Identifying causal structures which cannot support quantum correlations without fine-tuning

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I. MOTIVATION: BEYOND NON-LOCALITY

One of the most striking results of the twentieth century is Bell's theorem [1], which establishes that certain quantum correlations cannot be explained by any local classical theory (without resorting to conspicuous mechanisms like retrocausality or superdeterminism). This fact manifests as the *non-locality* of quantum theory. Non-locality is often regarded as the defining feature that makes certain quantum correlations non-classical. In this work we substantiate why a second characteristic – *fine-tuning* – deserves attention as a marker of non-classicality.

This can be better understood using the framework of *causality* and causal inference [2]. In this framework a causal structure is a directed acyclic graph (DAG) in which each node represents a variable and each directed edge from say nodes X to Y encodes the possibility of a direct causal influence of X on Y. Variables corresponding to observed nodes (triangles) appear in the data, latent nodes (circles) on the other hand represent unobserved common causes, the probability distributions over which are unknown. A probability distribution P is *classically compatible* with a DAG G if there exists a joint distribution Q over all the nodes satisfying the causal Markov condition: every variable is independent of its non-descendants given its parents, so that Q factorises as

$$Q(x_1, \ldots, x_n) = \prod_i Q(x_i|\mathrm{pa}(x_i))$$

and that Q marginalises to P over the observed variables. For the Bell causal structure (Figure 1), classical compatibility means $P(abxy) = \sum_\lambda Q(\lambda)\, Q(x)\, Q(y)\, Q(a|x\lambda)\, Q(b|y\lambda)$.

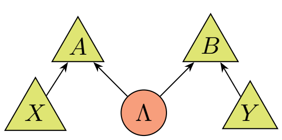

Figure 1: The Bell causal structure. X, Y are settings (independent of Λ), A, B are outcomes, Λ is a latent common cause.

What is fine-tuning? A causal explanation of P by a DAG G is *faithful* – equivalently, *not fine-tuned* – if every conditional independence in P arises from a corresponding d-separation relation in G. Fine-tuning occurs when the data displays a conditional independence that the causal structure does not force. For instance, consider a DAG in which the only paths connecting A and C are a direct causal path $A \to C$ and an indirect path $A \to B \to C$. If along those two paths equal and opposite correlations are generated between A and C that cancel out exactly, then A and C are independent despite no corresponding d-separation in G. Such conspiratorial cancellations have no structural justification and vanish under any generic perturbation of the causal statistical parameters of the DAG [4]. Faithfulness – the assumption that no such cancellations occur is a basic principle in causal discovery – the field that helps us discover which DAGs could be potential causal explanations of the observed correlations.

Fine-tuning in quantum mechanics. Wood and Spekkens [4] showed that the observed conditional independences of a Bell experiment – no-signalling relations between the two labs – uniquely identify the Bell DAG as the only faithful causal explanation of the observed independences, yet the Bell DAG cannot classically reproduce Bell-inequality-violating correlations. Any classical causal explanation of such correlations must therefore be fine-tuned. Moreover, if the two labs corresponding to Alice and Bob in a Bell experiment are made time-like separated then the correlations between Alice and Bob's outcomes can be generated entirely locally. Being time-like separated Alice and Bob however have the same labs as when they were space-like separated and this motivates the assumption that the same no-signalling independences between the two labs still hold. Thus, the same distributions as when the labs were space-like separated can be produced, and the same violations of Bell inequalities persist. Bell-inequality-violating correlations require fine-tuning independent of the spacetime separation. This demonstrates that resorting to fine-tuning for explaining Bell inequality violating correlations using the language of classical causality is a distinct and important facet of quantum non-classicality.

Following [5, 6], we refer to a causal structure as Interesting if it can support non-classical correlations. In this work, we show that the class of Interesting causal structures exhibits a sharp division. For some, the observed conditional independences that characterise the causal structure are equally well exhibited by a Non-Interesting causal structure – a causal structure that can never support non-classical correlations. Since both explanations are generated by the same observed conditional independences, our Theorem 1 guarantees that any non-classical quantum correlations in the original Interesting causal structure are perfectly classically explainable in the Non-Interesting causal structure without any fine-tuning. The non-classicality of the observed correlations was an artifact of having assumed the wrong causal explanation. For others, no such Non-Interesting alternative exists: the observed conditional independences uniquely force the Interesting causal structure, and *all non-classical correlations in it admit no faithful* classical causal explanations.

