The perturbative method for quantum correlations

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INTRODUCTION

The characterization of the set of quantum correlations $\mathcal{Q}$—the collection of probability distributions achievable by space-like separated parties sharing a quantum state—remains a foundational challenge in quantum information theory. Since Tsirelson's seminal work [1], the study of nonlocality — in several scenarios beyond the Bell case — has largely relied on convex geometric and algebraic methods [2–4], and hierarchies of semidefinite programs stemming from operator algebras [5–8]. However, despite having been thoroughly studied in the bipartite case [9], the geometric structure of $\mathcal{Q}$ in the multipartite setting remains unexplored.

In this paper, we propose a perturbative method rooted in the Lie group structure of unitary operations. We analyze the response of Bell functionals evaluations under infinitesimal unitary perturbations of quantum strategies. This dynamical perspective allows us to simplify the optimization problem around specific points of interest, namely the local deterministic correlations which form the vertices of the local polytope $\mathcal{L}$.

Our approach yields three primary contributions:

(i) A **dimension reduction theorem** showing that the second-order variation of a Bell functional around a deterministic point in the $(n, 2, d)$ scenario is governed by effective games with reduced output dimension $(d - 1)$.

(ii) A proof that in the $(n, 2, 2)$ scenario, the boundary of $\mathcal{Q}$ is flat around every local deterministic point, meaning — in a specific way stated in the paper — that there is no quantum extremal point in their immediate neighborhood.

(iii) A proposed pathway to resolve an open problem formulated by Gisin on the existence of Bell functionals requiring non-projective POVMs for some maximum local dimension [10] (see number 10).

Our result also highlights the fact that the dimension of a quantum learning ansatz is a critical resource for learning: even in the $(n, 2, 2)$ setting, where we know that qubit strategies suffice to maximize any Bell functional, we show that one cannot learn them efficiently using a quantum variational model, especially when using a *warm start* from the best classical correlation.

UNITARY PERTURBATIONS OF BELL FUNCTIONALS

We consider the Bell scenario denoted by $(n, 2, d)$, involving n parties, each choosing between 2 measurement settings with d possible outcomes. We index settings and outcomes with $\mathbb{Z}_2$ and $\mathbb{Z}_d$, respectively. A correlation p is a vector in $\mathbb{R}^{d^n \times 2^n}$ with components $p(a|x)$, where $x \in \mathbb{Z}_2^n$ and $a \in \mathbb{Z}_d^n$. A Bell functional $\vec{\beta}$ defines a linear form on correlations, representing the expected gain in a CHSH-type game [11].

We denote quantum strategies by $\Sigma = ((M^{(i)}_{x_i})_{1 \leq i \leq n, x_i \in \mathbb{Z}_2})$, where ρ is a shared state on $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ and $M^{(i)}_{x_i}$ are POVMs. We denote $P(\Sigma)$ the correlaion achieved by Σ. We also define the set $\mathcal{Q}_P(D)$ of correlations achievable by strategies which use projective measurements and a state ρ made up of n qu-D-its.

This set is especially important in the $(n, 2, 2)$ setting : Masanes' theorem states that in the $(n, 2, 2)$ scenario, the extremal points of $\mathcal{Q}$ are contained in $\mathcal{Q}_P(2)$: $\mathrm{Ext}(\mathcal{Q}) \subseteq \mathcal{Q}_P(2)$ [2].

We consider the unitary orbit of a strategy $\Sigma = (\rho, M)$ Let U_ρ act on the state and U_{i,x_i} act on the local measurement operators. We argue that one can fix the POVMs for input 0 and consider perturbations only on the second measurement (input 1) and the state. Let K_ρ and K_i be skew-Hermitian generators such that the unitaries are e^{-K_ρ} and e^{K_i}. We denote $\vec{K} = (K_\rho, K_1, \ldots, K_n)$.

We propose to evolve in the Heisenberg picture the *conditional Bell operator* for a fixed input string x:

$$B_x(M) = \sum_{a\in\mathbb{Z}_d^n} \beta_{a,x} M^{(1)}_{a_1|x_1} \otimes \cdots \otimes M^{(n)}_{a_n|x_n}.$$

The total Bell operator is $B(M) = \sum_x B_x(M)$.

Proposition 1 (Second-Order Response). *Let $\vec{K} = (K_\rho, K_1, \ldots, K_n)$. The evaluation of the Bell functional on the perturbed strategy admits the following second-order expansion:*

$$\vec{\beta} \cdot P(U_{\vec{K}} \cdot \Sigma) = \vec{\beta} \cdot P(\Sigma) + Tr\left(\rho \sum_x [K_{tot}, B_x]\right) + \frac{1}{2} Tr\left(\rho \sum_x \left([K_{tot}, B_x]_2 + [[x\cdot\vec{K}, K_\rho], B_x]\right)\right) + o(\|\vec{K}\|^2),$$

where $K_{tot} = K_\rho + x\cdot\vec{K}$ with $x\cdot\vec{K} = \sum_i x_i K_i$, and $[A,B]_2 = [A,[A,B]]$.

Note that a necessary condition for optimality is that the first order term vanishes for all $\vec{K}$, which yields a concise necessary optimality condition, reminescent of the work of Araújo et. al [12] using a different method.

For a quantum strategy realizing a *local deterministic* correlation $p_{\tilde{a}}$, the first-order term vanishes identically. The local optimality is thus determined entirely by the second-order term.

DIMENSION REDUCTION VIA SUBSET GAMES

The core technical result of this work is the simplification of the second-order term when the base strategy is a local deterministic point $p_{\tilde{a}}$. A local deterministic correlation $p_{\tilde{a}}$, defined by a family of functions $\tilde{a}_i$ for each players mapping inputs x_i to outputs $\tilde{a}_i(x_i)$, is a *one-flip maximum* if changing the output of a single party for a given input does not increase the Bell value. For QP strategies, we show that the effect of measurement evolution is limited.

Lemma 1 (Limited effect of measurement evolution (informal)). *When unitarily evolving a strategy realizing $p_{\tilde{a}}$, the measurement evolution contributes only on outcomes differing from $\tilde{a}(x)$ on one site. Moreover, if $D = d$ and $p_{\tilde{a}}$ is a one-flip maximum, this contribution is negative.*

A positive contribution can thereby only come from state evolution. We show that the second-order variation under state evolution depends on an operator $B_2(\Pi)$ which admits a direct sum decomposition based on subsets of parties $S \subseteq \{1,\ldots,n\}$ with $|S| \geq 2$.

For a subset S, we define a reduced Bell operator B_S (a *subset game*) with $d-1$ outcomes, corresponding to deviations from the deterministic strategy restricted to S.

This leads to our main criterion for local optimality:

Corollary 1 (Local optimality criterion). *Suppose $p_{\tilde{a}}$ is a strict one-flip maximum. If for all subsets S with $|S| \geq 2$, the subset game B_S has strictly negative value negative value, then $p_{\tilde{a}}$ is a local optimum in $\mathcal{Q}_P(d)$. Moreover if some subset game has strictly positive value, then $p_{\tilde{a}}$ is not a local optimum in $\mathcal{Q}_P(d)$.*

GEOMETRIC IMPLICATIONS: FLATNESS OF $\mathcal{Q}$ IN THE $(N,2,2)$ SCENARIO

Applying the dimension reduction theorem to the binary outcome case ($d = 2$) yields a striking geometric consequence.

In the $(n,2,2)$ scenario, the subset games become trivial, because the output dimension is $d-1 = 1$. We prove that if $p_{\tilde{a}}$ is a strict classical maximum, the associated scalars for all subset games are strictly negative, implying $p_{\tilde{a}}$ is a local optimum in $\mathcal{Q}_P(2)$

Since Masanes' theorem states that $\mathrm{Ext}(\mathcal{Q}) \subseteq \mathcal{Q}_P(2)$, and we have shown that no point in $\mathcal{Q}_P(2)$ can exceed the value of a strict classical maximum in its immediate neighborhood, it follows that in every direction around $p_{\tilde{a}}$, one does not encounter extremal points on the border of $\mathcal{Q}$, meaning the border is flat. To our knowledge, this is the first mathematical result on the geometry of the multipartite quantum set.

DISCUSSION AND OPEN PROBLEMS

Ansatz Dimension as a Learning Resource. Our results underscore the role of the *ansatz dimension* in variational quantum algorithms. Even in the $(n, 2, 2)$ case where the optimal violation is known to be achievable with qubits ($D = 2$), we show that a variational quantum circuit — using parametrized shared state preparation and local measurements — with local dimension 2 initialized near a classical point may fail to escape the local optimum. The convexity of $\mathcal{Q}$ suggests that increasing the local dimension of the ansatz can eliminate these local optima. This positions the dimension of the variational circuit as a resource for learning, independently on the target dimension.

New games exhibiting quantum gap. The local optimality criterion via subset games (Corollary 1) suggests a way to construct new games with a quantum gap. It suffices to consider a $(k, 2, d-1)$ game B with strictly positive value, and construct a game for which one of the subset games at the optimal classical correlation $p_{\tilde{a}}$ is B. Then, $p_{\tilde{a}}$ is not a local optimum for B in $\mathcal{Q}_P(d)$, and a fortiori the game has a quantum gap. It would be interesting to investigate if this construction can succeed and yield any nontrivial new type of game with a quantum gap.

POVMs vs. PVMs This work sheds new light on the open question posed by Gisin: does there exist a correlation achievable by POVMs of dimension D that is not in $\mathcal{Q}_P(D)$? Our analysis reveals a structural difference between PVMs and POVMs under perturbation. For PVMs, the measurement evolution terms in the second-order expansion vanish or are strictly negative due to orthogonality constraints. For general POVMs, non-orthogonal elements allow these terms to potentially contribute positively. We suggest that constructing a Bell functional where the subset games are negative but POVM perturbations can yield a positive gain would prove $\mathcal{Q}_P(D) \subsetneq \mathcal{Q}(D)$. Our perturbative machinery provides the explicit conditions required to search for such a functional.

CONCLUSION

By analyzing the dynamics of quantum correlations under unitary perturbations of strategies, we have uncovered new properties of the quantum correlation set $\mathcal{Q}$. The reduction of multipartite Bell optimization to subset games provides a powerful tool for analyzing local optimality. Our proof of the flatness of $\mathcal{Q}$ around deterministic points in the $(n, 2, 2)$ scenario is a first step towards understanding the geometry of the multipartite quantum set. Furthermore, the distinction we draw between projective and general POVM perturbations offers a concrete roadmap for resolving Gisin's problem.

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The set of quantum correlations, often denoted $\mathcal{Q}$, is the collection of all possible probability distributions on measurement outcomes achievable by space-like separated parties sharing a quantum state. Since the original work of Tsirelson [1], this set has mainly been studied through the means algebraic and convex geometry techniques.

We introduce a perturbative method using only common Lie-theoretic tools for the unitary group to analyze the response of the evaluations of Bell functionals under infinitesimal unitary perturbations of quantum strategies. Our main result shows that any $(n, 2, d)$ Bell operator near a classical deterministic correlation p_0 decomposes into a direct sum of $(k, 2, d-1)$ Bell operators, which we call *subset games*. We then derive three key insights: (1) in the $(n, 2, 2)$ case, if p_0 is classically optimal, it remains locally optimal even among 2-dimensional quantum strategies, implying in turn that the boundary of $\mathcal{Q}$ is flat around classical deterministic points; (2) it suggests a proof strategy for Gisin's open problem on correlations in $\mathcal{Q}(D)$ unattainable by projective strategies of the same dimension; and (3) it establishes the ansatz dimension as a critical resource for learning, even when the optimal solution admits a low-dimensional representation.

