
4. *if* $\circ$ *admits inverses (making* $\mathrm{Transf}(S)$ *a group) and* $\mathrm{Act}(A,S)$ *is closed under taking the inverse, then* $\mathrm{Act}(A',S)$ *is closed under taking the inverse;*
5. *if* $\mathrm{Transf}(S)$ *is a topological group (i.e. both* $\circ$ *and taking the inverse are continuous) that is compact and Hausdorff – which is in particular the case of* $\mathrm{UnitOp}(\mathcal{H}_S)$ *and* $\mathrm{UnitChan}(\mathcal{H}_S)$ *–, then* $\mathrm{Act}(A',S)$ *is closed under taking the inverse, even if* $\mathrm{Act}(A,S)$ *is not.*

Proof. The first, second, and fourth items are direct to prove. For the third, consider first the commutant set $\{\mathcal{F}\}'$ of a single transformation $\mathcal{F} \in \mathrm{Transf}(S)$. $\mathrm{Transf}(S)$ being Hausdorff means that the diagonal $\Delta := \{(\mathcal{T},\mathcal{T}) \,|\, \mathcal{T} \in \mathrm{Transf}(S)\}$ is a closed subspace of $\mathrm{Transf}(S) \times \mathrm{Transf}(S)$. Now, we have $\{\mathcal{F}\}' = (L_\mathcal{F}, R_\mathcal{F})^{-1}(\Delta)$, where $L_\mathcal{F}$ and $R_\mathcal{F}$ are left- and right-compositions by $\mathcal{F}$, respectively, which are continuous by assumption. Thus $\{\mathcal{F}\}'$ is closed. More general commutants, being intersections of commutants of singletons, are therefore closed as well.

For the fifth item, from the others we know that $\mathrm{Act}(A',S)$ is a closed submonoid of $\mathrm{Transf}(S)$; what remains to be shown is that any closed submonoid M of a compact topological group G is a subgroup, i.e. is closed under taking inverses. Fixing $g \in M$, by compactness we know that $(g^n)_{n\in\mathbb{N}}$ admits a converging subsequence, i.e. there exists a strictly increasing function $\sigma : \mathbb{N} \to \mathbb{N}$ such that $g^{\sigma(n)} \longrightarrow_{n\to\infty} \bar{g}$, with $\bar{g} \in M$ since M is topologically closed. Thus, using the continuousness of $\circ$ and of taking inverses, we obtain $M \ni g^{\sigma(n+1)-\sigma(n)-1} = g^{\sigma(n+1)} g^{-\sigma(n)} g^{-1} \longrightarrow_{n\to\infty} \bar{g}\bar{g}^{-1} g^{-1} = g^{-1}$, so by topological closure, $g^{-1} \in M$. □

Given a notion of complementary subsystems, it is natural to consider the *bi-complementary* subsystem A'' of A. The bi-complementation of subsystems, $A \mapsto A''$, has a natural interpretation as a closure of A, as the following fact illustrates.

Lemma 2.2. *Let* $\mathrm{Act}(A,S) \subseteq \mathrm{Transf}(S)$*. Then:*

$$\mathrm{Act}(A,S) \subseteq \mathrm{Act}(A'',S) = \mathrm{Act}(A'''',S) = \dots ; \tag{6a}$$

$$\mathrm{Act}(A',S) = \mathrm{Act}(A''',S) = \mathrm{Act}(A''''',S) = \dots . \tag{6b}$$

Proof. On the one hand, it is direct that $A \subseteq A''$ for any A. On the other hand, it is also direct that $X \subseteq Y \implies Y' \subseteq X'$, which, applied to the previous inequality, yields $A''' \subseteq A' \subseteq A'''$. □

Definition 2.3 (Bicom-closed subsystems)**.** *A subsystem* A *of* S *is* <u>*bicom-closed*</u>[6] *if* $A'' = A$*, or in other words,* $\mathrm{Act}(A,S)'' = \mathrm{Act}(A,S)$*.*

Lemma 2.3. *A is bicom-closed if and only if there exists a* B *such that* $A = B'$*.*

Proof. For the direct implication, take $B := A'$. For the reverse implication, using Lemma 2.2 we have $A'' = B''' = B' = A$. □

Bicom-closed subsystems display many appealing features that make it sensible to consider them the 'good subsystems'. First, at the conceptual level, they are by definition the subsystems with a satisfactory notion of complementarity, since they are the complement of their complement. Second, since a bicom-closed $\mathrm{Act}(A,S)$ is the commutant of another set by Lemma 2.3, we know directly by Lemma 2.1 that it is closed under many, if not all, of the typical algebraic or topological structures of $\mathrm{Transf}(S)$. Asking for bicom-closure is thus, in particular, a compact way to ensure closure under the relevant structures. In this paper, we will therefore focus on the study of bicom-closed subsystems.

Tenet 1. *A theory of subsystems can restrict itself to considering the bicom-closed ones.*

This was also stressed by Ref. [4]. In favour of this tenet, we can also add a final argument, which is that even if we hold it relevant to consider a subsystem A that is not bicom-closed, it would in practice be equivalent to consider instead its bicom-closure A''. Indeed, in a study of A's properties or perspective, the relevant data will be that of what 'does not matter' to A: e.g. what does not affect it, what can be traced out from it, or what can be lumped into equivalence classes. In our operational framework, this 'elsewhere' relative to A is obviously embodied by A'. Yet, since by Lemma 2.2 $A' = A'''$, A's elsewhere is the same as A'''s elsewhere. Therefore, it is reasonable to expect that any subsystem can, in practice, be replaced with its bicom-closure, thus yielding our Tenet 1.

Definition 2.4 (Bipartitions)**.** *Let* A *and* B *be two subsystems of* S*.* A *and* B *form a* <u>*bipartition*</u> *of* S *– denoted* $(A,B) \vdash S$ *– if and only if* $A = B'$ *and* $B = A'$*. (*A *and* B *are then necessarily bicom-closed by Lemma 2.2.)*

2.3 Operator subsystems

Going back to our presentation of three different flavours of quantum theory, the first one in terms of unitary operators ($\mathrm{Transf}(S) := \mathrm{UnitOp}(\mathcal{H}_S)$) yields a notion of a subsystem matching the most commonly adopted one, which is in terms of sub-C* algebras.

Definition 2.5. *A (finite-dimensional) *-algebra is a complex algebra equipped with an antilinear involution* $\dagger$ *satisfying* $(fg)^\dagger = g^\dagger f^\dagger$*. A sub-* algebra of it is a subalgebra closed under the dagger. A C* algebra is a *-algebra embeddable as a sub-* algebra of some space of linear operators on a Hilbert space,* $\mathrm{Lin}(\mathcal{H})$*, equipped with the adjoint as its dagger. A sub-* algebra of a C* algebra can thus be called a sub-C* algebra.*

[6] We can take this to mean, equivalently, 'bi-complementary-closed' or 'bi-commutant-closed'.

Proposition 2.1. *Let A be a subsystem in* $\mathrm{UnitOp}(\mathcal{H}_S)$*. A is bicom-closed if and only if there exists a sub-C* algebra $\mathcal{A}$ of* $\mathrm{Lin}(\mathcal{H}_S)$ *such that A corresponds to the unitary operators of $\mathcal{A}$, i.e.,*

$$\mathrm{Act}(A, S) = \mathcal{A} \cap \mathrm{UnitOp}(\mathcal{H}_S)\,. \tag{7}$$

Proof. Here, it is important to distinguish between commutants taken within $\mathrm{Unit}(\mathcal{H}_S)$ (which we will denote with a $'$), and commutants taken within $\mathrm{Lin}(\mathcal{H}_S)$ (which we will denote with a *). The crucial fact is that

$$\forall \mathcal{A} \text{ sub-C}^* \text{ algebra of } \mathrm{Lin}(\mathcal{H}_S), \quad \mathcal{A}^* \cap \mathrm{UnitOp}(\mathcal{H}_S) = (\mathcal{A} \cap \mathrm{UnitOp}(\mathcal{H}_S))'\,; \tag{8}$$

the forward inclusion is direct, and the reverse one stems from the fact that any finite-dimensional C* algebra is linearly spanned by its unitary elements, so that an element commuting with $\mathcal{A} \cap \mathrm{UnitOp}(\mathcal{H}_S)$ necessarily commutes with the whole of $\mathcal{A}$.