The problem this paper addresses. The Bell causal structure is not the only causal structure of interest: the Triangle, Bi-locality, GHZ, and many others have been actively studied. Whether the Wood-and-Spekkens fine-tuning result extends to these causal structures, and which structures admit a faithful classical alternative has not been systematically investigated. Here we provide a complete treatment across all structures with up to six nodes.

II. FRAMEWORK: INTERESTING STRUCTURES AND THE IC* ALGORITHM

Henson, Lal, and Pusey (HLP) [5] define a causal structure (DAG) as *Interesting* (Non-Algebraic [6]) if it imposes inequality constraints on classically compatible distributions beyond just the constraints from observed conditional independences – Bell's inequalities being the canonical example. Only Interesting DAGs can support non-classical correlations. However, every latent-free DAG, H is *Non-Interesting* (Algebraic [6]) – that is it cannot support any non-classical correlations. Hence, every quantum correlation in it has a perfect classical causal explanation and there is nothing non-classical to exhibit.

Here, we use the *IC* algorithm* [2, 3] to search for Non-Interesting alternatives to Interesting DAGs. IC* takes observed conditional independences as input and returns a *pattern*: a graph encoding all causal structures that faithfully explain those observed independences across all possible causal orderings of the variables. The key point for our results is: *if a latent-free Non-Interesting structure appears in the returned pattern, corresponding to the observed independences supported by the Interesting structure, for any possible causal ordering of the variables, then the non-classicality of the original Interesting structure is spurious.* Because IC* accepts only conditional independences, it is blind to the strength of the correlations – but this is precisely the right question, since we ask whether a faithful classical structure exists in principle, regardless of the strength of the observed correlations. On the other hand, if the Interesting causal structure is the only candidate exhibiting the corresponding observed independences, then such an Interesting structure admits genuine non-classicality, since just observing the conditional independences corresponding to it in an experiment imply that not all possible correlations supported by it can be explained faithfully.

Theorem 1. *Let G be an Interesting structure with non-classical quantum correlations and certain observed conditional independences. Let H be a Non-Interesting latent-free structure with exactly the same observed conditional independences. Then every quantum correlation non-classical in G is perfectly classically explainable in H, without any fine-tuning.*

The proof follows because one can show that the whole set of quantum distributions compatible with G lies inside the set of distributions classically compatible with H. We apply IC* to all Interesting causal structures from Appendix E of HLP [5] with up to six total nodes having observed conditional independences and check, for each, whether the pattern includes any Non-Interesting alternative under any possible causal ordering of the variables.

III. RESULTS

Spurious non-classicality. For five of the seven Interesting structures in Appendix E of [5] having observed conditional independences, IC* returns a pattern that includes a latent-free Non-Interesting alternative under at least one possible causal ordering of the variables. By Theorem 1, non-classical quantum correlations in these structures are *spurious*: they can be re-explained classically without any fine tuning by adopting the right latent-free Non-Interesting causal model. Critically, this conclusion is robust to all possible causal ordering choices – the latent-free alternative persists even when one exhausts all possible orderings of the variables, so it cannot be argued away by reassigning which variable occurs first. The non-classicality in such cases is entirely a consequence of having assumed an overly rich causal hypothesis.

Consider the particular Interesting structure (Figure 2, left) with observed conditional independence $\{C \perp\!\!\!\perp D\}$. Three distinct Interesting structures share this independence; IC* returns a pattern from which the single latent-free Non-Interesting DAG (Figure 2, right) is a valid faithful alternative for all three. A further Interesting structure with observed conditional independence $\{C \perp\!\!\!\perp E|D\}$ (Figure 3, left) is similarly covered by a different Non-Interesting alternative (Figure 3, right). In each case, any quantum correlations non-classical in the Interesting structure are classical in the latent-free Non-Interesting one, irrespective of how strong the quantum violation is.