INTRODUCTION

Quantum correlations and nonlocality The set of quantum correlations $\mathcal{Q}$ lies at the core of the study of quantum nonlocality. It is defined as the collection of measurement statistics — represented as high-dimensional vectors — achievable by space-like separated parties sharing a quantum state. First formalized by Tsirelson [1], it is strictly larger than its classical counterpart $\mathcal{L}$ of statistics explainable by local hidden variable models [2], demonstrating a quantum advantage in the correlations achievable by entangled parties. The comparative study of $\mathcal{Q}$ and $\mathcal{L}$ is mainly conducted through the means of violations of *Bell inequalities*, which are linear inequalities satisfied by every point in $\mathcal{L}$. The strict inclusion $\mathcal{L} \subsetneq \mathcal{Q}$ is a foundational result which enabled an experimental proof of the existence of entanglement and the nonlocal nature of quantum mechanics [3]. Although a full characterization of $\mathcal{Q}$ has been proven to be impossible (because $\mathrm{MIP}^* = \mathrm{RE}$ [4]), its understanding remains a central challenge in quantum foundations.

Nonlocal games A nonlocal game is an interactive protocol between a referee and a group of players. The referee sends each player a question, the players respond without communicating, and the referee decides whether they win based on their collective answers. The game is called nonlocal if the players can achieve a higher winning probability than what classical physics allows, by using an entangled quantum state, and deciding their answer using a measurement depending on the question. The zoology of nonlocal games provides a rich framework to explore the quantum-classical divide in correlations. For instance:

- The CHSH game [5] is the simplest Bell non-locality scenario, involving two parties, two measurement settings, and two outcomes. It demonstrates that quantum mechanics allows for correlations that violate classical bounds, with a maximal violation known as the Tsirelson bound.
- Multipartite games, such as the GHZ game [6], involve three or more parties and reveal stronger forms of non-locality. These games are essential for understanding the scalability of quantum advantage in multi-party settings.
- Graph-based games extend the framework to more complex interaction structures, where parties are arranged according to a graph, and correlations are constrained by the graph's topology. These games are particularly relevant for studying quantum networks and distributed quantum computing [7, 8].

Previous work Since Tsirelson's seminal work [1], the study of nonlocality has primarily relied on convex geometric and algebraic methods [9–11], as well as hierarchies of semidefinite programs derived from operator algebras [12–15]. While the bipartite case has been thoroughly studied [10, 11, 16], the geometric structure of the set of quantum correlations $\mathcal{Q}$ particularly in multipartite settings—remains poorly understood. In this context, variational methods—though widely used in hybrid quantum-classical optimization [17, 18] and quantum machine learning [19, 20]—have only been marginally applied to the optimization of nonlocal strategies [21, 22], since SDP hierarchies are in practice very efficient in accomplishing this task [12, 15, 23]. Nevertheless, a variational approach offers a promising theoretical framework to probe the structure of $\mathcal{Q}$, particularly near classical deterministic points. The dynamical behavior of quantum strategies under perturbations—and its implications for learnability—has received little attention. Although [10] introduces a trigonometric parametrization of the extremal points that enables differentiation, this approach remains limited to the bipartite scenario.

Proposed approach and results In this paper, we propose a perturbative method using only common Lie-theoretic tools on the unitary group. We analyze the response of the evaluation of a Bell functional under infinitesimal unitary perturbations of a quantum strategy. This dynamical perspective allows us to simplify the optimization problem around specific points of interest, namely the local deterministic correlations which form the vertices of the local polytope $\mathcal{L}$.

Our approach yields three primary contributions:

1. A dimension reduction theorem showing that the second-order variation of the evaluation of a Bell functional around a deterministic point in the $(n, 2, d)$ scenario is governed by effective *subset games* with reduced output dimension $(d-1)$.

2. A proof that in the $(n, 2, 2)$ scenario, the boundary of $\mathcal{Q}$ is flat around every local deterministic point, meaning — in a specific way stated later – that there is no quantum extremal point in their immediate neighborhood. To our knowledge, this is the first mathematical statement on the geometry of the multipartite quantum set.

3. A proposed pathway to resolve an open problem formulated by Gisin on the existence, for a maximum allowed local dimension, of quantum correlations requiring non-projective POVMs that maximally violate some Bell inequality [24] (see question 10). This question is fundamental for quantum theory: it asks if POVMs, objects that naturally arise in the mathematical axiomatic of measurements, generalizing PVMs, are in a sense *relevant* for quantum nonlocality. It has been answered positively under the restriction that the parties share a Bell state [25]. Exhibiting a $(n, 2, d)$ Bell operator for which all the subset game operators at the classical optimum p_0 are negative, but p_0 is not a local optimum in $\mathcal{Q}(d)$ –which could be achieved using the perturbative machinery we propose –would prove the existence of such correlations, with no constraint on the shared state.

Our findings also position again nonlocal games as a compelling toy model for studying quantum variational learning (see [21]), revealing fundamental limitations even in seemingly simple scenarios. Specifically, in the $(n, 2, 2)$ setting—where qubit strategies are known to suffice for maximizing any Bell functional—we demonstrate that efficient learning via variational quantum models using qubits is hard, particularly when initialized from the optimal classical correlation. Thus, the dimension of the quantum ansatz itself is a critical resource for learning, independent of the minimal dimension required to represent the optimal solution.

PRELIMINARIES

Following the usual convention of [26], we denote the Bell scenario of n space-like separated parties with m different measurement settings each having d possible outcomes by (n, m, d). In this work, we focus on the $(n, 2, d)$ scenario, for arbitrary n, and index the setting and outcome with $\mathbb{Z}_2$ and $\mathbb{Z}_d$, respectively.

Correlations The setting–outcome statistics of the n parties can be compiled in a real vector:

$$p = (p(a|x))_{a\in\mathbb{Z}_d^n, x\in\mathbb{Z}_2^n} \in \mathbb{R}^{d^n\times 2^n}, \tag{1}$$

where a and x are the strings of size n denoting respectively the outcome and the setting of each of the n players. We refer to such a real vector as a *correlation*, or *correlation point*.

Bell functional A Bell functional [2] $\vec{\beta}$ is a linear form on correlations. These functionals typically emerge as the expected gain in so called *linear games*, where each of the n parties receive a bit of the input string x from a referee, sampled according to a probability distribution π. Each party then produces a bit of the output bitstring a follwing a strategy realizing an n-partite correlation p, and sends it back to the referee who computes a reward $g(a, x) \in \mathbb{R}$. The associated Bell functional for the expected gain is then:

$$\vec{\beta}\cdot p = \mathbb{E}_{x\sim\pi}\left[\mathbb{E}_{a\sim p(\cdot|x)}\left[g(a,x)\right]\right] = \sum_{a\in\mathbb{Z}_d^n, x\in\mathbb{Z}_2^n} \pi(x)p(a|x)g(a,x). \tag{2}$$

The coordinates of the vector representing β in the canonical basis of $\mathbb{R}^{d^n\times 2^n}$ are $\beta_{a,x} = \pi(x)g(a, x)$, which makes it clear that every Bell functional can be represented as the expected gain of a linear game. Optimizing a strategy for a game then amounts to maximizing a Bell functional over the set of correlations attainable with the given resources.

Local correlations, local strategies A correlation is *local* if it is explainable by classical information shared between the n parties (the hidden variable) prior to the choice of measurement setting [2, 27]. Formally, we say that $p \in \mathbb{R}^{d^n \times 2^n}$ is a local correlation if there exists a (non-necessarily unique) *local strategy*, denoted $\Sigma_{\mathcal{L}} = (\Lambda, \mu, (p_\lambda^{(i)}(\cdot|x_i))_{1\leq i\leq n, 1\leq \lambda\leq \Lambda, x_i\in\mathbb{Z}_2})$, such that:

$$p(a|x) = \mathcal{P}(\Sigma_{\mathcal{L}})(a|x) \coloneqq \sum_{\lambda=1}^{\Lambda} \mu(\lambda) p_\lambda^{(1)}(a_1|x_1) \dots p_\lambda^{(n)}(a_n|x_n), \tag{3}$$

where $\Lambda \in \mathbb{N}$, μ is a probability distribution on $\{1, \dots, \Lambda\}$, for all $1 \leq i \leq n, 1 \leq \lambda \leq \Lambda, x_i \in \mathbb{Z}_2$, $p_\lambda^{(i)}(\cdot|x_i)$ is a probability distribution on $\mathbb{Z}_d$, and subscripts on strings denote the index.

The correlation p is *local deterministic* if $\Lambda = 1$, and $p^{(i)}(a_i|x_i) = \delta_{a_i \tilde{a}_i(x_i)}$ for some $\tilde{a}_i(x_i) \in \mathbb{Z}_d$. Hence, a local deterministic correlation is uniquely characterized by a family of n gates $\tilde{a} = (\tilde{a}_i : \mathbb{Z}_2 \to \mathbb{Z}_d)_{1\leq i\leq n}$, which we call a *local deterministic strategy*. We denote $\mathcal{P}(\tilde{a})$ the local deterministic correlation associated to $\tilde{a}$.

We denote by $\mathcal{L} \subseteq \mathbb{R}^{d^n \times 2^n}$ the set of local correlations. It is well established that $\mathcal{L}$ is a convex set, the extremal points of which are the local deterministic correlations, and is thus a polytope [28]. The faces are the locus of maximization of the Bell functionals. The maximization of a Bell functional defines a linear inequality satisfied by every point in $\mathcal{L}$, called a Bell inequality [5].

Quantum correlations, quantum strategies Quantum correlations are the correlations that the n parties can achieve by performing local measurements on a shared quantum state. Formally, we say that $p \in \mathbb{R}^{d^n \times 2^n}$ is a quantum correlation if there exists a *quantum strategy* $\Sigma_{\mathcal{Q}} = ((\mathcal{H}_i)_{1\leq i\leq n}, \rho, (M_{x_i}^{(i)})_{1\leq i\leq n, x_i\in\mathbb{Z}_2})$, where, for $1 \leq i \leq n, x_i \in \mathbb{Z}_2$, $\mathcal{H}_i$ is a Hilbert space, $M_{x_i}^{(i)} = \{M_{0|x_i}^{(i)}, M_{d-1|x_i}^{(i)}\}$ is a POVM, and ρ is a quantum state on $\mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_n$, such that:

$$p(a|x) = \mathcal{P}(\Sigma_{\mathcal{Q}}) \coloneqq \mathrm{Tr}\left(\rho M_{a_1|x_1}^{(1)} \otimes \dots \otimes M_{a_n|x_n}^{(n)}\right) \tag{4}$$

We denote by $\mathcal{Q}$ the set of quantum correlations, which is convex [1].

Extremal points, exposed points, and border For an arbitrary convex part C of $\mathbb{R}^N$, denote $\mathrm{Ext}(C)$ the set of *extremal points* of C, which are the points that cannot be expressed as a convex combination of different points in C. Extremal points are contained — in general, strictly — in the *border* ∂C of C, consisting of points no neighborhood of which lies fully in C. An extremal point is *exposed* if it is a unique maximum in C for some linear functional. Extremal points (or vertices) of a polytope are always exposed.