We move to the proof of the Proposition. We denote $\mathcal{A}$ to be the sub-C* algebra spanned by $\mathrm{Act}(A, S)$; by Von Neumann's bicommutant theorem in finite dimension, we have $\mathcal{A} = \mathrm{Act}(A, S)^{**}$. We can then compute,

$$\begin{aligned}
\mathrm{Act}(A, S)'' &:= (\mathrm{Act}(A, S)^* \cap \mathrm{UnitOp}(\mathcal{H}_S))' \\
&= (\mathrm{Act}(A, S)^{***} \cap \mathrm{UnitOp}(\mathcal{H}_S))' \\
&= (\mathcal{A}^* \cap \mathrm{UnitOp}(\mathcal{H}_S))' \\
&= \mathcal{A}^{**} \cap \mathrm{UnitOp}(\mathcal{H}_S) \\
&= \mathcal{A} \cap \mathrm{UnitOp}(\mathcal{H}_S)\,,
\end{aligned} \tag{9}$$

where in the last line we once again used Von Neumann's bicommutant theorem. We thus find that A is bicom-closed if and only if it is of the form $\mathcal{A} \cap \mathrm{UnitOp}(\mathcal{H}_S)$. □

As discussed in Section 1.3, framing subsystems using sub-C* algebras has a long history in the foundations of quantum physics, through algebraic quantum field theory as well as informational approaches.

2.4 Differences between operator subsystems and channel subsystems

Our focus in this paper, however, will be on the less-studied *channel subsystems*, by which we mean those based on the adoption of either $\mathrm{UnitChan}(\mathcal{H}_S)$ or $\mathrm{Chan}(\mathcal{H}_S)$ as our set of physical transformations. We shall in fact focus on the former of these. Before we move on to technical results, it is important to get a sense of the physical motivation for the channel-based approach, as well as some intuition of the differences it entails with the operator-based view.

The difference between a unitary operator and a unitary channel is that the latter includes no information about the global phase: it can easily be seen that for a given unitary channel $\mathcal{U}$, the operators U such that $\mathcal{U} = \hat{U}$ are related to one another by multiplication by an arbitrary phase factor, $U \mapsto e^{i\phi}U$. The non-physicalness of global phase thus makes it natural to conclude that – even notwithstanding the case of general channels, describing open quantum systems – unitary channels should be preferred over unitary operators to pin down the physical set of quantum transformations of closed systems. We encapsulate this into the following tenet.

Tenet 2. *Global phase is unphysical. Consequently, quantum transformations (of closed systems) are unitary channels, not unitary operators.*

What consequences does Tenet 2 have for a theory of subsystems? The critical shift is that, when seen at the level of operators, commutation will be replaced with *commutation up to a phase.* Take, for instance, the Pauli operators Z and X on a qubit. They do not commute, since $ZX = -XZ$; thus, in a theory based on $\mathrm{UnitOp}(\mathbb{C}^2)$, they cannot be seen as acting on two complementary subsystems. Yet the corresponding unitary channels, $\hat{Z} : \rho \mapsto Z\rho Z$ and $\hat{X} : \rho \mapsto X\rho X$, do commute; so that, in a theory based on $\mathrm{UnitChan}(\mathbb{C}^2)$, they indeed can be seen as acting on complementary subsystems. More precisely, we have the following seminal example.

Example 2. *In* $\mathrm{Transf}(S) = \mathrm{UnitChan}(\mathbb{C}^2)$*, the subsets*

$$\mathrm{Act}(A, S) := \left\{\hat{Z}\right\}'' = \left\{\hat{\mathbb{1}}, \hat{Z}\right\}, \tag{10a}$$

$$\begin{aligned}\mathrm{Act}(B, S) &:= \left\{\hat{Z}\right\}' \\ &= \left\{ \widehat{\begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}} \;\middle|\; \theta \in [0, 2\pi] \right\} \cup \left\{ \hat{U} \;\middle|\; U = \cos\phi X + \sin\phi Y,\, \phi \in [0, 2\pi] \right\} \\ &= \left\{ \hat{U} \;\middle|\; U = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix} X^l,\, \theta \in \mathcal{S},\, l \in \{0, 1\} \right\},\end{aligned} \tag{10b}$$

define a bipartition $(A, B) \vdash S$*.*

This is a special case of our upcoming structure theorem for bipartitions in unitary channels. One can indeed see that here, $\hat{Z} \in \mathrm{Act}(A, S)$ and $\hat{X} \in \mathrm{Act}(B, S)$. Similar subsystems could be defined in the theory of all channels, $\mathrm{Transf}(S) := \mathrm{Chan}(\mathbb{C}^2)$, by defining $\mathrm{Act}_{\mathrm{Chan}}(A, S) := \{\hat{Z}\}''$ and $\mathrm{Act}_{\mathrm{Chan}}(B, S) := \{\hat{Z}\}'$, with commutants taken in $\mathrm{Chan}(\mathbb{C}^2)$.

3 A characterisation of partitions in unitary channels

This section presents the main technical result of this paper: a study of bipartitions – in the sense of Definition 2.4 – in the theory of finite-dimensional unitary channels, with a characterisation of the form of those that are either the commutant or the bicommutant of a single unitary channel. In a nutshell, our theorem generalises Example 2: commutation up to a phase comes down to the appearance of a structure of clock- and shift-matrices, generalising the behaviour of the Pauli gates Z and X.

3.1 Unitaries commuting up to a phase

First, as we informally commented earlier, unitary channels commute if and only if the unitary operators describing their action commute up to a phase. We denote $\mathcal{S}$ for the group of reals modulo 2π, equipped with addition.

Proposition 3.1. *Unitary channels commuting Let* $\mathcal{U}, \mathcal{V} \in \mathrm{UnitChan}(\mathbb{C}^d)$ *be two unitaries channel. We pick (arbitrary) representatives* $U, V \in \mathrm{UnitOp}(\mathbb{C}^d)$ *such that* $\mathcal{U} = \hat{U}$ *and* $\mathcal{V} = \hat{V}$*. Then,*

$$\mathcal{U}\mathcal{V} = \mathcal{V}\mathcal{U} \iff \exists \phi \in \mathcal{S}, \quad UV = e^{i\phi} VU\,. \tag{11}$$

Proof. $\Longrightarrow$: Suppose $[\mathcal{U}, \mathcal{V}] = 0$. Showing that $\exists \phi, UV = e^{i\phi} VU$ is the same as showing that $\exists \phi, (VU)^\dagger UV = e^{i\phi} \mathbb{1}$, i.e. that all the eigenvalues of the unitary matrix $(VU)^\dagger UV$ are degenerate.