This situation has a direct parallel in the history of quantum foundations. Einstein, Podolsky, and Rosen argued that certain quantum correlations – the EPR correlations – were not truly non-classical, because a local hidden-variable model suffices to reproduce them: they were correct. It is only the stronger Bell-inequality-violating correlations that resist classical explanation. Our spurious structures occupy precisely this analogous position: correlations that appear non-classical within the Interesting causal hypothesis are, upon inspection, perfectly consistent with a local classical model – the Non-Interesting latent-free DAG. Just as EPR correlations dissolve into classicality once the right causal model is adopted, the spurious non-classicalities we identify disappear when one considers the full space of causal explanations. Crucially, the Non-Interesting alternatives persist under every possible causal ordering of the variables – so reassigning which variable occurs first provides no escape – and the non-classicality was not in the correlations themselves, but in the causal assumption.

Genuine non-classicality. For the remaining two Interesting structures, IC* returns only the original Interesting

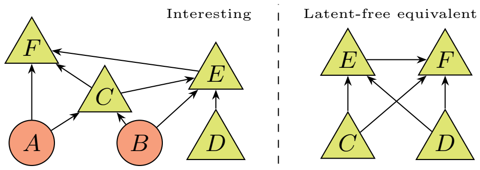

Figure 2: Spurious case 1. Left: Interesting structure with $\{C \perp\!\!\!\perp D\}$. Right: latent-free Non-Interesting alternative with the same observed conditional independence.

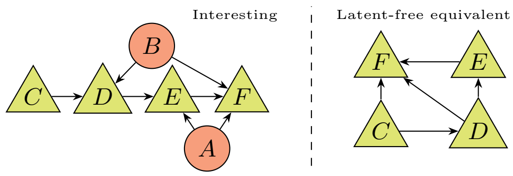

Figure 3: Spurious case 2. Left: Interesting structure with $\{C \perp\!\!\!\perp E|D\}$. Right: latent-free Non-Interesting alternative with the same conditional independence.

structure – no latent-free alternative exists under any causal ordering. Non-classical (quantum correlations) in these structures are *genuinely* irreducible: there is no causal structure of any kind that can explain them classically without fine-tuning. The Wood-and-Spekkens conclusion extends to each of these cases in full force.

What makes these structures genuinely interesting is that no restructuring of the causal model provides a faithful classical explanation: no additional edges, no alternative latent structure, no change of causal ordering, no appeal to time-like separations offers a rescue. The non-classicality here is a provable consequence of the logical structure of the causal relationships. These are the structures for which quantum mechanics is genuinely in tension with faithfulness, not as an accident of causal model choice, but as a feature of nature.

Further, the *Bi-locality structure* [11] (Figure 4, left) describes entanglement swapping: two independent latent sources A and B distribute entanglement, with one particle from each meeting at a central node D and the other reaching outer nodes G and H. The observed no-signalling conditional independences are so constraining that the IC* can only return the Bi-locality DAG itself. Similar is the result for the *GHZ structure [7]* (Figure 4, right). Any correlations non-classical in these structures have no faithful classical explanations anywhere and any experiment realising non-classical GHZ or Bi-locality correlations cannot be faithfully explained classically – regardless of the spacetime arrangement of the labs. Fine-tuning is not a modeling choice here; it is an unavoidable feature of nature. Building on the GHZ result, we prove the following general theorem.

Theorem 2. *There are no classically causal faithful explanations of Bell-inequality-violating correlations in any multi-partite Bell causal structure with* n *independent measurement settings. Their classical causal explanation necessarily requires fine-tuning.*

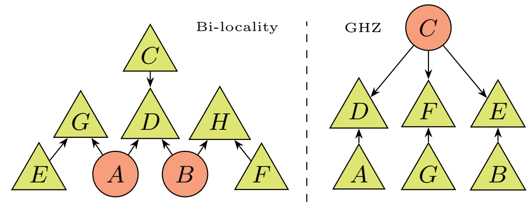

Figure 4: Genuine non-classicality. Left: Bi-locality structure. Right: GHZ structure. No Non-Interesting alternative exists for them.

Our proof of the above theorem extends Wood-and-Spekkens [4] fine-tuning result to all multi-partite Bell structures simultaneously, closing the question for that entire class.

IV. SUMMARY AND IMPORTANCE OF RESULTS

Our results establish a sharp structural division within the space of non-classical quantum correlations.