RESPONSE OF BELL FUNCTIONALS UNDER UNITARY PERTURBATIONS

We present a method for studying how the value of a Bell functional $\vec{\beta}$ evolves under unitary perturbations of a generic strategy. We propose to define, for each possible input bitstring $x \in \mathbb{Z}_2^n$, the *conditional Bell operator* $\mathcal{B}_x$, which, given a fixed set of POVMs, associates any state ρ to the expected value $\mathrm{Tr}(\rho\mathcal{B}_x)$ of the strategy conditioned on input x. Each of these conditional Bell operators can be evolved by perturbating the local measurements and also by evolving the state ρ, in the Heisenberg picture.

Definition 1. *Let $M = (M_{x_i}^{(i)})_{1\leq i\leq n, x_i\in\mathbb{Z}_2}$ (i.e. a tuple of arbitrary local POVMs), and $\vec{\beta}$ a Bell functional. For $x \in \mathbb{Z}_2^n$, define a conditional Bell operator as:*

$$\mathcal{B}_x(M) = \sum_{a\in\mathbb{Z}_d^n} \beta_{a,x} M_{a_1|x_1}^{(1)} \otimes \dots \otimes M_{a_n|x_n}^{(n)}. \tag{5}$$

The Bell operator $\mathcal{B}(M)$, giving the expected score $\vec{\beta} \cdot \mathcal{P}(M, \rho) = \mathrm{Tr}(\mathcal{B}(M)\rho)$ for the shared state ρ is then:

$$\mathcal{B}(M) = \sum_{x\in\mathbb{Z}_2^n} \mathcal{B}_x(M). \tag{6}$$

We now turn to describing an infinestimal unitary evolution of a generic strategy $\Sigma = (\rho, M)$. The *unitary orbit* of Σ is the set of strategies obtained by unitarily evolving the state ρ and the conditional POVMs for each party. More formally, the unitary orbit of Σ is the set $\{\vec{U} \cdot \Sigma = (\vec{U} \cdot \rho, \vec{U} \cdot M) \coloneqq (U_\rho \rho U_\rho^\dagger, (U_{i,x_i} M_{x_i}^{(i)} U_{i,x_i}^\dagger))_{1\leq i\leq n, x_i\in\mathbb{Z}_2}\}$ where $\vec{U}$ describes all tuples of unitaries acting on the state and local unitaries acting on the POVMs.

We argue that one can keep the POVMs $M_0^{(i)}$ fixed for our purpose. Indeed, its evolution can be absorbed into U_ρ when computing the Bell functional:

$$\mathrm{Tr}\Big(U_\rho \rho U_\rho^\dagger \mathcal{B}(\vec{U}\cdot M)\Big) = \mathrm{Tr}\left(\rho U_\rho^\dagger \sum_{a,x} \beta_{a,x} \bigotimes_{i=1}^n U_{i,x_i} M_{a_i|x_i}^{(i)} U_{i,x_i}^\dagger U_\rho\right) \tag{7}$$

$$= \mathrm{Tr}\left(\rho \left(\bigotimes_{i=1}^n U_{i,0}^\dagger U_\rho\right)^\dagger \sum_{a,x} \beta_{a,x} \bigotimes_{i=1}^n (U_{i,0}^\dagger U_{i,x_i}) M_{a_i|x_i}^{(i)} (U_{i,0}^\dagger U_{i,x_i})^\dagger \bigotimes_{i=1}^n U_{i,0}^\dagger U_\rho\right) \tag{8}$$

Hence, one can consider perturbations acting only on the second measurement for each party. The tuple $\vec{U} = (U_\rho, U_1, \ldots, U_n)$ compiles the local unitaries U_i to perturb the POVMs $M_{\cdot|1}^{(i)}$, and U_ρ to perturb the state ρ. We now present concise formulas for the first and second order response of the Bell functional under unitary evolution of Σ. This method can in principle be generalized to any order.

In all what follows, instead of Hamiltonians H, we use skew-Hermitian generators $K = iH$ to simplify notations, which is completely equivalent [29].

Proposition 1. *Let $\vec{U} \equiv (e^{-K_\rho}, e^{K_1}, \ldots, e^{K_n})$, where $K_i, 1 \le i \le n$ is a local skew-Hermitian operator, and K_ρ is a skew-Hermitian operator on the shared state. Then, the value of the Bell functional on Σ admits the following second order expansion.*

$$\vec{\beta}\cdot\mathcal{P}(\vec{U}\cdot\Sigma) = \vec{\beta}\cdot\mathcal{P}(\Sigma) + \mathrm{Tr}\left(\rho \sum_{x\in\mathbb{Z}_2^n} [K_\rho + x\cdot\vec{K}, \mathcal{B}_x(M)] + \frac{1}{2}\Big([K_\rho + x\cdot\vec{K}, \mathcal{B}_x(M)]_2 + [[x\cdot\vec{K}, K_\rho], \mathcal{B}_x(M)]\Big)\right) + o(||\vec{K}||^2), \tag{9}$$

where we denoted $x\cdot\vec{K} \equiv \sum_{i=1}^n x_i K_i^{(i)}$, $||\vec{K}||^2 \equiv -\sum_{i=1}^n \mathrm{Tr}\big(K_i^2\big) - \mathrm{Tr}\big(K_\rho^2\big)$, and $[A,B]_2 \equiv [A,[A,B]]$ the two-fold commutator.

For a local operator A, the superscript (i) means A acts on site i, i.e. $A^{(i)} = I\otimes\ldots\otimes A\otimes\ldots\otimes I$. However, since it is always clear that K_i acts on site i, we sometimes slightly abuse notations and drop the superscript.

Proof. We evolve the conditional Bell operators in the Heisenberg picture:

$$\vec{\beta}\cdot\mathcal{P}(\vec{U}\cdot\Sigma) = \mathrm{Tr}\left(e^{-K_\rho}\rho e^{K_\rho} \sum_{x\in\mathbb{Z}_2^n} \mathcal{B}_x(\vec{U}\cdot M)\right) \tag{10}$$

$$= \mathrm{Tr}\left(\rho \sum_{x\in\mathbb{Z}_2^n} e^{-K_\rho}\mathcal{B}_x(\vec{U}\cdot M) e^{-K_\rho}\right) \tag{11}$$

Let us explicit the evolution of $\mathcal{B}_x$ for a fixed bitstring x:

$$e^{K_\rho}\mathcal{B}_x(\vec{U}\cdot M)e^{-K_\rho} = e^{K_\rho} \sum_{a\in\mathbb{Z}_d^n} \beta_{a,x} \bigotimes_{i=1}^n e^{x_i K_i} M_{a_i|x_i}^{(i)} e^{-x_i K_i} e^{-K_\rho} \tag{12}$$

$$= e^{K_\rho} e^{x_1 K_1^{(1)} + \cdots + x_n K_n^{(n)}} \left(\sum_{a\in\mathbb{Z}_d^n} \beta_{a,x} \bigotimes_{i=1}^n M_{a_i|x_i}^{(i)}\right) e^{-x_1 K_1^{(1)} - \cdots - x_n K_n^{(n)}} e^{-K_\rho} \tag{13}$$

$$= e^{K_\rho} e^{x\cdot\vec{K}} \mathcal{B}_x(M) e^{-x\cdot\vec{K}} e^{-K_\rho} \tag{14}$$

$$= e^{K_\rho + x\cdot\vec{K} + \frac{1}{2}[K_\rho, x\cdot\vec{K}] + o(||\vec{K}||^2)} \mathcal{B}(M) e^{-(K_\rho + x\cdot\vec{K} + \frac{1}{2}[K_\rho, x\cdot\vec{K}]) + o(||\vec{K}||^2)} \tag{15}$$

$$= \mathcal{B}_x(M) + [K_\rho + x\cdot\vec{K} + \frac{1}{2}[K_\rho, x\cdot\vec{K}], \mathcal{B}_x(M)] + \frac{1}{2}[K_\rho + x\cdot\vec{K} + \frac{1}{2}[K_\rho, x\cdot\vec{K}], \mathcal{B}_x(M)]_2 + o(||\vec{K}||^2) \tag{16}$$

$$= \mathcal{B}_x(M) + [K_\rho + x\cdot\vec{K}, \mathcal{B}_x(M)] + \frac{1}{2}\Big([K_\rho + x\cdot\vec{K}, \mathcal{B}_x(M)]_2 + [[x\cdot\vec{K}, K_\rho], \mathcal{B}_x(M)]\Big) + o(||\vec{K}||^2), \tag{17}$$

where we used the BCH formula in the fourth line, and the exponential commutator expansion [30] in the fifth line. The desired result follows by summing the contributions of each conditional Bell operator in (10). □

As a corollary, note that we obtain the following necessary condition for optimality of a strategy, similar to that obtained in [31].

Corollary 1 (First order conditions of optimality). *If Σ is an optimal quantum strategy for $\vec{\beta}$, then:*

$$[\mathcal{B}(M), \rho] = 0 \tag{18}$$

$$\forall 1 \leq i \leq n, \mathrm{Tr}_{j\neq i}\left(\left[\sum_{x:x_i=0} \mathcal{B}_x(M), \rho\right]\right) = \mathrm{Tr}_{j\neq i}\left(\left[\sum_{x:x_i=1} \mathcal{B}_x(M), \rho\right]\right) = 0 \tag{19}$$

Proof. For Σ to be optimal, the first order term $\mathrm{Tr}\Big(\rho \sum_{x\in\mathbb{Z}_2^n}[K_\rho + x\cdot\vec{K}, \mathcal{B}_x(M)]\Big)$ must vanish for all choice of K_ρ and $K_i, 1 \leq i \leq n$. Moreover, one can write:

$$\begin{aligned}
\mathrm{Tr}\left(\rho \sum_{x\in\mathbb{Z}_2^n}[K_\rho + x\cdot\vec{K}, \mathcal{B}_x(M)]\right) &= \sum_{x\in\mathbb{Z}_2^n} \mathrm{Tr}\Big(\rho[K_\rho + x\cdot\vec{K}, \mathcal{B}_x(M)]\Big) && (20)\\
&= \sum_{x\in\mathbb{Z}_2^n} \mathrm{Tr}\Big(\rho(K_\rho + x\cdot\vec{K})\mathcal{B}_x(M)\Big) - \mathrm{Tr}\Big(\rho\mathcal{B}_x(M)(K_\rho + x\cdot\vec{K})\Big) && (21)\\
&= \sum_{x\in\mathbb{Z}_2^n} \mathrm{Tr}\Big(\mathcal{B}_x(M)\rho(K_\rho + x\cdot\vec{K})\Big) - \mathrm{Tr}\Big(\rho\mathcal{B}_x(M)(K_\rho + x\cdot\vec{K})\Big) && (22)\\
&= \sum_{x\in\mathbb{Z}_2^n} \mathrm{Tr}\Big([\mathcal{B}_x(M), \rho](K_\rho + x\cdot\vec{K})\Big). && (23)
\end{aligned}$$

Hence, if one sets $K_i = 0$ for $1 \leq i \leq n$, the first order term is:

$$\sum_{x\in\mathbb{Z}_2^n} \mathrm{Tr}([\mathcal{B}_x(M), \rho]K_\rho) = \mathrm{Tr}([\mathcal{B}(M), \rho]K_\rho) = 0. \tag{24}$$

We deduce that $[\mathcal{B}(M), \rho]$ is orthogonal to every skew-Hermitian matrix, which implies it is orthogonal to any operator, and necessarily $[\mathcal{B}(M), \rho] = 0$. This is expected: for ρ, M to be optimal for $\mathrm{Tr}(\mathcal{B}(M)\rho)$, ρ must be a convex combination of eigenvectors of $\mathcal{B}(M)$.