From $[\mathcal{U},\mathcal{V}]=0$, we have, for any $\rho \in \mathrm{Lin}(\mathbb{C}^d)$,

$$\begin{aligned}
\mathcal{U}\mathcal{V}(\rho) &= \mathcal{V}\mathcal{U}(\rho) \\
VU\rho(VU)^\dagger &= UV\rho(UV)^\dagger \\
\rho(VU)^\dagger(UV) &= (VU)^\dagger UV\rho \\
[\rho,(VU)^\dagger(UV)] &= 0
\end{aligned}$$

Let's call $M=(VU)^\dagger(UV)$, we have the condition that $\forall\rho\in\mathrm{Lin}(\mathbb{C}^d), [\rho, M]=0$. Let's show that this condition enforces $M=\lambda\mathbb{1}$. Unitarity of M will then impose $|\lambda|=1$. Take any $|\psi\rangle\in\mathbb{C}^d$ unit vector, then:

$$\begin{aligned}
M\left|\psi\right\rangle\left\langle\psi\right| &= \left|\psi\right\rangle\left\langle\psi\right| M \\
M\left|\psi\right\rangle\left\langle\psi\middle|\psi\right\rangle &= \left|\psi\right\rangle\left\langle\psi\right| M\left|\psi\right\rangle \\
M\left|\psi\right\rangle &= \lambda_\psi\left|\psi\right\rangle, \text{with } \lambda_\psi=\left\langle\psi\right| M\left|\psi\right\rangle \\
\text{Then, for } \left|\psi\right\rangle,\left|\phi\right\rangle \text{ orthogonal, call } \left|\phi+\psi\right\rangle &= \frac{\left|\psi\right\rangle+\left|\phi\right\rangle}{\sqrt{2}} \\
M\left|\phi+\psi\right\rangle &= \lambda_{\phi+\psi}\left|\phi+\psi\right\rangle \\
\text{but } M\left|\phi+\psi\right\rangle &= \frac{\lambda_\phi}{\sqrt{2}}\left|\phi\right\rangle+\frac{\lambda_\psi}{\sqrt{2}}\left|\psi\right\rangle \\
\text{Thus, by identification, } \lambda_{\phi+\psi} &= \lambda_\phi=\lambda_\psi
\end{aligned}$$

Hence any basis $\{|e_k\rangle\}$ is a set of eigenvectors of M with equal eigenvalues, proving that $M=\lambda\mathbb{1}$. Unitarity of M imposes λ to be a phase, thus, $\exists\phi/(VU)^\dagger(UV)=e^{i\phi}\mathbb{1}$. This concludes: $[\mathcal{U},\mathcal{V}]=0 \Longrightarrow \exists\phi/U.V=e^{i\phi}V.U$.

$\Longleftarrow$ is direct. □

Since the commutation of unitary channels comes down to the commutation of unitary operators up to a phase, we will focus on the latter.

Definition 3.1 (Generalised commutant)**.** *Given $U\in\mathrm{UnitOp}(\mathcal{H})$ and $\phi\in\mathcal{S}$, U's $e^{i\phi}$-commutant $\mathcal{C}_\phi(U)$ is*

$$\mathcal{C}_\phi(U) := \{V\in\mathrm{UnitOp}(\mathcal{H})\,|\,UV=e^{i\phi}VU\}\,, \tag{12}$$

and U's <u>generalised commutant</u> $\mathcal{C}(U)$ is

$$\mathcal{C}(U) := \bigcup_{\phi\in\mathcal{S}}\mathcal{C}_\phi(U)\,. \tag{13}$$

Given $A\subseteq\mathrm{UnitOp}(\mathcal{H})$, A's generalised commutant $\mathcal{C}(A)$ is

$$\mathcal{C}(A) := \bigcap_{U\in A}\mathcal{C}(U)\,. \tag{14}$$

Note that for any A, $\mathcal{C}(A)$ is a subgroup of $\mathrm{UnitOp}(\mathcal{H})$. We then have the following direct Corollary of Proposition 3.1.

Corollary 3.1. *Let A be a subsystem of S, in the theory of unitary channels; we denote $\tilde{A}:=\{U\in\mathrm{UnitOp}(\mathcal{H}_S)\,|\,\hat{U}\in\mathrm{Act}(A,S)\}$. Then*

$$\mathrm{Act}(A',S)=\{\hat{U}\,|\,U\in\mathcal{C}(\tilde{A})\}\,, \tag{15}$$

and A is bicom-closed if and only if $\mathcal{C}(\mathcal{C}(\tilde{A}))=A$.

In the following, with a slight abuse of notation, we will denote $A = \tilde{A}$.

The following proposition shows that there are in fact strong constraints on which phases can appear in $e^{i\phi}$-commutation, and on which operators can display it.

Proposition 3.2. *Let $U \in \mathrm{UnitOp}(\mathbb{C}^d)$ and $\phi \in \mathcal{S}$ such that $\mathcal{C}_\phi(U)$ is not empty. Then $e^{i\phi}$ is a d-th root of unity, i.e. $\phi = 2\pi \frac{k}{d}$ for some $k \in [\![0, d-1]\!]$, and U's spectrum (including the data of multiplicity) is invariant under multiplication by $e^{i\phi}$:*

$$\mathrm{Spec}(U) = e^{i2\pi\frac{k}{d}} \cdot \mathrm{Spec}(U)\,. \tag{16}$$

Proof. Taking $V \in \mathcal{C}_\phi(U)$, we get $\det(UV) = \det(e^{i\phi}VU) = e^{i\phi d}\det(VU) = e^{i\phi d}\det(UV)$. $\det(UV) \neq 0$ as UV unitary, thus $e^{i\phi d} = 1$ and ϕ is a d-th root of unity.

For the spectrum invariance, we have $U = e^{i2\pi\frac{k}{d}}VUV^\dagger$. Taking the characteristic polynomial $\chi_U(\lambda) := \det(\lambda\mathbb{1} - U)$ of U gives:

$$\chi_U(\lambda) = \det(V(\lambda\mathbb{1} - e^{i2\pi\frac{k}{d}}U)V^\dagger) = \det(\lambda\mathbb{1} - e^{i2\pi\frac{k}{d}}U) = \chi_U(\lambda e^{-i2\pi\frac{k}{d}})\,. \tag{17}$$

The equality $\chi_U(\lambda) = \chi_U(\lambda e^{-i2\pi\frac{k}{d}})$ implies the equality of the set of roots with multiplicity. Spec(U) being the set of roots of $\chi_U(\lambda)$ counted with multiplicity, we have Spec(U) $= e^{i2\pi\frac{k}{d}}$ Spec(U). □

Those two results show that commutation up to a phase for a unitary U reduces to standard commutation, except when U's spectrum exhibits some invariance, hence allowing for commutation up to a non trivial phase. This warrants the definition of a unitary's spectral period.

Definition 3.2 (Spectral period)**.** *Let $\mathcal{D}(d)$ be the set of integers dividing d, and $U \in \mathrm{UnitOp}(\mathbb{C}^d)$; U's spectral period is the element $b \in \{1, \dots, d\}$ defined by*

$$b := \max\{c \in \{1, \dots, d\} \,|\, \mathrm{Spec}(U) = e^{i2\pi\frac{1}{c}} \cdot \mathrm{Spec}(U)\}\,. \tag{18}$$

Note that b is in $\mathcal{D}(d)$, as are all elements of the set it maximises. $b = 1$ corresponds to the case in which U's spectrum features no particular invariance.

Lemma 3.1. *Let $U \in \mathrm{UnitOp}(\mathcal{H})$ with spectral period b. Then*

$$\mathcal{C}(U) = \bigcup_{m=0}^{b-1} \mathcal{C}_{2\pi\frac{m}{b}}(U) = \{V \in \mathrm{UnitOp}(\mathcal{H}) \,|\, \exists m \in [\![0, b-1]\!], UV = e^{i2\pi\frac{m}{b}}VU\}\,. \tag{19}$$

Proof. Suppose $\exists V, k \in \mathbb{N}^\star / UV = VUe^{i2\pi\frac{k}{d}}$. We denote $k^\star := \frac{d}{b}$. We have $\mathrm{Spec}(U) = e^{i2\pi\frac{k}{d}}\,\mathrm{Spec}(U)$. Let's then show that the only k's allowed are multiples of $k^\star$. By definition of b, we necessarily have that $k^\star \leq k$. Suppose ad absurdum that $k \neq pk^\star, p \in \mathbb{N}^\star$ ie that $\exists p \in \mathbb{N}^\star / k \in]\!]pk^\star, (p+1)k^\star[\![$. Using that $\mathrm{Spec}(U) = e^{i2\pi\frac{k^\star}{d}}\,\mathrm{Spec}(U)$, we also get that $\mathrm{Spec}(U) = e^{i2\pi\frac{k-p\cdot k^\star}{d}}\,\mathrm{Spec}(U), p \in \mathbb{N}^\star$. But then $0 < k - pk^\star < k^\star$ which is absurd. Thus k is necessarily an integer multiple of $k^\star$. Hence $\mathcal{C}(U) = \{V \in U(d) / \exists p \in \mathbb{N}, U.V = e^{i2\pi\frac{k^\star p}{d}}V.U\}$. As $\frac{k^\star}{d} = \frac{1}{b}$, $\mathcal{C}(U)$ is well given by Eq. 19. □

3.2 Clock and shift matrices

Clock and shift matrices play a central role in our classification. They should be thought of as generalisations of the Z and X Pauli matrices, trading anti-commutation for $e^{\frac{2\pi i}{d}}$-commutation.