Spurious vs. genuine non-classicality. Some quantum correlations appear non-classical only relative to a particular causal hypothesis. Whenever a Non-Interesting DAG faithfully reproduces the same conditional independences as the Interesting structure in question, the non-classicality of the observed correlations disappears upon switching causal models. This means the correlations themselves are not intrinsically non-classical – they are entirely consistent with classical physics, and the appearance of non-classicality was a consequence of an impoverished causal assumption. In this work we find examples of such structures (Figures 2,3).

By contrast, for the genuinely non-classical structures – Bell, Bi-locality, GHZ, all multi-partite Bell structures and *some other structures from Appendix E of [5]* – no causal ordering, no latent structure, and no causal model can explain the correlations classically without fine-tuning.

Fine-tuning as a tool for causal discovery. Our work reiterates a fundamental limitation of IC*: it is blind to correlation strength. We propose supplementing it with Fraser's algorithm [10], showing that correlation strength must be taken into account in causal discovery – a lesson extending beyond quantum foundations to causal inference in statistics and machine learning.

Connections to causal inference. Our work thus contributes both to quantum foundations – by establishing fine-tuning as a fundamental marker of non-classicality across a wide class of structures – and to causal inference, where knowing whether a causal model can be falsified by inequality constraints rather than only by independence tests has direct implications for causal discovery and instrumental variable analyses, tightening the connection between quantum resources and the structure of causal explanations.

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1 Abstract

Bell's eponymous theorem implies that Quantum Mechanics is incompatible with local causality. This leads to tension between Relativity and Quantum Theory because certain quantum correlations cannot be explained locally. Non-classicality of such quantum correlations is often realized to arise because of their non-local nature. Making use of the framework of classical causality and causal discovery algorithms, Wood and Spekkens have shown that for the Bell causal structure Bell inequality violating correlations cannot be explained causally without resorting to fine-tuning. Here we follow Wood and Spekkens and use the framework of classical causality to study the non-classicality of observed correlations for other arbitrary causal structures. For our results we apply the IC^* algorithm to observed conditional independences corresponding to certain causal structures and show that there are several other causal structures apart from the Bell structure which possess non-classical correlations that need not necessarily be non-local, but their causal explanation necessarily requires fine-tuning. We show that requiring fine-tuning to be explained classically is another interesting feature of the non-classicality of such correlations apart from just the fact that they could be non-local. Moreover, we show that the Bi-locality, the GHZ, and the general multi-partite Bell causal structures are also examples of such structures. On the other hand for a number of causal structures, which exhibit non-classical quantum correlations, we show that we can always explain such non-classical quantum correlations perfectly classically– using instead another causal structure which has the same observed conditional independences. The non-classicality of the observed correlations was an artifact of having assumed the wrong causal explanation. This work also validates the observation that causal discovery algorithms that try to reproduce a causal hypothesis for a given set of data must also look into properties of the data other than just the observed conditional independences present in it. A candidate for such a property can be the strength of correlations that are allowed to be possible. Since causal discovery finds applications in Machine Learning and Artificial Intelligence, our work may therefore be of relevance to these fields as well.

2 Introduction

Studying causal relations- i.e., what causes what and how extensively, has been of interest since centuries. Nevertheless this study of causal relations- causality, has been shrouded with confusion since then [1]. Only recently has the concept of causality been given a rigorous mathematical framework [2, 3, 4]. Causal relations unlike correlations are asymmetric dependencies of some variables on others. For example certain negative behavioural changes in an individual can be correlated with depressing environmental factors but it is the environmental factors that *cause* the behavioural changes and not the other way round. Causal relations between variables can be understood by intervening on them unlike correlations which do not require any external interventions.

Causal discovery is the study responsible for generating possible causal models representing the properties corresponding to some observed data, usually conditional independences. Causal discovery algorithms are responsible to answer this problem. They return possible causal models corresponding to the particular observed conditional independences in the data. Causal discovery algorithms constitute a major active area of research in the fields of Machine Learning and Artificial Intelligence. There are several causal discovery algorithms like IC^*, IC (Inductive Causation) [2, 5, 6], PC (Peter-Clarke) [3], CI (Causal Inference) [7]. However, in this paper we use only the IC^* algorithm to study several conditional independences exhibited by certain causal scenarios.