Now, if one sets $K_\rho = 0$, and $K_j = 0$ for all $j \neq i$, i fixed, the first order term becomes:

$$\sum_{x\in\mathbb{Z}_2^n} \mathrm{Tr}([\mathcal{B}_x(M), \rho]x_iK_i) = \mathrm{Tr}\left(\left[\sum_{x:x_i=1} \mathcal{B}_x(M), \rho\right] K_i\right) = 0. \tag{25}$$

We deduce that $[\sum_{x:x_i=1} \mathcal{B}_x(M), \rho]$ is orthogonal to every operator acting on the i-th subsystem, or equivalently $\mathrm{Tr}_{j\neq i}\left([\sum_{x:x_i=1} \mathcal{B}_x(M), \rho]\right) = 0$. To obtain the same equation with the sum on x such that $x_i = 0$, we simply insert $[\mathcal{B}(M), \rho] = 0$ in the equality:

$$\mathrm{Tr}_{j\neq i}\left(\left[\sum_{x:x_i=0} \mathcal{B}_x(M), \rho\right]\right) = \mathrm{Tr}_{j\neq i}\left([\mathcal{B}(M), \rho]\right) - \mathrm{Tr}_{j\neq i}\left(\left[\sum_{x:x_i=1} \mathcal{B}_x(M), \rho\right]\right) = 0. \tag{26}$$

Note : While unitary perturbations are not sufficient to reconstruct the full neighborhood of a quantum strategy. note that this framework can still allow to recover generic POVM perturbations. One can reparametrize the correlations of local dimension D by the Naimark dilation [32] $\Pi_{\cdot|x}$ of the measurements $M_{\cdot|x}$ used to realize them. If $\Sigma = (\rho, M)$ is a quantum strategy, the dilated strategy $D(\Sigma) = (\rho \otimes |0\rangle\langle 0|^{\otimes n}, \Pi)$ (where each party holds an extra auxiliary qu-D-it in state $|0\rangle$), is then a D^2-dimensional projective strategy such that $\mathcal{P}(\Sigma) = \mathcal{P}(D(\Sigma))$. Unitary perturbations of $D(\Sigma)$ not acting on the auxiliary space should then correspond to the whole neighborhood of Σ. However, we do not make use of this idea in this paper. □

UNITARY PERTURBATIONS OF LOCAL DETERMINISTIC STRATEGIES

Given a deterministic correlation $p_{\tilde{a}} = \mathcal{P}((\tilde{a}_j)_{1\leq j\leq n})$, how do quantum correlations in its neighborhood compare with respect to a functional $\vec{\beta}$? This question can be adressed by considering all the quantum strategies realizing a specific local deterministic correlation, and using the tools from the previous section to obtain the behaviour of $\vec{\beta}$ under perturbations of these strategies. As we show in this section, the response of $\vec{\beta}$ under unitary perturbations takes a much simpler form when the base point is a local deterministic point. The interest of this problem is threefold. First, especially in the $n22$ scenario, as we will see in the next section, it informs us on the geometry of $\mathcal{Q}$ around local deterministic points. Second, it investigates the limitations of optimizing quantum strategies via gradient-based methods. Finally, comparing the behaviour of $\vec{\beta}$ under perturbations of projective strategies, and those of non-projective POVM strategies of same local dimension D might reveal the existence of correlation points which can be obtained with the latter type, but not with the former type of strategy. hether such points exist for some strategy dimension D and a number of outputs d is an open problem formulated by Gisin in [24]. For example, one could find a functional for which a correlation point p is locally optimal among correlations realized by projective strategies, but not among those realizable with POVMs of local dimension D. Let us define formally what we mean by projective strategies.

Definition 2 (QP correlations and strategies)**.** *We say that a quantum strategy* $\Sigma_{\mathcal{Q}} = ((\mathcal{H}_i)_{1\leq i\leq n}, \rho, (M^{(i)}_{x_i})_{1\leq i\leq n, x_i\in\mathbb{Z}_2})$ *is a qu-D-it projection (QP) strategy if:*

- *for all* $1 \leq i \leq n$, $\mathcal{H}_i = \mathbb{C}^D$,
- *for all* $1 \leq i \leq n$, $x_i \in \mathbb{Z}_2$, $M^{(i)}_{x_i}$ *is a projective valued measurement, characterized by projector elements* $(\Pi^{(i)}_{a_i|x_i})_{a_i\in\mathbb{Z}_d}$ *with* $\sum_{a_i=0}^{d-1} \Pi_{a_i|x_i} = I_D$ *and* $\Pi_{a_i|x_i}\Pi_{a'_i|x_i} = 0$ *for* $a_i \neq a'_i$.

The set of all QP correlations one can realize using a QP strategy with qu-D-its in the n2d scenario is denoted $\mathcal{Q}_P(D)$*, and is a subset of* $\mathcal{Q}$.

If $\mathcal{Q}(D)$ denotes the set of correlations realizable using a quantum strategy of local dimension D at most, the problem then restates as : do we have $\mathcal{Q}_P(D) \subsetneq \mathcal{Q}(D)$ for some D? We do not answer this precise question, but we give new elements pointing to a positive answer, and a potential strategy to address this conjecture using the tools described in the previous section.

Let us first characterize all the strategies realizing a local deterministic point $p_{\tilde{a}}$.

Lemma 1. *Let* $p_{\tilde{a}}$ *be a local deterministic correlation point, and* $\Sigma = (\rho, M)$ *a quantum strategy. Then* $\mathcal{P}(\Sigma) = p_{\tilde{a}}$ *if and only if* $\rho = \sum_{j=1}^k p_j |\psi_j\rangle\langle\psi_j|$*, such that for all* $1 \leq j \leq k$ $M_{\tilde{a}_1(x_1)\ldots\tilde{a}_n(x_n)|x} |\psi_j\rangle = |\psi_j\rangle$*, and consequently, for* $a \in \mathbb{Z}_d^n$, $M_{a|x} |\psi_j\rangle = 0$ *if* $a \neq \tilde{a}_1(x_1)\ldots\tilde{a}_n(x_n)$*, where* $M_{a|x} = M^{(1)}_{a_1|x_1} \otimes \ldots \otimes M^{(n)}_{a_n|x_n}$*. In particular, for all* $x \in \mathbb{Z}_2^n, \mathcal{B}_x(M)\rho = \beta_{\tilde{a}(x),x}\rho$.

Proof. First suppose that $\rho = |\psi\rangle\langle\psi|$ is a pure quantum state. First, by definition of $p_{\tilde{a}}$, we know that for all $x \in \mathbb{Z}_2^n$:

$$\mathcal{P}(\Sigma)(\tilde{a}_1(x_1)\ldots\tilde{a}_n(x_n)|x) = 1 \tag{27}$$

$$|\langle\psi| M^{(1)}_{\tilde{a}_1(x_1)|x_1} \otimes \ldots \otimes M^{(n)}_{\tilde{a}_n(x_n)|x_n} |\psi\rangle|^2 = 1 \tag{28}$$

$$||M^{(1)}_{\tilde{a}_1(x_1)|x_1} \otimes \ldots \otimes M^{(n)}_{\tilde{a}_n(x_n)|x_n} |\psi\rangle||^2 \geq 1 \qquad \text{by Cauchy-Schwarz} \tag{29}$$

Now, note that since $M_{\tilde{a}(x)|x} \equiv M^{(1)}_{\tilde{a}_1(x_1)|x_1} \otimes \ldots \otimes M^{(n)}_{\tilde{a}_n(x_n)|x_n}$ is a POVM element, its maximal eigenvalue $\lambda_{\max}$ must be lower than 1. Hence, the last inequality forces $\lambda_{\max} = 1$ and $M_{\tilde{a}(x)|x} |\psi\rangle = |\psi\rangle$. For a general density operator $\rho = \sum_{j=1}^k p_j |\psi_j\rangle\langle\psi_j|$, with $\sum_{j=1}^k p_j = 1$,we have:

$$\mathcal{P}(\Sigma)(\tilde{a}_1(x_1)\ldots\tilde{a}_n(x_n)|x) = \sum_{j=1}^k p_j |\langle\psi_j| M_{\tilde{a}(x)|x} |\psi_j\rangle|^2 = 1, \tag{30}$$

implying $|\langle\psi_j| M_{\tilde{a}(x)|x} |\psi_j\rangle|^2 = 1$ for all $1 \leq j \leq k$, which by the previous discussion concludes. □

The following energy-weighted sum rule will be crucial in the study of the response of $\vec{\beta}$ under perturbations of Σ. It forms a connection between the spectral structure of A and the evolution under X of A up to second order when the starting point is an eigenvector of A.

Lemma 2 (Second order energy weighted sum rule for observables)**.** *Let A be any observable, and $\lambda_1, \ldots, \lambda_N$, $A_1, \ldots, A_N$ be such that $A = \sum_{m=1}^N \lambda_m A_m$ and $\sum_{m=1}^N A_m = I$. Let ρ be a density operator such that $A\rho = \rho A = \lambda\rho$ for some eigenvalue λ of A. Let X be skew-Hermitian.*

$$\operatorname{Tr}([X, A]\rho) = 0 \tag{31}$$

$$\operatorname{Tr}([X, A]_2\rho) = 2\operatorname{Tr}\big((A - \lambda)X\rho X^\dagger\big) = 2\sum_{m=1}^{N}(\lambda_m - \lambda)\operatorname{Tr}\big(A_m X\rho X^\dagger\big). \tag{32}$$

This is in particular the case if $A_1, \ldots, A_N$ are the orthogonal projectors on the eigenspaces of A.

Proof. We simply develop the commutators:

$$\begin{aligned}
\operatorname{Tr}([X, A]\rho) &= \operatorname{Tr}(XA\rho) - \operatorname{Tr}(AX\rho) && (33)\\
&= \operatorname{Tr}(XA\rho) - \operatorname{Tr}(\rho AX) && (34)\\
&= \lambda(\operatorname{Tr}(X\rho) - \operatorname{Tr}(\rho X)) = 0 && (35)\\
\operatorname{Tr}([X, A]_2\rho) &= \operatorname{Tr}\big((X^2A - 2XAX - AX^2)\rho\big) && (36)\\
&= \operatorname{Tr}\big(\lambda X^2\rho\big) - 2\operatorname{Tr}(XAX\rho) + \operatorname{Tr}\big(\lambda X^2\big) && (37)\\
&= 2(\operatorname{Tr}(\lambda X\rho X) - \operatorname{Tr}(AX\rho X)) && (38)\\
&= 2\operatorname{Tr}\big((A - \lambda I)X\rho X^\dagger\big) \qquad X^\dagger = -X && (39)\\
&= 2\operatorname{Tr}\left(\left(\sum_{m=1}^{N}\lambda_m A_m - \lambda\sum_{m=1}^{N}A_m\right)X\rho X^\dagger\right) && (40)\\
&= 2\sum_{m=1}^{N}(\lambda_m - \lambda)\operatorname{Tr}\big(A_m X\rho X^\dagger\big). && (41)
\end{aligned}$$

□

Translating to our setting, this yields the following formula.