Definition 3.3 (Clock and shift matrices)**.** *Let $d \in \mathbb{N}^*$; we denote $\omega := e^{\frac{2\pi i}{d}}$. The shift matrix $\Sigma_1^{(d)}$ and the clock matrix $\Sigma_3^{(d)}$ of dimension d (which we will denote as Σ_1 and Σ_3 when d is unambiguous) are the unitary matrices of size d defined by*

$$\Sigma_1 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \qquad \Sigma_3 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \omega & 0 & \cdots & 0 \\ 0 & 0 & \omega^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \omega^{d-1} \end{pmatrix}. \tag{20}$$

Note that for $d = 1$ we have $\Sigma_1^{(1)} = \Sigma_3^{(1)} = (1)$, and that for $d = 2$, $\Sigma_1^{(2)} = X$ and $\Sigma_3^{(2)} = Z$. The shift and clock matrices precisely $e^{\frac{2\pi i}{d}}$-commute, as

$$\Sigma_3^{(d)} \Sigma_1^{(d)} = e^{\frac{2\pi i}{d}} \Sigma_1^{(d)} \Sigma_3^{(d)} . \tag{21}$$

A non-trivial spectral period for a unitary indicates that, up to a change of basis, it can be built out of a Σ_3. (Note that this could equivalently be done with a Σ_1, as the clock and shift matrices are equivalent up to a change of basis.)

Proposition 3.3 (Characterisation of unitaries from their spectral period)**.** *Let $d \in \mathbb{N}^*$, $\mathcal{H}$ a Hilbert space of dimension d, and $U \in \mathrm{UnitOp}(\mathcal{H})$ with spectral period b. Then there exist $r \in \mathbb{N}^*$, a list of dimensions $(d_1, \ldots, d_r)$ satisfying*

$$d = \sum_{m=1}^{r} d_m \cdot b \,, \tag{22}$$

a set of phases $(\phi_m)_{m=1,\ldots,r}$, and an adapted form of $\mathcal{H}$, $\mathcal{H} \cong \bigoplus_{m=1}^{r} \mathbb{C}^b \otimes \mathbb{C}^{d_m}$, such that in this adapted form,

$$U = \bigoplus_{m=1}^{r} \Sigma_3^{(b)} \otimes e^{i\phi_m} \mathbb{1}_{d_m} . \tag{23}$$

Furthermore, the picked r is minimal for U if and only if the ϕ_m's are all different modulo $\frac{2\pi}{b}$.

One can remark that for $b = 1$, i.e. when U features no particular invariance, Proposition 3.3 reduces to the standard spectral theorem.

Proof. From the definition of b, $\mathrm{Spec}(U) = e^{i2\pi\frac{1}{b}}\,\mathrm{Spec}(U)$. Hence, $\mathrm{Spec}(U)$ is a set of d phases (unitarity of U), with this periodicity constraint. Introduce $\nu = e^{i2\pi\frac{1}{b}}$ and $p/d = bp$. We have: $\mathrm{Spec}(U) = \bigcup_{j=1}^{p} S_j$, with $S_j = e^{i\psi_j}\{\nu^l, l = 0, ..., b-1\}$. The set of phases $\{\psi_j\}_{j=1}^{p}$ keeps the redundant phases, *i.e.* contains p phases. Suppose we have$i \neq j$ and $\psi_i - \psi_j = 0[\frac{2\pi}{b}]$. Then $S_i = S_j$, the two sets are degenerate. Thus let's consider the subset of independent phases $\{\phi_m\}_{m=1}^{r} \subseteq \{\psi_j\}_{j=1}^{p}$ such that $m \neq n \implies \phi_m - \phi_n \neq 0[\frac{2\pi}{b}]$. The r parameter depends on U, and we have $1 \leq r \leq p$. The case $r = p$ is the non degenerate one, in which all the eigenvalues of U are distinct. For a phase ϕ_m, let's call

d_m the number of phases in $\{\psi_j\}_{j=1}^p$ such that $\psi_j = \phi_m[\frac{2\pi}{b}]$. This is the degeneracy of the phase ϕ_m, and hence of the set S_m. We now work exclusively with the d_m degenerated phases ϕ_m. In a nutshell, keeping in mind that $p = \frac{d}{b}$, $r \in [1,p]$, $\nu = e^{i2\pi\frac{1}{b}}$, and $m \neq n \implies \phi_m - \phi_n \neq 0[\frac{2\pi}{b}]$, we have shown that:

$$\mathrm{Spec}(U) = \bigcup_{m=1}^{r} e^{i\phi_m}\{\nu^l, l = 0, ..., b-1\}$$

Let's define E_m the subspace generated by the eigenvectors associated to the eigenvalues in the set $S_m = e^{i\phi_m}\{\nu^l, l = 0, ..., b-1\}, \nu = e^{i2\pi\frac{1}{b}}$. From the independency of the ϕ_m, $S_m \cap S_n = \emptyset$, and the different E_m's are orthogonal subspaces, as associated to different eigenvalues. Then we can see the E_m as independent sectors with well-defined phase ϕ_m, the spectrum being made of r independent loops. Eigenvalues in subspace S_m are all degenerated d_m times.

Conclusion: U in its diagonal basis is formed of r sectors of size $d_m \cdot b$, with $b\sum_{m=1}^r d_m = bp = d$. Each sector is characterized by a global phase ϕ_m. Its eigenvalues are $e^{i\phi_m}\{\nu^l, l = 0, ..., b-1\}$, each degenerated d_m times. □

3.3 The commutant and bicommutant of a unitary channel

We now have the tools to state this paper's main technical result: a full structural elucidation of the commutant and bicommutant sets of a given unitary channel $\mathcal{U} = \hat{U}$ in finite dimension. First, by Corollary 3.1, we know that studying $\{\mathcal{U}\}'$ and $\{\mathcal{U}\}''$ amounts to characterising $\mathcal{C}(U)$ and $\mathcal{C}(\mathcal{C}(U))$, so we focus on the latter problem. We then have the following Theorem.

Theorem 3.1. *Let $U \in \mathrm{UnitOp}(\mathcal{H})$ with $\mathcal{H}$ of dimension d, with spectral period b. Using Proposition 3.3, we move to an adapted form of $\mathcal{H}$ for U, $\mathcal{H} \cong \bigoplus_{m=1}^r \mathbb{C}^b \otimes \mathbb{C}^{d_m}$, in which (23) holds. Then,*

$$\begin{aligned}\mathcal{C}(U) = \Bigg\{ \bigoplus_{m=1}^{r} \left(\mathrm{Diag}\left(V_0^{(m)}, \dots, V_{b-1}^{(m)}\right) \cdot \left(\left(\Sigma_1^{(b)}\right)^l \otimes \mathbb{1}_{d_m}\right)\right), \\ \Bigg| V_0^{(m)}, \dots, V_{b-1}^{(m)} \in \mathrm{UnitOp}(\mathbb{C}^{d_m}) \forall m, \quad 0 \leq l \leq b-1 \Bigg\},\end{aligned} \tag{24a}$$

$$\begin{aligned}\mathcal{C}(\mathcal{C}(U)) = \Bigg\{ \left(\bigoplus_{m=1}^{r} \left(\Sigma_3^{(b)}\right)^k \otimes e^{i\theta_m}\mathbb{1}_{d_m}\right), \\ \Bigg| 0 \leq k \leq b-1, \quad \theta_1, \dots, \theta_r \in \mathcal{S} \Bigg\}.\end{aligned} \tag{24b}$$

Proof. See Appendix A. □

A few remarks are in order to lend some intuition. First, one can note that in the case $b = 1$, i.e. when U features no particular periodicity, we have $\mathcal{C}(U) = \mathcal{C}_0(U) = \{U\}'$, and Theorem 3.1 simply characterises the commutant and bi-commutant (not up to a phase) of U, in a basis in which it is diagonal (moving to the latter basis is the point of Proposition 3.3 in that case). Indeed, in that case, it is not possible to commute with U up to a phase, as already shown by Lemma 3.1.