Any causal hypothesis about some cause and its effects needs a strong experimental backing. This can be done by studying the probability distributions obtained in an actual realization of such an experiment. These probability distributions might possess certain conditional independences and correlations which we would like to be explained by our hypothesis causally. If our hypothesis is able to explain such conditional independences and correlations consistently then indeed our understanding of what is causing the observed effect is along the right direction. Often though, we want to test our belief in a certain causal scenario by asking questions such as - "What are the possible causal scenarios that can possess the particular set of conditional independences observed in the experiment ?" . In answering such questions we fall back on the causal discovery algorithms such as the IC^*.

Asking these questions for certain scenarios is the subject of this paper. We consider certain causal scenarios for which we know that d-separation [8] relations alone are certainly not the only constraints restricting the set of probability distributions that are Markov (compatible) with respect to them. Such scenarios have been termed as Interesting previously in the literature [9]. Analogously, causal scenarios for which d-separation relations alone are the only constraints restricting the set of probability distributions that are Markov (compatible) with respect to them have been termed as Non-Interesting [9]. The Interesting scenarios are the potential candidates for exhibiting non-classical correlations. The motivation here is to check whether all the Interesting scenarios are the only possible scenarios that can model the corresponding observed conditional independences without any fine-tuning or are there some Non-Interesting causal scenarios that can also potentially explain the respective conditional independences without resorting to fine-tuning. If so, can the quantum correlations[1] which are not classically explainable in an Interesting scenario be actually explainable classically using another Non-Interesting scenario which gives rise to the same observed conditional independences as the Interesting one?

We input the observed conditional independences corresponding to these Interesting causal scenarios into the IC^* algorithm. Since conditional independences are the only required input that the algorithm needs, we cannot test those Interesting causal scenarios which do not have

[1] Throughout this paper we consider only those quantum correlations which have the only observed conditional independences that are generated from the observed d-separation relations in the causal scenario.

any conditional independences because for all such scenarios (Interesting or Non-Interesting), the IC^* algorithm will just return a completely connected graph. For many of the scenarios that we test the algorithm returns that the observed conditional independences can be modelled by both an Interesting and a Non-Interesting scenario. These Interesting scenarios support [10, 11] quantum correlations which cannot be modeled in any classical way if we restrict to explaining them using the Interesting scenario itself without resorting to any fine-tuning. But as we show, we can model such quantum correlations in a perfectly classical way by changing our causal explanation and resorting to the Non-Interesting scenario being the correct causal model. This is because the Non-Interesting scenario can also model the observed conditional independnences without fine-tuning. In Section 7 we provide explicit examples of such quantum correlations.

Nevertheless, for some scenarios, the algorithm returns that the observed conditional independences can be explained by only an Interesting causal scenario. Any quantum correlations that are not classically explainable using these Interesting scenarios will also not be classically explainable by *any other causal scenario*. Amongst examples of such scenarios we find the Bi-locality [12, 13] scenario of entanglement swapping and the GHZ scenario. Building on from the GHZ scenario [14, 15], we also prove a general theorem, Theorem 3, showing that scenarios with an arbitrary number of independent parties, i.e., *multi-partite Bell scenarios*, are also examples of such scenarios.

Thus, certain correlations corresponding to these causal scenarios cannot be explained using classical causality or equivalently using any classical theory without resorting to fine-tuning. The important thing to note is that such non-classical correlations need not be non-local, their explanation just requires a particular fine-tuning. And thus being fine-tuned is another important characteristic of these non-classical correlations apart from them being potentially non-local.

Previously the same has been studied for the Interesting Bell scenario [16] by Wood and Spekkens. They showed that all causal explanations of Bell inequality violating correlations require fine-tuning. Causal discovery algorithms cannot differentiate between Bell inequality violating and Bell inequality respecting correlations. This is because the causal discovery algorithms take only the conditional independences as inputs so necessarily they cannot distinguish between probability distributions possessing different possible strengths of correlations. That the strength of correlations observed in an experiment also plays an important role in our causal explanation of it is another of their conclusions.

Hence, there are scenarios beyond the Bell scenario which not only can just support non-classical correlations, but they are the only possible scenarios which can explain the corresponding experimentally observed conditional independences. That is, modelling these particular conditional independences will necessarily lead to non-classical correlations. On the other hand, there also exist scenarios which support non-classical correlations, but these non-classical correlations can be modelled classically by explaining them using a different (Non-Interesting) scenario.