Lemma 3. *Let Σ be a quantum strategy realizing a local deterministic correlation $p_{\tilde{a}}$. Then Σ is a critical point of its unitary orbit for any Bell functional $\vec{\beta}$. More precisely, if we denote $\vec{U} = (e^{K_1}, \ldots, e^{K_n}, e^{-K_\rho})$, and $\Pi_{a,x}$ the projector on the eigenspace of $\mathcal{B}_x(M)$ with eigenvalue $b_{a,x}$, then:*

$$\vec{\beta}\cdot\mathcal{P}(\vec{U}\cdot\Sigma) = \vec{\beta}\cdot\mathcal{P}(\Sigma) + \sum_{x,a}(b_{a,x} - \beta_{\tilde{a}(x),x})\operatorname{Tr}\Big(\Pi_{a,x}(x\cdot\vec{K} + K_\rho)\rho(x\cdot\vec{K} + K_\rho)^\dagger\Big) + o(||\vec{K}||^2) \tag{42}$$

Proof. For all $x \in \mathbb{Z}_2^n$, since Σ is local deterministic, we have $\mathcal{B}_x(M)\rho = \rho\mathcal{B}_x(M) = \beta_{\tilde{a}(x),x}\rho$. Thus, by (31), the first order term $\operatorname{Tr}\big([x\cdot\vec{K} + K_\rho, \mathcal{B}_x(M)]\rho\big)$ of (9) vanishes, as well as the second order term $[[x\cdot\vec{K} + K_\rho], \mathcal{B}_x(M)]$. We obtain the desired result by applying (32) to $[x\cdot\vec{K} + K_\rho, \mathcal{B}_x(M)]_2$. □

We now turn to the specific case of quantum projective strategies.

In this case, the eigenvalues of $\mathcal{B}_x(M)$ can be directly related to $\vec{\beta}$, as they are exactly the numbers $\beta_{a,x}, a \in \mathbb{Z}_d^n$. The unitary response of $\vec{\beta}$ can be split into a main term depending on the evolved state $K_\rho\rho K_\rho^\dagger$, and a remaining term involving measurement basis variations, which has support only on outcomes differing from $\tilde{a}(x)$ on only one site.

Definition 3. *(One-flip maximum) A local deterministic correlation $p_{\tilde{a}}$ is a one-flip maximum (resp. strict one-flip maximum) for $\vec{\beta}$ if and only if for all $1 \le j \le n$, and for all $\tilde{a}' = (\tilde{a}'_k)_{1\le k\le n}$ such that $\tilde{a}'_k = \tilde{a}_k$ for all $k \ne j$, $\vec{\beta}\cdot p_{\tilde{a}} \ge \vec{\beta}\cdot p_{\tilde{a}'}$ (resp. $\vec{\beta}\cdot p_{\tilde{a}} > \vec{\beta}\cdot p_{\tilde{a}'}$).*

Note that being a one-flip maximum is a necessary condition for classical optimality.

Lemma 4. *(Limited effect of measurement evolution for QP strategies) Let $\Sigma = (\rho, \Pi)$ be a QP strategy of local dimension $D \ge d$.. For $a \in \mathbb{Z}_d^n$, $x \in \mathbb{Z}_2^n$, denote $w(a) := |\{1 \le i \le n \mid a_i \ne 0\}|$, and $\Pi_{a|x} \equiv \Pi^{(1)}_{a_1|x_1} \otimes \ldots \otimes \Pi^{(n)}_{a_n|x_n}$. Then, using the same notation as in Proposition 1:*

$$\vec{\beta}\cdot\mathcal{P}(\vec{U}\cdot\Sigma) = \vec{\beta}\cdot\mathcal{P}(\Sigma) + \operatorname{Tr}\big(\mathcal{B}_2(\Pi)K_\rho\rho K_\rho^\dagger\big) + R(\Pi, \vec{K}, \rho) + o(||\vec{K}||^2) \tag{43}$$

where $R(\Pi,\vec{K},\rho)\equiv\sum_{x,a:w(a-\tilde{a}(x))=1}(\beta_{a,x}-\beta_{\tilde{a}(x),x})\operatorname{Tr}\big(\Pi_{a|x}(x_jK_j+K_\rho)\rho(x_jK_j+K_\rho)^\dagger\big)$ *only involves ouptut strings with* $w(a-\tilde{a}(x))=1$*, and* $\mathcal{B}_2(\Pi)=\sum_{x,a:w(a-\tilde{a}(x))\geq 2}(\beta_{a,x}-\beta_{\tilde{a}(x),x})\Pi_{a|x}$

Moreover, if $D=d$ *and* $p_{\tilde{a}}$ *is a one-flip maximum, then* $R(\Pi,\vec{K},\rho)\leq 0$*.*

Proof. Recall that for a QP strategy, $\mathcal{B}_x(M)=\sum_{a\in\mathbb{Z}_d^n}\beta_{a,x}\Pi_{a|x}$, which is a spectral decomposition of $\mathcal{B}_x(M)$, since the projectors $\Pi_{a|x}$ partition the identity for a fixed x. Thus, we can use Lemma 3 with $\Pi_{a,x}=\Pi_{a|x}$, which yields, up to second order:

$$\vec{\beta}\cdot\mathcal{P}(\vec{U}\cdot\Sigma)-\vec{\beta}\cdot\mathcal{P}(\Sigma)=\sum_{x,a}(\beta_{a,x}-\beta_{\tilde{a}(x),x})\operatorname{Tr}\Big(\Pi_{a|x}(x\cdot\vec{K}+K_\rho)\rho(x\cdot\vec{K}+K_\rho)^\dagger\Big)\tag{44}$$

$$=\sum_{x,a}(\beta_{a,x}-\beta_{\tilde{a}(x),x})\operatorname{Tr}\Big(\Pi_{a|x}(x\cdot\vec{K}+K_\rho)\rho(x\cdot\vec{K}+K_\rho)^\dagger\Pi_{a|x}\Big)\qquad \Pi_{a|x}=\Pi_{a|x}^2\tag{45}$$

$$=\sum_{x,a}(\beta_{a,x}-\beta_{\tilde{a}(x),x})\operatorname{Tr}\Big((\Pi_{a|x}(x\cdot\vec{K})\Pi_{\tilde{a}(x)|x}+\Pi_{a|x}K_\rho\Pi_{\tilde{a}(x)|x})\rho(\Pi_{a|x}(x\cdot\vec{K})\Pi_{\tilde{a}(x)|x}+\Pi_{a|x}K_\rho\Pi_{\tilde{a}(x)|x})^\dagger\Big)\tag{46}$$

Where in the last line, we used $\Pi_{\tilde{a}(x)|x}\rho=\rho\Pi_{\tilde{a}(x)|x}=\rho$. Note that most of the contributions of $x\cdot\vec{K}$ are null. Indeed, for $a\in\mathbb{Z}_d^n, x\in\mathbb{Z}_2^n$:

$$\Pi_{a|x}(x\cdot\vec{K})\Pi_{\tilde{a}(x)|x}=\sum_{i=1}^n x_i\bigotimes_{j=1}^n\Pi^{(j)}_{a_j|x_j}K_i^{\delta_{ij}}\Pi^{(j)}_{\tilde{a}_j(x_j)|x_j}\tag{47}$$

Note that $\Pi^{(j)}_{b|x_j}\Pi^{(j)}_{b'|x_j}=0$ for $b\neq b'$. If $w(a-\tilde{a}(x))\geq 2$, at least one of the overlaps $\Pi^{(j)}_{a_j|x_j}\Pi^{(j)}_{\tilde{a}_j(x_j)|x_j}$ where a and $\tilde{a}(x)$ differ at j appear, making the overlap equal to 0. Hence, if $w(a-\tilde{a}(x))\geq 2$, $\Pi_{a|x}(x\cdot\vec{K})\Pi_{\tilde{a}(x)|x}=0$. The strings a such that $w(a)=1$ are denoted by k^j, for $1\leq k\leq d$, $1\leq j\leq n$, where k^j is the string equal to 0 on all sites but on j where $k^j_j=k$. Consequently, the second order term splits as:

$$\sum_{x,a:w(a-\tilde{a})\geq 2}(\beta_{a,x}-\beta_{\tilde{a}(x),x})\operatorname{Tr}\Big(\Pi_{a|x}K_\rho\rho K_\rho^\dagger\Big)+R(\Pi,\vec{K},\rho))=\operatorname{Tr}\Big(\mathcal{B}_2(\Pi)K_\rho\rho K_\rho^\dagger\Big)+R(\Pi,\vec{K},\rho))\tag{48}$$

where $R(\Pi,\vec{K},\rho))\equiv\sum_{k,j}\sum_x(\beta_{\tilde{a}(x)+k^j,x}-\beta_{\tilde{a}(x),x})\operatorname{Tr}\big(\Pi_{\tilde{a}(x)+k^j|x}(x_jK_j+K_\rho)\rho(x_jK_j+K_\rho)^\dagger\big)$, as announced.

Suppose now that $D=d$. Then, since $\Pi_{\tilde{a}(x)|x}$ is of rank 1, $\Pi_{\tilde{a}(x)|x}\rho=\rho\Pi_{\tilde{a}(x)|x}=\rho$ implies:

$$\rho=\Pi_{\tilde{a}(x)|x}=\bigotimes_{j=1}^n\Pi^{(j)}_{\tilde{a}_j(x_j)|x_j},\tag{49}$$

independently of $x\in\mathbb{Z}_d^n$. In particular, for all $1\leq j\leq n$, $\Pi^{(j)}_{\tilde{a}_j(1)|1}=\Pi^{(j)}_{\tilde{a}_j(0)|0}$. This allows us to write :

$$R(\Pi,\vec{K},\rho)=\operatorname{Tr}\left(\sum_{k,j}\bigotimes_{i=1}^{j-1}\sum_x(\beta_{\tilde{a}(x)+k^j,x}-\beta_{\tilde{a}(x),x})\Pi^{(i)}_{\tilde{a}_i(0)|0}\otimes\Pi^{(j)}_{\tilde{a}_j(x_j)+k|x_j}\otimes\bigotimes_{i=j+1}^n\Pi^{(i)}_{\tilde{a}_i(0)|0}(x_jK_j+K_\rho)\rho(x_jK_j+K_\rho)^\dagger\right)\tag{50}$$

$$=\sum_{k,j}\operatorname{Tr}\Big(R_{k,j,0}K_\rho\rho K_\rho^\dagger\Big)+\sum_{k,j}\operatorname{Tr}\Big(R_{k,j,1}(K_j+K_\rho)\rho(K_j+K_\rho)^\dagger\Big)\tag{51}$$

Where we set $R_{k,j,b}=\bigotimes_{i=1}^{k-1}\Pi^{(i)}_{\tilde{a}_i(0)|0}\otimes\Big(\sum_{x:x_j=b}\beta_{\tilde{a}(x)+k^j,x}-\beta_{\tilde{a}(x),x}\Big)\Pi^{(j)}_{\tilde{a}_j(x_j)+k|x_j}\bigotimes_{i=k+1}^n\Pi^{(i)}_{\tilde{a}_i(0)|0}$. The hypothesis that $p_{\tilde{a}}$ is a one flip maximum (resp. strict one flip maximum) is equivalent to $R_{k,j,b}\leq 0$ (resp. $R_{k,j,b}<0$) for all $k\in\mathbb{Z}_d, j\in\{1,\ldots,n\}, b\in\{0,1\}$, which implies $R\leq 0$.

