ENTANGLEMENT IN THE DICKE SUBSPACE

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Introduction

Entanglement lies at the heart of quantum theory as a fundamental form of non-classicality and also as a key resource in many applications across many areas, from teleportation and cryptography to metrology and measurement-based computation [HHHH09]. Yet deciding whether a given mixed state is entangled is computationally intractable in general (NP-hard) [Gur03, Gha10]. In the multipartite setting, *genuine multipartite entanglement* underpins protocols such as secret sharing and multiparty teleportation [HBB99, CZG06].

A standard approach to bypass the hardness of the entanglement detection problem is to use *necessary separability criteria*: violating any such test certifies entanglement. The most prominent example is the *Peres–Horodecki* (PPT) criterion [Per96, Hor97]. While PPT detects entanglement in many regimes, there exist PPT-entangled (that are also bound entangled [Smo01]) states that are nevertheless undistillable and central to several open problems in entanglement theory [DSS+00].

We focus on bosonic systems of identical particles, where permutation symmetry reduces the effective degrees of freedom and can sharpen entanglement tests. Writing P_π for the permutation operator associated with $\pi \in S_n$, bosonic states are supported on the symmetric subspace $\vee^n \mathbb{C}^d := \{\psi \in (\mathbb{C}^d)^{\otimes n} \mid P_\pi \psi = \psi, \forall \pi \in S_n\}$. These symmetry constraints make bosonic entanglement both experimentally relevant and mathematically tractable, motivating a systematic study of PPT and PPT entanglement in bosonic (and, in particular, diagonally symmetric) subspaces [MMRRP+25].

Bosonic Entanglement and Extendibility

In general, bosonic mixed states of n qudits are either fully separable, or have entanglement depth n (i.e. they cannot be written as a mixture of states where at most $n-1$ parties are entangled). Moreover, any fully separable bosonic state is of the form $\sum_{k=1}^{K} p_k \, |x_k\rangle\langle x_k|^{\otimes n}$. This separable bosonic state also satisfies the PPT criterion applied to *any* bipartition of the system. It is also known that for bosonic systems of 2 or 3 qubits, PPT provides a necessary and sufficient condition for separability [ESBL02]. For the remaining cases, entangled states that satisfy PPT across each bipartition have been constructed [ATSL12]. For symmetric states, some special entanglement criteria have been studied [ESBL02].

Bosonic Extendibility. A bosonic state $\rho \in \mathcal{B}((\mathbb{C}^d)^{\otimes n})$ is called *r-PPT extendible* if there exists a bosonic state $\tilde{\rho} \in \mathcal{B}((\mathbb{C}^d)^{\otimes (n+r)})$ such that (i) $\mathrm{Tr}_{[r]}(\tilde{\rho}) = \rho$, and (ii) for every partial transpose on up to r additional subsystems, the resulting operator is positive semidefinite (in particular, $\tilde{\rho} \succeq 0$). This notion is tailored to bosonic states and differs from standard DPS extendibility.

The nested sets of k-extendible states provide systematic outer approximations of separable states and are connected to quantum de Finetti theorems [CKMR07, KR05] and other applications in multipartite entanglement theory [BC12]. In particular, a separable bosonic state $\sum_{k=1}^{K} p_k \, |x_k\rangle\langle x_k|^{\otimes n}$ admits, for every r, the natural extension $\sum_{k=1}^{K} p_k \, |x_k\rangle\langle x_k|^{\otimes (n+r)}$ whose n-body marginals coincide with ρ. Extendibility also arises in the contexts of security of quantum key distribution [MCL06], capacities of quantum channels [NH09].

Entanglement Witnesses. Since all the aforementioned families of quantum states are convex, duality plays a crucial role in entanglement theory. The set of bosonic separable states is dual to the set of entanglement witnesses that are defined as self-adjoint $W \in \mathcal{B}((\mathbb{C}^d)^{\otimes n})$ satisfying $\forall x \in \mathbb{C}^d, \langle x^{\otimes n} | W | x^{\otimes n} \rangle \geq 0$. The set of *decomposable witnesses* forms the dual set for PPT states, and any such witness is incapable of detecting PPT entangled states.

Sum of Squares and Positivstellensätze

Deciding whether a polynomial is globally non-negative is a central and computationally difficult problem in optimization theory. A tractable sufficient certificate is *sum-of-squares* (SOS): if $p = \sum_{k=1}^{K} f_k^2$, then $p(x) \geq 0$ for all $x \in \mathbb{R}^d$. Hilbert famously characterized the exceptional cases where non-negativity coincides with SOS [Hil88];

outside these regimes, there exist positive polynomials that are not SOS (e.g., the Motzkin/Robinson examples [Mot67, Rob73]). A canonical example in three variables is

$$p(x,y,z) = x^2y^4 + x^4y^2 + z^6 - 3x^2y^2z^2,$$

which satisfies $p \geq 0$ but is not SOS. To systematically tighten SOS certificates, one can use Positivstellensätze: for instance, test whether $p(x)\|x\|^{2r}$ is SOS for some r. This yields a semidefinite hierarchy for polynomial positivity; by a theorem of Reznick, every positive definite form is detected at some finite level [Rez95].