It is when U has a non-trivial spectral period that things get interesting, with some unitaries commuting with it up to a non-trivial phase. The situation is then, first, that U can be seen as a series of m-labelled blocks, each acting on a space $\mathbb{C}^b \otimes \mathbb{C}^{d_m}$; in each it is equal to the (b-dimensional) clock matrix on the left-hand factor, multiplied by a certain phase (this is the content of Proposition 3.3).

Second, as formalised by (24a), an element of $\mathcal{C}(U)$ acts in each block as a product of two unitaries: one that is a certain power l of the d_m times degenerate shift matrix, and one that is an arbitrary block diagonal unitary, each block acting freely on a degenerate eigensubspace of U. As an additional constraint, l has to be the same across blocks.

Third, as formalised by (24b), an element of $\mathcal{C}(\mathcal{C}(U))$ acts on each block as a certain power k of the clock matrix on the left-hand factor, multiplied by an arbitrary phase. Here again, k has to be the same across blocks.

In the case $b = d = 2$, Proposition 3.3, (24a), and (24b) precisely yield Example 2. This shows that in dimension 2, and up to a change of basis, this example constitutes the only non-trivial (i.e. not reducible to commutation of the underlying unitary operators) instance of commutation of unitary channels.

All in all, Theorem 3.1 shows that commutation up to a phase is a tightly constrained behaviour at the structural level: it may only arise when the unitaries at hand can be block-decomposed into collections of clock or shift matrices, interplaying in a very specific way. Nonetheless, it does feature a family of non-trivial examples, leading to instances of subsystems in the theory of unitary channels that cannot be understood in terms of the traditional C*-algebraic subsystems, predicated on the use of unitary operators and 'strict' commutation between those.

4 What can be measured in a subsystem? The strange case of 'blind agents'

Equipped with a solid understanding of the mathematical structure of commutants and bi-commutants in unitary channels, we turn to an exploration of the physical features they yield for channel-based subsystems. Of particular interest is the kind of measurements that they allow for: we show how some seem to correspond to 'opaque subsystems', possessed by 'blind agents', who can non-trivially act on the world, yet cannot perform any measurement.

4.1 What is measurable in a channel subsystem

In a theory that only involves deterministic operations, the status of measurements is not straightforward. In the context of our study here, this means that while our definition of a subsystem transparently ties it to what an agent can deterministically do to it, it is not as direct to read out what this agent can *measure* on it, since the obtention of a measurement outcome is a non-deterministic transformation.[7]

[7] A way around this issue would be to include nondeterministic transformations in $\mathrm{Transf}(S)$, e.g. trace-non-increasing CP maps in the quantum case. Since our main focus is on the theory of unitary channels, in which this inclusion is unnatural, we refrain from adopting this approach. However it seems clear that it would lead to conclusions similar to ours in this section: i.e. to the existence of subsystems – defined within a theory including all trace-non-increasing CP maps – that only include trace-preserving CP maps, thus not featuring any measurements. This is the case, for instance, for the subsystem $\{\hat{Z}\}''$ of the set of trace-non-increasing CP maps on a qubit (with commutants taken within this theory), in close parallel with our results here.

Since subsystems defined in the theory of unitary operators match the C*-algebraic approach by Proposition 2.1, it is very natural to posit that what can be measured in that case are simply the C* algebra's observables – indeed, in the historically adopted approach, the direction of inference is reversed: the subsystem is primarily defined not by its closed dynamics as here, but by its observables. For an argument connecting dynamics to observables in that context, see Ref. [28]. However, in an approach based on channels (unitary or not), there is no direct mathematical clue to the right notion.

Our approach here will be to define measurability through the inclusion of ancillary systems and the resort to commutants. The idea is that the measurements Alice can perform on her subsystem A are the ones that can be implemented by 1/ bringing in an ancillary system E in her entire possession (i.e. initialised however she wants), 2/ implementing a transformation on AE, and 3/ measuring the final state of E. In that context, the open question for us is to pin down the extent of Alice's possible operations in step 2; indeed her subsystem was originally defined only as a subset of transformations of the original system S, while here we are talking about transformations of SE. This is where commutants come in: we shall define the set of transformations available to Alice on SE as the commutant, within the transformations of SE, of transformations of A'.

In our versions of quantum theory, this takes the following form. For a subset X of some vector space, we denote $X \otimes \mathbb{1} := \{\mathcal{U} \otimes \mathbb{1} \,|\, \mathcal{U} \in X\}$.

Definition 4.1 (Subsystem extension)**.** *In one of our three quantum theories, let A be a bicom-closed subsystem of S, defined by* $\mathrm{Act}(A,S) \subseteq \mathrm{Transf}(S)$. *We take an ancillary system E and define* $\mathrm{Transf}(SE)$, *corresponding to the Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_E$.*

A's E-extension is the subsystem AE of SE defined by[8]

$$\mathrm{Act}(AE,SE) := \Big(\mathrm{Act}(A,S)'_{\mathrm{Transf}(S)} \otimes \mathbb{1}_E\Big)'_{\mathrm{Transf}(SE)} \,. \tag{25}$$

It is directly bicom-closed by Lemma 2.3. The interpretation is that AE is the set of transformations available to an agent in possession of both systems A and E.

4.2 Characterising subsystems extensions in unitary channels

The following theorem is proven in Appendix B.

Theorem 4.1 (Subsystem extensions in unitary channels)**.** *In the theory of unitary channels, consider a subsystem* $\mathrm{Act}(A,S) \subseteq \mathrm{Transf}(S) = \mathrm{UnitChan}(\mathcal{H}_S)$ *arising as the bicommutant of a singleton; i.e.* $\mathrm{Act}(A,S)$ *is of the form (24b). Take an ancillary system E, corresponding to Hilbert space $\mathcal{H}_E$.*

Then, defining

$$\forall k \in \{0,\dots,b-1\}, \quad O_k := \left\{ \bigoplus_{m=1}^{r} \left(\left(\Sigma_3^{(b)}\right)^k \otimes \mathbb{1}_{d_m} \otimes (U_m)_E \right) \,\middle|\, \forall m \in \{1,\dots,r\}, U_m \in \mathrm{UnitOp}(\mathcal{H}_E) \right\}, \tag{26}$$

we have

$$\mathrm{Act}(AE,SE) = \bigcup_{k=0}^{b-1} \left\{ \hat{U} \,\middle|\, U \in O_k \right\} . \tag{27}$$

[8]To avoid any ambiguity, we specify as a subscript the overall set in which each commutant is taken.

This can be understood as stating that the allowed operations on SE are given by 1/ picking a k, and 2/ in each m-block, performing a $(\Sigma_3^{(b)})^k$ on the appropriate subfactor on that block, and an operation U_m on E, with the latter possibly coherently controlled on the value of m.

If we look in particular at the case $r = 1$, we find that the channels in $\mathrm{Act}(AE, SE)$ are all tensor products of a channel on S (which is some power of Σ_3) and an (arbitrary) channel on E. This is quite striking: these channels feature no interaction at all between S and E, and thus allow no information transfer from A to the ancilla E, on which Alice performs measurements. This can be seen more concretely in our seminal qubit example.