The reciprocal does not necessarily hold, but note that for $(k,j)\neq(k',j')$, $R_{k,j,b}R_{k',j',b'}=0$, so $R<0$ implies $R_{k,j,0}+R_{k,j,1}<0$ for all k,j (take $\vec{K},\rho)=0$ and K_ρ arbitrary, which allows $K_\rho\rho K_\rho^\dagger$ to describe any pure state).

□

This already highlights a difference between PVMs and POVMs: for PVMs, the effect of conditional measurement variations is very limited, as they only act at second order through the projectors $\Pi_{\tilde{a}(x)|x+k^j|x}$, and not at all if $D=d$ and $p_{\tilde{a}}$ is a one-flip maximum. For POVMs there is no such restriction a priori. This is due to the overlaps between POVM elements, which forbid to conclude that most terms in (47) vanish.

We now prove a dimension reduction result for deciding if a given local deterministic strategy is a local optimum in $\mathcal{Q}_P(d)$, showing how structured the optimization problem in $\mathcal{Q}_P(d)$ is compared to $\mathcal{Q}(d)$. In short, we show that this problem reduces to finding a $\mathcal{Q}_P(d-1)$ strategy with positive gain for a set of non-local games with $d-1$ outcomes constructed from $\vec{\beta}$.

Lemma 5 (Dimension reduction via subset games). *For all $k \in \mathbb{N}$, $c \in \mathbb{Z}_k^n$, define the support of c as $S(c) = \{1 \leq i \leq n \mid c_i \neq 0\}$, and for $S = \{s_1, \ldots, s_r\} \subseteq \{1, \ldots, n\}$, denote the restriction of c to S as $c_S = c_{s_1} \ldots c_{s_n}$.*

For $S = \{s_1, \ldots, s_r\} \subseteq \{1, \ldots, n\}$, $x \in \mathbb{Z}_2^n$, $\alpha \in \mathbb{Z}_d^{S,x} \equiv \{\alpha \in \mathbb{Z}_d^{|S|} \mid \forall 1 \leq i \leq |S|, \alpha_i \neq \tilde{a}_{s_i}(x_{s_i})\}$, denote $\alpha^{x,S}$ the unique string in $\mathbb{Z}_d^n$ such that $\alpha_S^{x,S} = \alpha$, and $\alpha_j^{x,S} = \tilde{a}_j(x_j)$ for all $j \notin S$. Then, define, for all $\xi \in \mathbb{Z}_2^{|S|}$:

$$\beta_{\alpha,\xi}^S \coloneqq \sum_{x : x_S = \xi} \beta_{\alpha^{x,S},x} - \beta_{\tilde{a}(x),x}. \tag{52}$$

Up to relabeling of the output strings, $\vec{\beta}^S$ defines a $|S|2(d-1)$ functional, to which one can associate a Bell operator $\mathcal{B}^S(M) \coloneqq \sum_{\alpha,\xi} \beta_{\alpha,\xi}^S M_{\alpha|\xi}$, which we call a subset game. Let $\Sigma = (\rho, \Pi) \in \mathcal{Q}_P(d)$ such that $\mathcal{P}(\Sigma) = p_{\tilde{a}}$. Then, $\mathcal{B}_2(M)$ admits a direct sum decomposition into subset games.

$$\mathcal{B}_2(\Pi) = \sum_{\substack{S \subseteq \{1,\ldots,n\} \\ |S| \geq 2}} I_S \otimes \mathcal{B}^S(\Pi^S), \tag{53}$$

where $I_S = \bigotimes_{j \notin S} \Pi^{(j)}_{\tilde{a}_j(0)|0}$ is the projector used by the parties that are inactive for the subset S.

In other words, to investigate second order variations of $\vec{\beta} \cdot p$ around a local deterministic correlation $p_{\tilde{a}}$, one can equivalently study each of the $(k, 2, d-1)$ functional defined by the average gain that the subset S of $1 \leq k \leq n$ players obtains by changing their projective strategies —forbidding projectors that can yield the outputs prescribed by $\tilde{a}(x)$.

Proof. Let $x, x' \in \mathbb{Z}_2^n$, and $a, a' \in \mathbb{Z}_d^n$ such that $S(a - \tilde{a}(x)) \neq S(a' - \tilde{a}(x'))$. Then, there exists $1 \leq j \leq n$ such that $a_j = \tilde{a}_j(x_j)$ and $a'_j \neq \tilde{a}_j(x'_j)$. Recall that $\mathcal{P}(\Sigma) = p_{\tilde{a}}$ forces $\Pi_{\tilde{a}_j(0)|0} = \Pi_{\tilde{a}_j(1)|1}$. Thus, $\Pi_{a|x}\Pi_{a'|x} = \bigotimes_{i=1}^n \Pi^{(i)}_{a_i|x_i}\Pi^{(i)}_{a'_i|x'_i} = 0$ because $\Pi_{\tilde{a}_j|x_j}\Pi_{\tilde{a}'_j|x'_j} = \Pi_{\tilde{a}_j(x_j)|x_j}\Pi_{\tilde{a}'_j|x'_j} = \Pi_{\tilde{a}_j(x'_j)|x'_j}\Pi_{a'_j|x'_j} = 0$.

We deduce the following direct sum decomposition for $\mathcal{B}_2(\Pi)$, sorting the output strings a by their support S (where they do not match with $\tilde{a}(x)$):

$$\mathcal{B}_2(\Pi) \coloneqq \sum_{x,a : w(a-\tilde{a}(x)) \geq 2} (\beta_{a,x} - \beta_{\tilde{a}(x),x})\Pi_{a|x} = \sum_{\substack{S \subseteq \{1,\ldots,n\} \\ |S| \geq 2}} \sum_{\xi \in \mathbb{Z}_2^{|S|}} \sum_{\alpha \in \mathbb{Z}_d^{S,x}} \sum_{x : x_S = \xi} (\beta_{\alpha^{x,S},x} - \beta_{\tilde{a}(x),x})\Pi_{\alpha^{x,S}|x}, \tag{54}$$

where the terms in the first sum are in direct sum by the previous remark. Observe that up to to permutation of the tensor product, we have:

$$\begin{aligned}
\Pi_{\alpha^{x,S}|x} &= \bigotimes_{i \in S} \Pi^{(i)}_{\alpha_i^{x,S}|x_i} \bigotimes_{i \notin S} \Pi^{(i)}_{\alpha_i^{x,S}|x_i} && (55)\\
&= \bigotimes_{i=1}^{|S|} \Pi^{(s_i)}_{\alpha_i|\xi_i} \bigotimes_{i \notin S} \Pi^{(i)}_{\tilde{a}_i(x_i)|x_i} && (56)\\
&= \bigotimes_{i=1}^{|S|} \Pi^{(s_i)}_{\alpha_i|\xi_i} I_S, && (57)
\end{aligned}$$

where $I_S \equiv \bigotimes_{i \notin S} \Pi^{(i)}_{\tilde{a}_i(x_i)|x_i} = \bigotimes_{i \notin S} \Pi^{(i)}_{\tilde{a}(0)|0}$ because $\mathcal{P}(\Sigma) = p_{\tilde{a}}$.

Reinjecting into (54), we obtain, again up to a permutation of the factors depending only on S:

$$\mathcal{B}_2(\Pi) = \sum_{\substack{S\subseteq\{1,\dots,n\}\\|S|\geq 2}} I_S \otimes \sum_{\xi\in\mathbb{Z}_2^{|S|}} \sum_{\alpha\in\mathbb{Z}_d^{S,x}} \sum_{x:x_S=\xi} (\beta_{\alpha^{x,S},x} - \beta_{\tilde{a}(x),x}) \bigotimes_{i=1}^{|S|} \Pi^{(i)}_{\alpha_i|\xi_i} \tag{58}$$

$$= \sum_{\substack{S\subseteq\{1,\dots,n\}\\|S|\geq 2}} I_S \otimes \sum_{\xi,\alpha} \beta^S_{\alpha,\xi} \bigotimes_{i=1}^{|S|} \Pi^{(s_i)}_{\alpha_i|\xi_i} \tag{59}$$

$$= \sum_{\substack{S\subseteq\{1,\dots,n\}\\|S|\geq 2}} I_S \otimes \mathcal{B}^S(\Pi^S), \tag{60}$$

where we denoted $\Pi^S = (\Pi^{(s_1)}, \dots, \Pi^{(s_{|S|})})$ the measurement strategy of the players in S, and $\mathcal{B}^S$ the Bell operator associated to $\vec{\beta}^S$. Since the last equation is a direct sum, and I_S is a projector, the spectrum of $\mathcal{B}_2(\Pi)$ is, up to multiplicities, the union of the spectras of $\mathcal{B}^S(\Pi^S)$ for all S. This guarantees that the LHS of (**??**) cannot exceed the RHS, and, since any projective measurement strategy Π^S for the players of a subset S can be extended to a projective measurement strategy Π for all the players, we must have the equality. □

If none of these games admit a $\mathcal{Q}_P(d-1)$ strategy with positive gain, then $p_{\tilde{a}}$ is a local optimum in $\mathcal{Q}_P(d)$ for $\vec{\beta}$.

Corollary 2 (Local optimality criterion via subset games)**.** *For $S \subseteq \{1,\dots,n\}$, define the active subspace of the subset game $\mathcal{B}_S$ as $V_S := \bigoplus_{(\alpha,\xi)\in A_S} \mathrm{Im}(\Pi_{\alpha|\xi})$, with $A_S := \{(\alpha,\xi), \alpha \in \mathbb{Z}_d^{|S|}, \xi \in \mathbb{Z}_2^{|S|} \mid \forall 1 \leq i \leq n, \alpha_i \neq \tilde{a}_i(\xi_i)\}$. Suppose that $p_{\tilde{a}}$ is a strict one-flip maximum, and that for all $S \subseteq \{1,\dots,n\}$ with $|S| \geq 2$, and all quantum projective strategy $\Sigma = (\Pi,\rho)$ of local dimension d such that $\mathcal{P}(\Sigma) = p_{\tilde{a}}$, $\mathcal{B}_S(\Pi^S)|_{V_S} \prec 0$. Then, $p_{\tilde{a}}$ is a local optimum in $\mathcal{Q}_P(d)$. Moreover, if there exists such a Σ and S, $|\psi_S\rangle \in V_S$ such that $\langle\psi_S|\,\mathcal{B}_S(\Pi^S)\,|\psi_S\rangle > 0$, then $p_{\tilde{a}}$ is not a local optimum in $\mathcal{Q}_P(d)$.*

This property is not obvious for two reasons. First, it requires to write the neighborhood of $p_{\tilde{a}}$ from the union of the neighborhood of its preimages, which requires a crucial compacity hypothesis. Second, we need to deal with degenerate directions where the second order term is zero, and guarantee that higher order terms are not positive.