Main Results

The *qudit Dicke states*, $|D_\alpha\rangle \in (\mathbb{C}^d)^{\otimes n}$ are bosonic pure states defined as the equal superposition of the computational states with Hamming weight (denoted by h) $\alpha \in \mathbb{N}^d$:

$$|D_\alpha\rangle := \frac{1}{\sqrt{\binom{n}{\alpha}}} \sum_{i\in[d]^n \,:\, h(i)=\alpha} |i_1 i_2 \cdots i_n\rangle .$$

The Hamming weight of the multi-index i is a vector counting how many times the letter $x \in [d]$ appears in i. Dicke states, by construction, enjoy two different symmetries: a *local* diagonal-unitary symmetry and *global* permutation invariance,

$$\mathrm{diag}(e^{\mathrm{i}\varphi_1}, \ldots, e^{\mathrm{i}\varphi_d})^{\otimes n} |D_\alpha\rangle = e^{i\varphi\cdot\alpha} |D_\alpha\rangle \quad \text{and} \quad P_\pi |D_\alpha\rangle = |D_\alpha\rangle .$$

We consider multipartite mixed states $\rho \in \mathcal{B}((\mathbb{C}^d)^{\otimes n})$ satisfying both symmetries,

$$U^{\otimes n}\rho\,\overline{U}^{\otimes n} = \rho \quad \text{and} \quad P_\pi \rho P_\pi^\dagger = \rho. \tag{1}$$

This class of states, that we call *diagonally symmetric*, is precisely the set of states that are diagonal in the Dicke basis:

$$\rho = \sum_{\alpha\in\mathbb{N}^d, |\alpha|=n} p_\alpha |D_\alpha\rangle\!\langle D_\alpha| , \quad p_\alpha \geq 0, \sum_{\alpha\in\mathbb{N}^d, |\alpha|=n} p_\alpha = 1.$$

States of this form are referred to as the *diagonally symmetric states* (or *mixtures of Dicke states*, abbreviated DS). These form a crucial subclass of bosonic states and have received much attention in the bipartite case [TAQ+18, Yu16]. For bipartite *mixtures of Dicke states*, and the separability and PPT properties have a one-to-one correspondence to *completely positive matrices* and *doubly non-negative matrices*. As these two sets coincide for $d \leq 4$, the PPT and separability notions coincide. Surprisingly, also, in any number of qubits, the situation is identical; there is no PPT entanglement. Recently, a strong conjecture was proposed in the recent work [RPAR+25] that for a multi-partite mixture of Dicke states in $d = 3$ and $d = 4$, all separable states are PPT. Overall, the situation seemed far from understood and a rigorous and well-understood mathematical framework to explicitly describe PPT and indecomposable maps was lacking.

In this work, we provide a comprehensive framework for studying *entanglement in mixtures of Dicke states*, by showing that the entanglement properties of these states can be translated into well-known objects in tensor analysis, semi-algebraic geometry, and are intimately connected to some tools in polynomial optimization theory.

Tensor/Polynomial Parametrizations. The n-partite matrices $X \in \mathcal{M}_d^{\otimes n}$ satisfying the the two symmetries in Eq. (1) are called *diagonally symmetric* matrices. For the linear subspace of self-adjoint matrices, we introduce the following symmetric tensor-based parametrization, $Q : \mathrm{Herm}[\mathcal{M}_d^{\otimes n}] \to \vee^n\mathbb{R}^d$ defined as $Q[X]_i := \langle i|X|i\rangle$. Moreover, we also define an associated *real* polynomial $p_X(x) := \langle x^{\otimes n}|X|x^{\otimes n}\rangle$. If X is a quantum state, the tensor satisfies $\forall i \in [d]^n$, $Q_i[X] \geq 0$ and the normalization $\sum_{i\in[d]^n} Q_i[X] = 1$.

Entanglement Properties. We find an exact correspondence between the entanglement properties of mixtures of Dicke states and properties of tensors. The first main result concerns the *separability* of these states: a mixture of Dicke state is separable if and only if the tensor Q is *completely positive* (denoted as $\mathrm{CP}_d^{(n)}$), i.e. it can be written as a convex sum of tensors of the form $v^{\otimes n}$ for v with non-negative entries ($v \geq 0$). The notion of completely positive tensors generalizes the one of completely positive matrices, which is already connected to bipartite entanglement theory [Yu16, TAQ+18, GNP25].