Example 3. *Going back to our Example 2, suppose we bring in an ancilla E of arbitrary dimension in the possession of Alice. Her possible operations are then given by*

$$\begin{aligned} \mathrm{Act}(AE, SE) = &\{\mathbb{1}_S \otimes \mathcal{U}_E \mid \mathcal{U} \in \mathrm{UnitChan}(\mathcal{H}_E)\} \\ &\cup \{\hat{Z}_S \otimes \mathcal{U}_E \mid \mathcal{U} \in \mathrm{UnitChan}(\mathcal{H}_E)\}\,. \end{aligned} \tag{28}$$

Thus, Alice can in fact measure nothing about S. (We could make this statement more formal by characterising the POVMs she can build out of her unitary channels, but the conclusion is already transparent given that the latter are all factorisable.) This is in spite of her system A being non-trivial, since her set of possible operations is non-trivial, as it includes the $\hat{Z}$ unitary channel.[9]

4.3 The strange case of 'blind agents'

We thus reached our second major result at the mathematical level: a theory of quantum subsystems construed as subsets of unitary channels (as opposed to unitary operators) leads to the appearance of subsystems on which no measurements are possible – subsystems that, in fact, cannot unitarily exchange information with an environment. If we think of these subsystems as being in the possession of corresponding agents, then we are dealing with *blind agents*, who may non-trivially affect their system, yet may not learn anything about it.

This result, which is unescapable from a mathematical point of view, calls for a conceptual and physical clarification. How could these subsystems, or these agents, arise concretely, on a physically instantiated system? How could their operations' restrictions be understood naturally? How could their 'blindness' find a logical explanation?

To these riddles, we unfortunately have not found satisfactory answers so far.

This leaves us – and the reader – with two possible attitudes concerning the conclusions to be drawn from our findings. The first would be to conclude that subsystems based on subsets of unitary channels, or equivalently on the commutation of unitary operators up to a phase, make no sense, and should not be taken seriously. While perfectly admissible, we want to stress that this conclusion sits oddly with Tenet 2, or in other words with the widely accepted idea that global phase is a mathematical fiction that should be disregarded in serious investigations of quantum theory. Indeed, this idea is what led us to channel-based subsystems.

[9] Additionally, Alice cannot even check, through a measurement, whether she herself performed a $\hat{Z}$ operation. Still this operation is not trivial: a more powerful agent – for instance having access to all operations on S – could indeed figure out whether she performed it. Agents that can implement some transformations but not check whether they implemented them already arise in the C* algebraic theory of subsystems, when the subsystems correspond to non-factor C* algebras; see Section 2.4 in Ref. [1]. However these agents are not blind, they still have access to a non-trivial set of measurements.

Thus, a proponent of this first attitude – one, at least, with interest in providing clear operational definitions of what subsystems are – should come up with operational principles that rule out blind agents (or even partially blind ones, which is the general case). On that front, our view is that current justifications do not cut the deal. For instance, in the traditional approach adopting C* algebras and interpreting them as containing the accessible observables, it is not clear where the closure under multiplication is coming from. The closest to an operational justification we can think of is to instead interpret them through their unitary operators and appeal to their closure under bicommutants – this is the content of Proposition 2.1; but this justification crucially depends on the meaningfulness of global phase.

The other possible attitude is to bite the bullet, and to hold that these subsystems, proven by bicom-closure to be very robust at the operational level, make sense; so that it might be meaningful, under suitable circumstances, to refer to them in foundational enquiries – for instance, to hold them to define causal relata in causal modelling. Of course, a proponent of this attitude should, as we said, provide concrete instances of situations in which this is indeed meaningful. Despite our failure at doing so, it is not clear to us either that such instances cannot be found; maybe they just require us to strip away some hidden conceptual prejudice, blinkering our view in favour of more traditional subsystems.

As can be seen, here we chose to remain agnostic as to which conclusion should be preferred. At this stage, we find it more fruitful to present the problem, and to leave it to future work to take stands concerning it.

5 Conclusion and outlook

This paper was dedicated to deriving the concrete consequences, for quantum theory, of the introduction of an operational definition of subsystems as proposed in the abstract in Refs. [2, 3, 4]. We showed that, while that definition recovers the traditionally adopted C* algebraic subsystems in the unitary-operator version of quantum theory, the unitary-channel version features other, more exotic types of subsystems, which at the operator level can be understood as displaying commutation up to a global phase. We displayed the general form of a subclass of such subsystems, namely those arising as the commutant or bicommutant of a certain unitary channel. Finally, we showed that some instances admit no unitary interaction with an environment, and thus no measurements, and discussed the possible views on this odd state of affairs.

As stated above, the main open question left by our work is about the status of these subsystems corresponding to blind agents: one would need either clear operational principles banning them, or concrete examples of situations in which it makes sense to consider them. If the latter are provided – i.e. if these subsystems prove themselves as relevant parts of the Universe to consider – then it would make it particularly interesting to consider what their incorporation into our theories can teach us. If we hold them to be legitimate causal relata in a theory of causal modelling, for instance, then what can be said about the causal influences exerted on or by them? If we hold them to yield a legitimate, extended notion of locality, then what can it tell us about the potential splitting of physical degrees of freedom into spacetime? These subsystems' surprising features in terms of possible measurements, in particular, promise to lead to counterintuitive – but maybe instructive – consequences in those contexts.

Another direction left open is the consideration of bicom-closed subystems in the context of the quantum theory of open systems, on in other words in a set of global transformations $\mathrm{Transf}(S) = \mathrm{Chan}(\mathcal{H}_S)$ including all quantum channels, and not only the unitary ones. This would entail two challenges, one conceptual and one mathematical. The first would be whether subsystems in a theory involving noise can be taken to hold fundamental relevance, since the presence of that noise could usually be ascribed to contingent reasons (such as experimental imperfections) and not to fundamental, physically relevant ones. The second is, more mundanely, that commutants in a theory of quantum channels seem – at least to us – quite hard to keep track of. For instance, it is not clear to us whether, given an arbitrary noisy channel $\mathcal{C}$, there exists a compact and structurally illuminating way to derive the set of all channels commuting with $\mathcal{C}$.

A Proof of Theorem 3.1

Firstly, let's prove Eq. 24a.

Proof. For $U = \oplus_{m=1}^r \Sigma_3^{(b)} \otimes e^{i\phi_m} \mathbb{1}_{d_m}$, introduce its eigenvectors. As the degenerate eigenvalues are of the form $e^{i\phi_m}\nu^k$, with $\nu = e^{i2\pi\frac{1}{b}}$, let's use three indices: $m \in [\![1, r]\!]$ referring to the phase sector, $k \in [\![0, b-1]\!]$ referring to the clock position; $i \in [\![1, d_m]\!]$ taking into account the possible degeneracy of the sector. The set of eigenvectors is then: $\{|m, k, i\rangle, m \in [\![1, r]\!], k \in [\![0, b-1]\!], i \in [\![1, d_m]\!]\}$. Let's search for the structure of the $V \in \mathcal{C}(U)$. V is such that $\exists l \in [\![0, b-1]\!], UV = \nu^l VU$. Hence, we have:

$$
\begin{aligned}
UV |m, k, i\rangle &= \nu^l VU |m, k, i\rangle \\
U(V |m, k, i\rangle) &= e^{i\phi_m}\nu^{k+l}(V |m, k, i\rangle)
\end{aligned}
$$

That shows that $V |m, k, i\rangle, i \in [\![1, d_m]\!]$ are eigenvectors of U with eigenvalue $e^{i\phi_m}\nu^{k+l}$. This eigenvalue is associated to the degenerate eigenvectors $\{|m, k+l, i\rangle, i \in [\![1, d_m]\!]\}$. Thus, $V \cdot \text{Span}(\{|m, k, i\rangle, i \in [\![1, d_m]\!]\}) \subseteq \text{Span}(\{|m, k+l, i\rangle, i \in [\![1, d_m]\!]\})$. However those two spaces have same dimension d_m, hence are equal:

$$
\forall m, V \cdot \text{Span}(\{|m, k, i\rangle, i \in [\![1, d_m]\!]\}) = \text{Span}(\{|m, k+l, i\rangle, i \in [\![1, d_m]\!]\})
$$