Lemma 6. *Consider two separated topological spaces X and Y, with X compact. Let $f : X \to Y$ be a continuous map. Then, for all $y \in Y$, $\bigcup_{x\in f^{-1}(y)} f(V_x)$ (where V_x denotes any neighborhood of x), is a neighborhood of y in $f(X)$.*

Proof. Note that $Z = X - \bigcup_{x\in f^{-1}(y)} V_x$ is compact, since it is closed in the compact X. Hence, $f(Z)$ is compact, and thus closed since Y is separated, and does not contain y, implying that $f(X) - f(Z)$ is an open set, which contains y since $f(X)$ does. To conclude, notice that $f(X) - f(Z) \subseteq f(X - Z) = f(\bigcup_{x\in f^{-1}(y)} V_x) = \bigcup_{x\in f^{-1}(y)} f(V_x)$. □

Proof of Corollary 2. Let $\Sigma = (\rho, \Pi)$ be a $\mathcal{Q}_P(d)$ strategy such that $\mathcal{P}(\Sigma) = p_{\tilde{a}}$. For all $\varepsilon > 0$, the set

$$B_\varepsilon \equiv \{\vec{U}(\vec{K}) \cdot \Sigma, ||\vec{K}|| < \varepsilon\}, \tag{61}$$

where $\vec{U}(\vec{K}) = (e^{K_1}, \dots, e^{K_n}, e^{-K_\rho})$, is a neighborhood of Σ in $\mathcal{Q}_P(d)$, which is compact. Hence, by Lemma 6, it suffices to show that for $||\vec{K}||$ small enough, $\vec{\beta} \cdot \mathcal{P}(\vec{U}(\vec{K}) \cdot \Sigma) < \vec{\beta} \cdot \mathcal{P}(\Sigma)$. Recall Lemma 4:

$$\vec{\beta} \cdot \mathcal{P}(\vec{U} \cdot \Sigma) = \vec{\beta} \cdot \mathcal{P}(\Sigma) + \mathrm{Tr}\big(\mathcal{B}_2(\Pi) K_\rho \rho K_\rho^\dagger\big) + R(\Pi, \vec{K}, \rho)) + o(||\vec{K}||^2). \tag{62}$$

By Lemma 4, since p_0 is a one-flip maximum, $R(\Pi, \vec{K}, \rho)) \leq 0$. By Lemma 5, since $B^S(\Pi^S) \prec 0$ for all subset of parties S, we deduce that $\mathrm{Tr}\big(\mathcal{B}_2(\Pi) K_\rho \rho K_\rho^\dagger\big) \leq 0$. Now suppose that the second order term is zero. Then, we must have $R(\Pi, \vec{K}, \rho)) = 0$ and $\mathrm{Tr}\big(\mathcal{B}_2(\Pi) K_\rho \rho K_\rho^\dagger\big) = 0$. Since $\mathcal{B}_2(\Pi) \leq 0$, the latter implies that $\mathcal{B}_2(\Pi) K_\rho \rho K_\rho^\dagger = 0$, so the support of $K_\rho \rho K_\rho^\dagger$ is in $\ker \mathcal{B}_2(\Pi)$.

Since the PVMs in Π are not degenerate, let us denote $\Pi_{a|x} = |a|x\rangle\langle a|x|$ and show that $\ker \mathcal{B}_2(\Pi) = \mathrm{Vect}(|a|0^n\rangle, w(a - \tilde{a}(0^n)) \leq 1)$. Let $|\psi\rangle \in \ker \mathcal{B}_2(\Pi)$, and write $|\psi\rangle = \sum_{a\in\mathbb{Z}_d^n} c_a |a|0^n\rangle$. By definition $\mathcal{B}_2(\Pi)\,|\psi\rangle = 0$, but since by hypothesis $I_S \otimes \mathcal{B}_S(\Pi^S) \leq 0$ for all $S \subseteq \{1,\dots,n\}$, we deduce that $I_S \otimes \mathcal{B}_S(\Pi^S)\,|\psi\rangle = 0$. Let us compute the latter:

$$I_S \otimes \mathcal{B}_S(\Pi^S)\,|\psi\rangle = \sum_{a\in\mathbb{Z}_d^n} c_a I_S \otimes \mathcal{B}_S(\Pi^S)\,|a|0^n\rangle = \sum_{a\in\mathbb{Z}_d^n} c_a \left(I_S \bigotimes_{i\notin S} |a_i|0\rangle \right) \otimes \left(\mathcal{B}_S(\Pi^S) \bigotimes_{i\in S} |a_i|0\rangle \right). \tag{63}$$

Now, by definition of I_S and $B_S(\Pi^S)$, $I_S \bigotimes_{i\notin S} |a_i|0\rangle$ is non zero if and only if $a_i = \tilde{a}_i(0)$ for all $i \notin S$, and $\mathcal{B}_S(\Pi^S) \bigotimes_{i\in S} |a_i|0\rangle$ is non zero if and only if $a_i \neq \tilde{a}_i(0)$ for $i \in S$. Hence, $I_S \otimes \mathcal{B}_S(\Pi^S)\,|a|0^n\rangle \neq 0$ if and only if $S(a - \tilde{a}(0^n)) = S$. Hence, the last equation simplifies to:

$$I_S \otimes \mathcal{B}_S(\Pi^S)\,|\psi\rangle = \bigotimes_{i\notin S} |\tilde{a}_i(0)|0\rangle \otimes \mathcal{B}_S(\Pi^S)\,|\psi_S\rangle\,, \tag{64}$$

where $|\psi_S\rangle = \sum_{a:S(a-\tilde{a}(0^n))=S} c_a\,|a_S\rangle$. Since $\mathcal{B}_S(\Pi^S)$ is strictly negative on V_S and $|\psi_S\rangle \in V_S$ by definition, we must have $|\psi_S\rangle = 0$, and thus $c_a = 0$ for all a such that $S(a - \tilde{a}(0^n)) = S$. Since this is true for all S with $|S| \geq 2$, only the coefficients c_a with $w(a - \tilde{a}(0^n)) \leq 1$ remain, which proves that $\ker \mathcal{B}_2(\Pi) = \mathrm{Vect}(|a|0^n\rangle\,, w(a - \tilde{a}(0^n)) \leq 1)$.

We now use that $R(\Pi, \vec{K}, \rho) = 0$, which rewrites, using the same notation as in (50):

$$\sum_{k,j} \mathrm{Tr}\big(R_{k,j,0} K_\rho \rho K_\rho^\dagger\big) + \sum_{k,j} \mathrm{Tr}\big(R_{k,j,1}(K_j + K_\rho)\rho(K_j + K_\rho)^\dagger\big) = 0. \tag{65}$$

Since $p_{\tilde{a}}$ is a one-flip maximum, $R_{k,j,b}$ is negative, meaning the support of $K_\rho \rho K_\rho^\dagger$ must also lie in the kernel of $R_{k,j,b}$. Moreover, since the optimum is strict, the factor of $R_{k,j,b}$ on the k-th site, $\sum_{x:x_j=b}(\beta_{\tilde{a}(x)+k^j,x} - \beta_{\tilde{a}(x),x})\Pi_{\tilde{a}_k(x)|b}$, is strictly negative on the subspace $V_{\{k\}}$. By an analogous reasoning, taking $b = 0$, we deduce that the coefficients c_a of a state $|\psi\rangle$ in the support of $K_\rho \rho K_\rho^\dagger$ for $w(a - \tilde{a}(0)) = 1$ must be zero, hence $K_\rho \rho K_\rho^\dagger \propto |\tilde{a}(0^n)|0^n\rangle\langle\tilde{a}(0^n)|0^n|$, which means $K_\rho \rho \propto |\tilde{a}(0^n)|0^n\rangle\langle\phi|$ for some state $|\phi\rangle$. Now note that since $\mathcal{P}(\Sigma) = p_{\tilde{a}}$ and the PVMs are not degenerate, we must have $\rho = \Pi_{\tilde{a}(0^n)|0^n}$ by Lemma 1. Combined with the previous, we conclude that $K_\rho \rho \propto \rho$. We now turn to the second term of (65). Again, since $R_{k,j,1} \preceq 0$ for all k, j, the summand is null for all k, j. Note that $R_{k,j,1}\rho = \rho R_{k,j,1} = 0$, and hence, denoting $K_\rho \rho = i\lambda\rho$ for some real λ:

$$R_{k,j,1}(K_j + K_\rho)\rho(K_j + K_\rho)^\dagger = R_{k,j,1}(K_j \rho K_j^\dagger + \lambda(\rho K_j^\dagger + K_j \rho) + K_\rho \rho K_\rho^\dagger) = R_{k,j,1} K_j \rho K_j^\dagger + \lambda R_{k,j,1} K_j \rho \tag{66}$$

$$0 = \mathrm{Tr}\big(R_{k,j,1}(K_j + K_\rho)\rho(K_j + K_\rho)^\dagger\big) = \mathrm{Tr}\Big(R_{k,j,1} K_j \rho K_j^\dagger\Big) + \lambda\,\mathrm{Tr}(\rho R_{k,j,1} K_j) \tag{67}$$

$$= \mathrm{Tr}\Big(R_{k,j,1} K_j \rho K_j^\dagger\Big), \tag{68}$$

where we used trace cyclicity in the second to last line. Again, this induces $R_{k,j,1} K_j \rho K_j^\dagger = 0$, and thus:

$$0 = R_{k,j,1} K_j \rho K_j^\dagger = \bigotimes_{i=1}^{j-1} \Pi^{(i)}_{\tilde{a}_j(0)|0} \otimes \left(\sum_{x:x_j=1} (\beta_{\tilde{a}(x)+k^j,x} - \beta_{\tilde{a}(x),x}) \Pi^{(j)}_{\tilde{a}_j(1)+k|1} K_j \Pi_{\tilde{a}_j(0)|0} K_j^\dagger \right) \otimes \bigotimes_{i=j+1}^{n} \Pi^{(i)}_{\tilde{a}_i(0)|0}. \tag{69}$$

Since $p_{\tilde{a}}$ is a strict one-flip maximum, $\sum_{x:x_j=1}(\beta_{\tilde{a}(x)+k^j,x} - \beta_{\tilde{a}(x),x}) < 0$, so $\Pi_{\tilde{a}_j(1)+k|1} K_j \Pi_{\tilde{a}_j(0)|0} K_j^\dagger = 0$. This is true for all $1 \leq k \leq d-1$, but the projectors $\Pi_{\tilde{a}_j(1)+k|1} K_j \Pi_{\tilde{a}_j(0)}, k \in \mathbb{Z}_d$ form a partition of the identity, so $K_j \Pi_{\tilde{a}_j(0)|0} K_j^\dagger \propto \Pi_{\tilde{a}_j(0)|0}$, which is equivalent to $K_j \rho K_j^\dagger \propto \rho$,

We deduce that for all $x \in \mathbb{Z}_2^n$, $(x \cdot \vec{K} + K_\rho)\rho(x \cdot \vec{K} + K_\rho)^\dagger \propto \rho$. By the same reasoning as earlier, we deduce that $(x \cdot \vec{K} + K_\rho)\rho = i\mu_x \rho = \rho(x \cdot \vec{K} + K_\rho)$ for some real μ_x. Finally, we deduce that if the second order term is not strictly negative, then:

$$\vec{\beta} \cdot \mathcal{P}(\vec{U} \cdot \Sigma) = \mathrm{Tr}\left(\sum_x \mathcal{B}_x(\Pi) e^{x\cdot\vec{K}+K_\rho} \rho e^{-x\cdot\vec{K}-K_\rho} \right) = \mathrm{Tr}\left(\sum_x \mathcal{B}_x(\Pi) e^{i\mu_x} \rho e^{-i\mu_x} \right) = \vec{\beta} \cdot \mathcal{P}(\Sigma), \tag{70}$$

which concludes. □

THE (N, 2, 2) QUANTUM SET IS FLAT AROUND EVERY LOCAL DETERMINISTIC CORRELATION

We conclude with an unexpected consequence for the geometry of $\mathcal{Q}$ in the $(n, 2, 2)$ scenario, namely that the faces around any local deterministic point are flat. We make this statement formal in the following theorem.