Structure of the Paper

We first introduce the mathematical framework of classical causality that is necessary to understand our results in Section 3. Then we explain what are non-classical correlations and Interesting scenarios in Sections 4, 5. In Section 6 we provide a very brief overview of the IC^* algorithm, moving on to our results in Section 7. Section 8 presents our main theorem 3 which states that there are no causal explanations of Bell inequality violating correlations in any multi-partite Bell scenario with n independent settings. Finally, Section 11 provides a summary of our conclusions.

3 Preliminaries

3.1 Causal Structures using Classical Bayesian Networks

The area of Causal Inference is concerned with finding potential causal explanations for observed events. For example, imagine that we want to find out what is the causal relationship between three events: a particular disease, a drug treatment for it and financial expenditures. Figure 1 depicts two possible causal structures between these three events. In Figure 1(a) we hypothesize that the disease causes one to undergo the drug treatment and the drug treatment can possibly lead to heavy medical expenditures, or that a patient might just consult a doctor several times which results in his expenditures increasing significantly, but the doctor does not put him on a drug treatment, if deems his immune system to be particularly strong. In Figure 1(b), on the other hand, we hypothesize that financial expenditures leads to drug treatment which in turn leads to the disease, or that financial expenditures lead to the disease directly. There is an

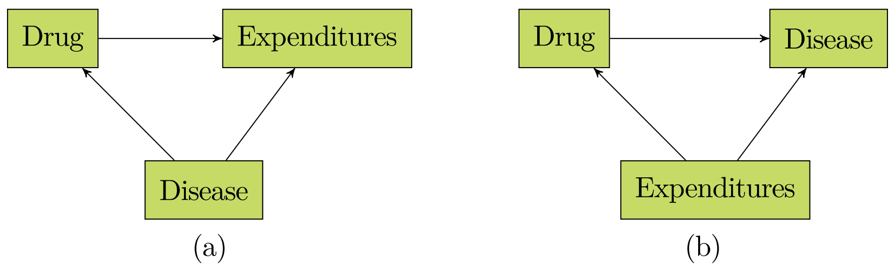


Figure 1: Two hypothesis for the causal relationships between observing a disease, its drug treatment and financial expenditures. By intervening on the experiment we can attest that (a) is the correct explanation, but we cannot attest that if we only passively look at the correlations between those events.

easy way through which we can check that Figure 1(a) is the correct causal hypotheses: if we consider a person who recently buys a property and a house, his expenditures would be quite increased but that would not give him the disease under consideration.

This approach assumes that we can manipulate the experiment by intervening and controlling one of its variables, such as the "Expenditures" variable in Figure 1, to achieve a desired outcome. However, this might not be always feasible, as ethical, situational or technical limitations may prohibit such interventions. In situations where manipulation of variables is not possible, researchers can still draw conclusions by passively observing the correlations between events of interest.

Each causal structure has its own limitations on the selection of probability distributions that can be explained by it through passive observation. These constraints can be tested against the observed probability distribution to determine if the causal structure is a valid hypothesis for the observed phenomena. In Figure 1 it happens that both causal structures impose the same constraints on observed probability distributions[2] so they are not distinguishable by passive observations.

Hence causal discovery algorithms will return both these causal scenarios as possible explanations of the observed conditional indpendences. If we would further need to choose one scenario as the best possible explanation from the bunch of scenarios returned by causal discovery algorithms we would need to resort to other assumptions, like a causal order on the variables.

[2] Any probability distribution that is compatible with the causal structure in Figure 1 (a) is also compatible with the causal structure in Figure 1 (b) and vice-versa.

3.2 Directed Acyclic Graphs

We use the mathematical framework where causal structures are represented by directed acyclic graphs (DAG).

Definition 1 (Directed Graph)**.** A directed graph G consists of two things-:

- a set of nodes A
- a set of directed edges $E \subseteq A \times A$.

A directed path in a directed graph is a sequence of nodes $X^1, X^2, X^3 .... X^n$ such that $X^i \to X^{i+1}$ for $i = 1, ..., n$.

Having defined what a directed path is, we can now define directed acyclic graphs (DAGs).