Let's derive the consequences on V's structure. Firstly, V acts independently on each phase sector, which identifies to $\mathbb{C}^{d_m \cdot b}$, with $\mathcal{H} \cong \oplus_{m=1}^r \mathbb{C}^{d_m \cdot b}$. Hence V decomposes in U's adapted basis as: $V = \bigoplus_{m=1}^r \tilde{V}_m$, $\tilde{V}_m \in \text{UnitOp}(\mathbb{C}^{d_m \cdot b})$. Secondly, each unitary block $\tilde{V}_m$ has a particular structure, as it connects eigensubspaces of $\Sigma_3^{(b)} \otimes e^{i\phi_m}\mathbb{1}_{d_m}$: for all $k \in [\![0, b-1]\!]$, we have $\tilde{V}_m \,\text{Span}(\{|m, k, i\rangle, i \in [\![1, d_m]\!]\}) = \text{Span}(\{|m, k+l, i\rangle, i \in [\![1, d_m]\!]\}$. This feature reveals an underlying shift matrix in the structure of $\tilde{V}_m$. Hence $\tilde{V}_m$ is necessarily of the form: $\tilde{V}_m = \text{Diag}(V_j^{(m)})_{j \in [\![0, b-1]\!]} \cdot ((\Sigma_1^{(b)})^l \otimes \mathbb{1}_{d_m})$, with $\forall m, j, V_j^{(m)} \in \text{UnitOp}(\mathbb{C}^{d_m})$. Let's check that indeed the commutation relation holds well:

$$
\begin{aligned}
\Sigma_3^{(b)} \otimes \mathbb{1}_{d_m} \cdot \tilde{V}_m &= \Sigma_3^{(b)} \otimes \mathbb{1}_{d_m} \cdot (\text{Diag}(V_j^{(m)})_{j \in [\![0, b-1]\!]} \cdot (\Sigma_1^{(b)})^l \otimes \mathbb{1}_{d_m}) \\
&= \text{Diag}(V_j^{(m)})_{j \in [\![0, b-1]\!]} \cdot (\Sigma_3^{(b)} \otimes \mathbb{1}_{d_m} \cdot (\Sigma_1^{(b)})^l \otimes \mathbb{1}_{d_m}) \\
&= \text{Diag}(V_j^{(m)})_{j \in [\![0, b-1]\!]} \cdot (\Sigma_3^{(b)} \cdot (\Sigma_1^{(b)})^l) \otimes \mathbb{1}_{d_m} \\
&= \nu^l \,\text{Diag}(V_j^{(m)})_{j \in [\![0, b-1]\!]} \cdot (\Sigma_1^{(b)})^l \cdot \Sigma_3^{(b)}) \otimes \mathbb{1}_{d_m} \\
&= \nu^l \tilde{V}_m \cdot \Sigma_3^{(b)} \otimes \mathbb{1}_{d_m}
\end{aligned}
$$

Commutation used from line 2 to line 3 comes from $\Sigma_3^{(b)} \otimes \mathbb{1}_{d_m} = \text{Diag}(\nu^j \mathbb{1}_{d_m})_{j \in [\![0, b-1]\!]}$.

Thus the $(d_m b) \times (d_m b)$ matrices $\tilde{V}_m$ is specified by an integer l, which is independent of m, and a set of b unitaries of size d_m, that can be chosen freely. Hence,

$$
\mathcal{C}(U) = \{\bigoplus_{m=1}^{r} \text{Diag}(V_j^{(m)})_{j \in [\![0, b-1]\!]} \cdot ((\Sigma_1^{(b)})^l \otimes \mathbb{1}_{d_m}), \text{ with } (V_j^{(m)})_{j \in [\![0, b-1]\!]} \in \text{UnitOp}(\mathbb{C}^{d_m})^b, l \in [\![0, b-1]\!]\}
$$

, as given by Eq. 24a. $\square$

Secondly, let's show Eq. 24b.

Proof. Take $G \in \mathcal{C}(\mathcal{C}(U))$, commutation up to a phase of G with all the matrices of the form
$\bigoplus_{m=1}^{r} \mathrm{Diag}(V_j^{(m)})_{j\in[\![0,b-1]\!]}$, with $(V_i^{(m)})_{i\in[\![0,b-1]\!]} \in \mathrm{UnitOp}(\mathbb{C}^{d_m})^b$ imposes that G is of the form:
$\bigoplus_{m=1}^{r} \mathrm{Diag}(e^{i\psi_i^{(m)}}\mathbb{1}_{d_m})_{i\in[\![0,b-1]\!]} = \bigoplus_{m=1}^{r} \mathrm{Diag}(e^{i\psi_i^{(m)}})_{i\in[\![0,b-1]\!]} \otimes \mathbb{1}_{d_m}$.

Then consider commutation up to phase ψ with $A = \bigoplus_{m=1}^{r} \Sigma_1^{(b)} \otimes \mathbb{1}_{d_m}$. In a given m-th sector, denoting ψ_i for $\psi_i^{(m)}$ for short, we obtain:

$$\forall k \in [0, b-1], \psi_k = \psi_0 + k\psi,$$
$$\exists p \in [0, b-1], \psi = 2\pi\frac{p}{b}$$

Meaning that the phases are related together. We obtain that

$$\begin{aligned}\mathrm{Diag}(e^{i\psi_l^{(m)}}\mathbb{1}_{d_m})_{l\in[0,b-1]} &= e^{i\psi_0^{(m)}}\,\mathrm{Diag}(e^{i2\pi\frac{p\cdot l}{b}}\mathbb{1}_{d_m})_{l\in[0,b-1]}\\ &= e^{i\psi_0^{(m)}}\left(\Sigma_3^{(b)}\right)^p \otimes \mathbb{1}_{d_m}\end{aligned}$$

Again, p is independent of the sector m, but all values in $[\![0, b-1]\!]$ are allowed. Hence, the general form is, as given by Eq. 24b, the following:

$$\mathcal{C}(\mathcal{C}(U)) = \left\{ \left(\oplus_{m=1}^{r} \left(\Sigma_3^{(b)}\right)^k \otimes e^{i\theta_m}\mathbb{1}_{d_m}\right), \middle|\, 0 \le k \le b-1,\, \theta_1, \dots, \theta_r \in \mathcal{S} \right\}. \qquad \square$$

B Proof of Theorem 4.1

Proof. Consider a unitary channel $\hat{U}_{SE} \in \mathrm{Act}(\mathrm{A}, \mathrm{SE})$. As the unitary U_{SE} acts non trivially on SE and commutes up to a phase with unitaries of the form $V_S \otimes \mathbb{1}_E$, $V_S \in \mathcal{C}(U)$, it's natural to look at the decomposition of its action on S and E. This is given by the Schmidt operator decomposition, which can be seen as a particular case of the Schmidt decomposition for pure state [29]. See Ref. [30] for further details. Let's introduce notations for the Schmidt operator decomposition of $U_{SE} \in \mathrm{UnitOp}(\mathcal{H}_S \otimes \mathcal{H}_E)$. Remind that $d = \dim(\mathcal{H}_S)$, and call $d_E = \dim(\mathcal{H}_E)$. Then, there exists a positive integer $\Omega(U_{SE}) \le \min(d, d_E)^2$ called the Schmidt operator rank, two sets of Hilbert-Schmidt orthogonal operators $\{f_S^i\}_{i=1}^{\Omega(U_{SE})} \subset \mathrm{Lin}(\mathcal{H}_S)$ and $\{N_E^i\}_{i=1}^{\Omega(U_{SE})} \subset \mathrm{Lin}(\mathcal{H}_E)$ such that:

$$U_{SE} = \sum_{i=1}^{\Omega(U_{SE})} f_S^i \otimes N_E^i \tag{29}$$

Let's now use this convenient rewriting to characterize the elements f_S^i. Restrictions comes from the fact that $\hat{U}_{SE} \in \mathrm{Act}(\mathrm{A}, \mathrm{SE})$.