Theorem 1. *Let $p_{\tilde{a}}$ be a local deterministic correlation, and P be a 2-dimensional plane such that $p_{\tilde{a}} \in \mathcal{Q} \cap P$. Then, there exists $r_P > 0$ such that* $\mathrm{Ext}(Q) \cap P \cap B(p_{\tilde{a}}, r_P) = \{p_{\tilde{a}}\}$.

We first show the following proposition based on the dimension reduction lemma from the previous section.

Lemma 7. *Let $p_{\tilde{a}}$ be a local deterministic correlation which is a unique classical maximum for a $(n, 2, 2)$ Bell functional $\vec{\beta}$. Then, $p_{\tilde{a}}$ is a local optimum for $\vec{\beta}$ in $\mathcal{Q}_P(2)$.*

For the proof of Theorem 1, we will need the following Theorem by Masanes [9]

Theorem 2. *In the $(n, 2, 2)$ setting,* $\mathrm{Ext}(\mathcal{Q}) \subseteq \mathcal{Q}_P(2)$.

Proof. $p_{\tilde{a}}$ is in particular a strict one-flip maximum, so by Corollary 2, it remains to show that the subset games are strictly negative on their active subspace. However, since $d = 2$ here, the subset games are scalars. More precisely, for all $S \subseteq \{1, \ldots, n\}$ with $|S| \geq 2$, $\mathbb{Z}_2^{x,S}$ consists of a single string which is the complementary of $\tilde{a}(x)_S$, and its lift $\alpha^{x,S}$ is equal to $\tilde{a}(x)$ outside of S.

$$\mathcal{B}_S(\Pi^S) = \sum_{\xi \in \mathbb{Z}_2^{|S|}} \sum_{x : x_S = \xi} (\beta_{\alpha^{x,S},x} - \beta_{\tilde{a}(x),x}) = \sum_x \beta_{\alpha^{x,S},x} - \beta_{\tilde{a}(x),x} \tag{71}$$

Note that this is exactly the score increase, compared to $p_{\tilde{a}}$, of the local deterministic correlation realized by fixing the output of each player i to $\tilde{a}_i(x_i)$ if $i \in S$, and $\tilde{a}_i(x_i) \oplus 1$ if $i \notin S$. It is strictly negative since $p_{\tilde{a}}$ is a strict classical optimum. □

This is surprising : we know since that every extremal point of $\mathcal{Q}$ can be realized by a $\mathcal{Q}_P(2)$ correlation, so every Bell functional admits a maximum at a correlation in $\mathcal{Q}_P(2)$. Hence, if one could find extremal points of Q arbitarily close de $p_{\tilde{a}}$ (which is the case if $\mathcal{Q}$ is curved around $p_{\tilde{a}}$), one could choose $\vec{\beta}$ such that $p_{\tilde{a}}$ is the strict optimal classical correlation, but the optimal quantum correlation is a quantum extremal point p close from $p_{\tilde{a}}$. If p is sufficiently close, one expects that, slightly perturbating $p_{\tilde{a}}$ towards p inside $\mathcal{Q}_P(2)$ achieves a larger score than $p_{\tilde{a}}$. This intuition is formalized in the proof of Theorem 1.

Proof. Let P be a two dimensional plane such that $p_{\tilde{a}} \in \mathcal{Q}_P \equiv \mathcal{Q} \cap P$. $\mathcal{L}$ is a polytope having $p_{\tilde{a}}$ as a vertex, so $\mathcal{L}_P \equiv \mathcal{L} \cap P$ is a polygon and also has $p_{\tilde{a}}$ as a vertex, which is adjacent to two edges $E_1 = F_1 \cap P$ and $E_2 = F_2 \cap P$ of $\mathcal{L}_P$, where F_1 and F_2 are two faces of $\mathcal{L}$. Let H_1 and H_2 be the half spaces delimited by F_1 and F_2 such that $H_1 \cap \mathcal{L} = F_1$, and $H_2 \cap \mathcal{L} = F_2$. Since $\mathcal{L}_P \subseteq \mathcal{Q}_P$, it is clear that $H_1 \cup H_2 \cap \partial \mathcal{Q}_P$ is a neighborhood of $p_{\tilde{a}}$ in $\partial \mathcal{Q}_P$ (where ∂ denotes the border). We now treat $H_1 \cap \partial \mathcal{Q}_P$ and $H_2 \cap \partial \mathcal{Q}_P$ separately and show that if $p \in H_1 \cap \partial \mathcal{Q}_P$ is sufficiently close from $p_{\tilde{a}}$, then p is not extremal (the reasoning for H_2 is the same).

If $H_1 \cap \partial \mathcal{Q}_P = H_1 \cap \partial \mathcal{L}_P$, then every point in $H_1 \cap \partial \mathcal{Q}_P$ lies on an edge and is hence not extremal (except for the extremities).

Otherwise, we claim it is possible to find a Bell functional $\vec{\beta}$ such that $p_{\tilde{a}}$ is a unique classical maximum for $\vec{\beta}$, and, if one defines $H_{\vec{\beta}} = \{p \in \mathbb{R}^{2^n \times d^n} \mid \vec{\beta} \cdot p \geq \vec{\beta} \cdot p_{\tilde{a}}\}$, and its associated separating hyperplane $F_{\vec{\beta}}$ then $(H_{\vec{\beta}} - F_{\vec{\beta}}) \cap \partial \mathcal{Q}_P$ is still a neighborhood of $p_{\tilde{a}}$ in $\partial \mathcal{Q}_P$. Let $\vec{\beta}_1$ be a Bell functional such that $F_1 = \{p \in \mathcal{L} \mid \vec{\beta}_1 \cdot p = \vec{\beta}_1 \cdot p_{\tilde{a}}\}$. Since $p_{\tilde{a}}$ is a vertex of $\mathcal{L}$, one can also find a Bell functional $\vec{\beta}_{\tilde{a}}$ such that $p_{\tilde{a}}$ is a unique classical maximum for $\vec{\beta}_{\tilde{a}}$. Then, for all $\varepsilon > 0$, $p_{\tilde{a}}$ is a unique classical maximum for $\vec{\beta}_1 + \varepsilon \vec{\beta}_{\tilde{a}}$, defining a hyperplane $F_\varepsilon = \{p \in \mathbb{R}^{2^n \times d^n} \mid \vec{\beta}_\varepsilon \cdot p = \vec{\beta}_\varepsilon \cdot p_{\tilde{a}}\}$. The interior oriented angle $\theta(\varepsilon)$ between E_1 and the line $F_\varepsilon \cap P$ is then clearly continuous in ε, and always positive since otherwise some part of $\mathcal{L} - \{p_{\tilde{a}}\}$ lies above F_ε making $p_{\tilde{a}}$ suboptimal for $\vec{\beta}_\varepsilon$. We also have $\theta(0) = 0$ by defintiion. Moreover, since $H_1 \cap \partial \mathcal{L}_P \neq H_1 \cap \partial \mathcal{Q}_P$, the interior oriented angle θ between E_1 and the convex curve $H_1 \cap \partial \mathcal{Q}_P$ must be strictly positive. This means there must exist ε_0 such that $0 < \theta(\varepsilon_0) < \theta$, in which case $\vec{\beta}_{\varepsilon_0}$ satisfies the desired property. By Lemma 7, $p_{\tilde{a}}$ is a local optimum in $\mathcal{Q}_P(2)$ for $\vec{\beta}_{\varepsilon_0}$. This is equivalent to saying that there exists $r_{P,1}$ such that $\mathcal{Q}_P(2) \cap (H_{\varepsilon_0} - F_{\varepsilon_0}) \cap B(p_{\tilde{a}}, r_{P,1}) = \emptyset$. For $r_{P,1}$ small enough, since $\partial \mathcal{Q}_P \cap (H_{\varepsilon_0} - F_{\varepsilon_0})$ is a neighborhood of $p_{\tilde{a}}$ in $\partial \mathcal{Q}_P$, this implies $\mathcal{Q}_P(2) \cap \partial \mathcal{Q}_P \cap B(p_{\tilde{a}}, r_{P,1}) = \emptyset$. Theorem 2 concludes. □

DISCUSSION AND OPEN QUESTIONS

By analyzing the dynamics of quantum correlations under unitary perturbations of strategies, we have uncovered new properties of the quantum correlation set $\mathcal{Q}$. The reduction of multipartite Bell optimization to subset games provides a powerful tool for analyzing local optimality. Our proof of the flatness of $\mathcal{Q}$ around deterministic points in the $(n, 2, 2)$ scenario is a first step towards understanding the geometry of the multipartite quantum set. Furthermore, the distinction we draw between projective and general POVM perturbations offers a concrete roadmap for resolving Gisin's problem.

Ansatz Dimension as a Learning Resource. Our results underscore the role of the *ansatz dimension* in variational quantum algorithms. Indeed, we show that classical optimum correlations are always locally optimal in the set of qubit projective strategies. This means that even in the $(n, 2, 2)$ case where the optimal violation is known to be achievable with qubits ($D = 2$), a variational quantum circuit — using parametrized shared state preparation and local measurements — with local dimension 2 initialized near a classical point may fail to escape the local optimum. The convexity of $\mathcal{Q}$ and of Bell functionals suggests that increasing the local dimension of the ansatz can eliminate these local optima. This positions the dimension of the variational circuit as a resource for learning, independently on the target dimension.

New games exhibiting quantum–classical gap. The local optimality criterion via subset games (Corollary 2) suggests a way to construct new games with a quantum–classical gap. It suffices to consider a $(k, 2, d-1)$ game B with strictly positive value, and construct a game for which one of the subset games at the optimal classical correlation $p_{\tilde{a}}$ is B. Then, $p_{\tilde{a}}$ is not a local optimum for B in $\mathcal{Q}_P(d)$, and a fortiori the game has a quantum gap. It would be interesting to investigate if this construction can succeed and yield any nontrivial new type of game with a quantum–classical gap.

POVMs vs. PVMs This work sheds new light on the open question posed by Gisin: does there exist a correlation achievable by POVMs of dimension D that is not in $\mathcal{Q}_P(D)$? Our analysis reveals a structural difference between PVMs and POVMs under perturbation. For PVMs, the measurement evolution terms in the second-order expansion vanish or are strictly negative due to orthogonality constraints. For general POVMs, non-orthogonal elements allow these terms to potentially contribute positively. We suggest that constructing a Bell functional where the subset games are negative but POVM perturbations can yield a positive gain would prove $\mathcal{Q}_P(D) \subsetneq \mathcal{Q}(D)$. Our perturbative machinery provides the explicit conditions required to search for such a functional.

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