For the PPT property, we show that partial-transpose positivity in the DS subspace is equivalent to positive semidefiniteness of a family of matrices obtained from $Q[X]$ by taking *even slices* (Fig. 1, right) and applying a *balanced flattening* (Fig. 1, left). We call this family the *slice-flattenings* of $Q[X]$ and denote it by $\mathrm{SF}(Q[X])$.

Moreover, on the dual side, the associated polynomial is non-negative, $p_X(x) \geq 0$ if and only if X is an entanglement witness. We prove that an entanglement witness is *decomposable* if and only if $p_X(x)$ is a sum-of-squares polynomial.

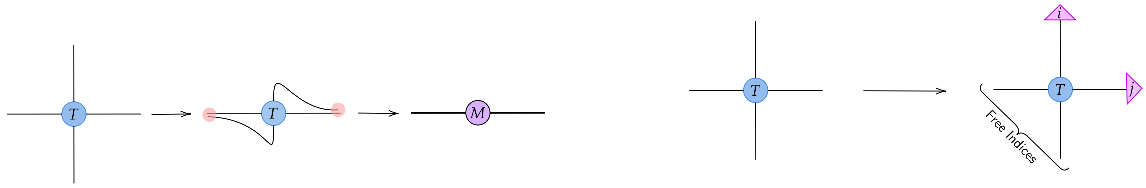

FIGURE 1. The left figure represents a balanced flattening of a tensor $T \in \vee^4 \mathbb{R}^d$ and the right figure, (i,j)-slice of the tensor $T \in \vee^4 \mathbb{R}^d$ to obtain a tensor in $T \in \vee^2 \mathbb{R}^d$

$$\begin{array}{ccccccc}
Q[X] \in \mathrm{CP}_d^{(n)} & \subseteq & \mathrm{SF}(Q[X]) \succeq 0 & \subseteq & p_X \in \mathrm{SOS} & \subseteq & p_X \geq 0 \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
X \in \mathrm{Sep}_d^{(n)} & \subseteq & X \in \mathrm{PPT}_d^{(n)} & \subseteq & X \in \mathrm{Dec}_d^{(n)} & \subseteq & X \in \mathrm{EntW}_d^{(n)}
\end{array}$$

FIGURE 2. The different notions of positivity for Hermitian matrices in the DS subspace (bottom row) and the corresponding class of real symmetric tensors/polynomials (top row).

Largest PPT is enough. We show that for mixtures of Dicke states, PPT across a bipartition of size $(k, n-k)$ already implies PPT across every smaller bipartition $(1, n-1), \ldots, (k, n-k)$. In particular, it suffices to test PPT on the *largest* bipartition, resolving the conjecture of [RPAR+25]. In the tensor framework, PPT for the bipartition $(k, n-k)$ is equivalent to positive semidefiniteness of certain slice-flattenings (SF) of sizes larger than $n-2k$.

PPT entanglement. Using the tensor/polynomial correspondence, we construct *indecomposable bosonic* witnesses for all $d \geq 3$ and $n \geq 3$. For $d = 3$ and $n = 3$, we introduce a witness X such that

$$\langle x^{\otimes n} | X | x^{\otimes n} \rangle = x_1^4 x_2^2 + x_1^2 x_2^4 + x_3^6 - 3x_1^2 x_2^2 x_3^2.$$

This polynomial is nonnegative but not SOS, hence X is an entanglement witness that is *indecomposable*. We provide analogous families for all $d \geq 3$ and $n \geq 3$ and, dually, detect PPT entanglement in these regimes, giving counterexamples to the conjecture in [RPAR+25].

	$n=2$	$n=3$	$n \geq 4$
$d=2$	No	No	No
$d=3$	No	Yes	Yes
$d=4$	No	Yes	Yes
$d \geq 5$	Yes	Yes	Yes

[TAQ+18, Yu16]
This article
[RPAR+25, GNP25]

TABLE 1. Is there PPT entanglement in mixtures of Dicke states of n parties and local dimension d?

Bosonic Extendibility. We first show that a mixture of Dicke states has a bosonic PPT r-extension if and only if it admits such an extension that is itself a mixture of Dicke states. We derive an exact correspondence between the polynomial *SOS hierarchy* and PPT extendibility witnesses in the diagonally symmetric subspace. In particular, a matrix X in the diagonally symmetric subspace detects PPT r-extendibility if the real polynomial $p_X(x)\|x\|^{2r}$ is SOS.

On the dual side, we give a polynomial-sized SDP for the extendibility hierarchy of mixtures of Dicke states: checking r-bosonic PPT extendibility of an n-partite bosonic state reduces to testing positivity of a matrix of maximum size $\binom{\lfloor \frac{n+r}{2} \rfloor - d - 1}{d-1} \approx \frac{(\lfloor \frac{n+r}{2} \rfloor - 1)^d}{(d-1)!}$. This extends the analogous bipartite results of [GNP25, BL25].

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