Firstly, as $\hat{U}_{SE} \in \mathrm{Act}(\mathrm{A}, \mathrm{SE})$, take $\hat{V}_S \in \mathrm{Act}(\mathrm{B}, \mathrm{S})$, and remind that $\mathrm{Act}(\mathrm{B}, \mathrm{S})$ is here specified by unitaries in $\mathcal{C}(U)$. We have:

$$[\hat{U}_{SE}, \hat{V}_S \otimes \mathbb{1}_E] = 0 \iff \exists \phi \in \mathcal{S} \;/\; U_{SE} \cdot V_S \otimes \mathbb{1}_E = e^{i\phi} V_S \otimes \mathbb{1}_E \cdot U_{SE}$$

Expanding U_{SE} according to Eq.29 and identifying i-th elements thanks to orthogonality of E components gives:

$$\forall V_S \in \mathcal{C}(U), \exists \phi \in \mathcal{S} / \forall i \in [\![1, \Omega(U_{SE})]\!], f_S^i V_S = e^{i\phi} V_S f_S^i \tag{30}$$

Hence elements f_S^i must commute up to a i independent phase with all unitaries $V_S \in \mathcal{C}(U)$. This phase can depends on V_S. This condition allows us to derive the explicit form of the elements $\{f_S^i\}$. Let's recall 24a for convenience

$$\mathcal{C}(U) = \Bigg\{ \bigoplus_{m=1}^{r} \left(\mathrm{Diag}\left(V_0^{(m)}, \ldots, V_{b-1}^{(m)}\right) \cdot \left(\left(\Sigma_1^{(b)}\right)^l \otimes \mathbb{1}_{d_m} \right) \right), \\ \Bigg| V_0^{(m)}, \ldots, V_{b-1}^{(m)} \in \mathrm{UnitOp}(\mathbb{C}^{d_m}) \forall m, \quad 0 \leq l \leq b-1 \Bigg\},$$

1. Commutation up to a phase of f_S^i with all matrices in $\mathcal{C}(U)$ for which $l = 0$, ie of the form
$\bigoplus_{m=1}^r \mathrm{Diag}((V_k^{(m)})_{k\in[0,b-1]})$, with $(V_k^{(m)})_{k\in[0,b-1]} \in \mathrm{UnitOp}(\mathbb{C}^{d_m})^b$, imposes that f_S^i has this structure: $\bigoplus_{m=1}^r D^{m,i} \otimes \mathbb{1}_{d_m}$, with $D^{m,i} = \mathrm{Diag}(\lambda^{(i,m,n)})_{n\in[0,b-1]}$, where $(\lambda^{(i,m,n)})_{n\in[0,b-1];m\in[1,r]}$ a set of any scalars characterizing f_S^i.

2. Commutation up to phase ϕ with $\bigoplus_{m=1}^r \Sigma_1^{(b)} \otimes \mathbb{1}_{d_m}$ implies $\forall m \in [1,r], D^{m,i}\Sigma_1^{(b)} = e^{i\phi}\Sigma_1^{(b)} D^{m,i}$. Hence,

$$\forall n \in [0, b-1], \lambda^{(i,m,n)} = \lambda^{(i,m,0)} e^{in\phi}$$
$$\exists k \in [0, b-1], \phi = 2\pi\frac{k}{b}$$

Let $\lambda^{(i,m)} := \lambda^{(i,m,0)}$. We have

$$\begin{aligned} D^{m,i} &= \mathrm{Diag}(\lambda^{(i,m,n)})_{n\in[0,b-1]} \\ &= \lambda^{(i,m,0)}\, \mathrm{Diag}(e^{i2\pi\frac{k\cdot n}{b}})_{n\in[0,b-1]} \\ &= \lambda^{(i,m)} (\Sigma_3^{(b)})^k \end{aligned}$$

k has no dependency on the $m-th$ sector, as it comes from the global phase ϕ, but all values in $[0, b-1]$ are allowed. Hence the general form for f_S^i is:

$$f_S^i = \bigoplus_{m=1}^{r} (\Sigma_3^{(b)})^k \otimes \lambda^{(i,m)} \mathbb{1}_{d_m} \tag{31}$$

Reinserting it back in U_{SE} gives:

$$\begin{aligned} U_{SE} &= \sum_{i=1}^{\Omega(U_{SE})} f_S^i \otimes N_E^i \\ &= \sum_{i=1}^{\Omega(U_{SE})} \left(\bigoplus_{m=1}^{r} (\Sigma_3^{(b)})^k \otimes \lambda^{(i,m)} \mathbb{1}_{d_m}\right) \otimes N_E^i \end{aligned}$$

Now remind the following properties on the $\otimes$ product: for $A \in M_{m\times n}(\mathbb{K}), B \in M_{p\times q}(\mathbb{K}), C \in M_{r\times s}(\mathbb{K})$:

1. $\otimes$ on the right is distributive with respect to $\oplus$: $(A \oplus B) \otimes C = (A \otimes C) \oplus (B \otimes C)$.[10]
2. Associativity : $(A \otimes B) \otimes C = A \otimes (B \otimes C)$

[10]However it's not distributive on the left. $A \otimes (B \oplus C) \neq (A \otimes B) \oplus (A \otimes C)$. Yet, there exists a canonical isomorphism such that $A \otimes (B \oplus C) \cong (A \otimes B) \oplus (A \otimes C)$.

Using those and linearity of the $\otimes$ product gives:

$$\begin{aligned} U_{SE} &= \sum_{i=1}^{\Omega(U_{SE})} \bigoplus_{m=1}^{r} ((\Sigma_3^{(b)})^k \otimes \lambda^{(i,m)} \mathbb{1}_{d_m} \otimes N_E^i) \\ &= \sum_{i=1}^{\Omega(U_{SE})} \bigoplus_{m=1}^{r} (\Sigma_3^{(b)})^k \otimes (\mathbb{1}_{d_m} \otimes \lambda^{(i,m)} N_E^i) \\ &= \bigoplus_{m=1}^{r} (\Sigma_3^{(b)})^k \otimes \mathbb{1}_{d_m} \otimes (\sum_{i=1}^{\Omega(U_{SE})} \lambda^{(i,m)} N_E^i) \end{aligned}$$

Let's now call $U_m = \sum_{i=1}^{\Omega(U_{SE})} \lambda^{(i,m)} N_E^i$. Imposing unitarity of U_{SE} gives:

$$U_{SE} U_{SE}^\dagger = U_{SE}^\dagger U_{SE} = \mathbb{1}_{SE}$$

$$\bigoplus_{m=1}^{r} \mathbb{1}_b \otimes \mathbb{1}_{d_m} \otimes U_m U_m^\dagger = \bigoplus_{m=1}^{r} \mathbb{1}_b \otimes \mathbb{1}_{d_m} \otimes U_m^\dagger U_m = \mathbb{1}_{SE} = \bigoplus_{m=1}^{r} \mathbb{1}_b \otimes \mathbb{1}_{d_m} \otimes \mathbb{1}_E$$

Direct identification imposes:

$$\forall m \in [\![1, r]\!], U_m \in \mathrm{UnitOp}(\mathcal{H}_E) \tag{32}$$

Altogether U_{SE} is given by:

$$U_{SE} = \bigoplus_{m=1}^{r} (\Sigma_3^{(b)})^k \otimes \mathbb{1}_{d_m} \otimes (U_m)_E \tag{33}$$

with $k \in [\![0, b-1]\!]$, and $\forall m \in \{1, \ldots, r\}, U_m \in \mathrm{UnitOp}(\mathcal{H}_E)$, which is the proposition.

$\square$

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