Definition 2 (Directed acyclic graphs (DAGs))**.** A directed acyclic graph (DAG) is a directed graph that does not have any directed cycles.

Below we introduce some definitions related to DAGs that will be helpful later on.

Definition 3 (Children, Parents, Descendants, Ancestors)**.** Let X be a node of a DAG G.
If Y is another node of G such that there is a directed edge $X \to Y$, then Y is called a *child* of X, and X is called a *parent* of Y. The set of all children of X is denoted $CH(X)$, and the set of all parents of X is denoted $\mathrm{PA}(X)$. The *descendants* of X in G are all the nodes in G that can be reached from X by a directed path. Conversely, all the nodes that have X as a descendant are called *ancestors* of X. The set of all nodes that are not *descendants* of X in G are called *non-descendants* of X. Note that the set *ancestors* of X is a subset of the set of *non-descendants* of X.

For example consider the DAG of Figure 2. In it we have that $\mathrm{PA}(B) = \{A, D, C, F\}$, and that E is a descendant of D, even if it is not its child. F is a non-descendant of A even though F is not an ancestor of A. This is a valid DAG since it does not contain any directed path both beginning and ending at the same node (i.e a directed cycle). Also, note that we draw the nodes as triangles.

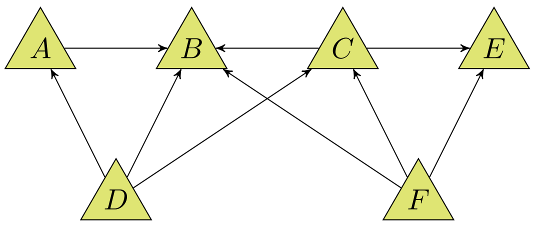


Figure 2: First example of a directed acyclic graph (DAG). The probability distributions that are classically compatible with this DAG are those that can be decomposed as in Equation (2).

3.3 Classical Bayesian Networks

Each node in a directed acyclic graph (DAG) can represent an event of interest, making the DAG a representation of a causal structure. An edge $X \to Y$ indicates the possibility of a direct causal influence of X on Y. If all events of interest are described by classical random variables, the notion that a probability distribution can be causally explained by a certain DAG is captured by the *(Local) Markov condition*.

Definition 4 (**Markov Condition**)**.** Let G be a DAG with nodes A. A probability distribution P over the variables A is said to be *Markov with respect to* G iff it can be factorized as:

$$P(A) = \prod_i P(x^i|\mathrm{PA}(x^i)) \tag{1}$$

Where $A = \{x^1, \ldots\ldots, x^n\}$ and $\mathrm{PA}(x^i)$ is the set of parents of the node x^i in G. Further, the probabilities $P(x^i|PA(x^i))$ are called the causal statistical parameters for the respective node x^i.

If P is Markov with respect to G, we also say that it is "classically compatible" with G. If P cannot be factorized as in Equation (1) then it is *not* "classically compatible" with the DAG G.

As a first example of this definition, a joint probability distribution P_{ABCDEF} over the random variables A, B, C, D, E and F is Markov with respect to the DAG of Figure 2 if it can be decomposed as:

$$P_{ABCDEF}(a,b,c,d,e,f) = P_{A|D}(a|d) P_{B|A,D,C,F}(b|a,d,c,f) P_{C|D,F}(c|d,f) P_D(d) P_{E|C,F}(e|c,f) P_F(f) \tag{2}$$

Any probability distribution over A,B,C,D,E,F which can be factorised as in Equation 2 is calssically compatible with the DAG of Figure 2. Equivalently, any probability distribution over A,B,C,D,E,F which cannot be factorised as in Equation 2 is classically incompatible with this DAG. Therefore, through the Markov condition, each DAG imposes constraints on the probability distributions that are classically compatible with it.

3.4 d-separation: A graphical criterion for conditional independence

If P is a probability distribution over a certain set of random variables A, we say that the variables of the subset $X \subseteq A$ are *conditionally independent* of the variables of the subset $Y \subseteq A$ given the subset $Z \subseteq A$ if P can be factorized as $P(X,Y|Z) = P(X|Z)P(Y|Z)$ [3]. This is denoted by $X \perp\!\!\!\perp Y|Z$.

A DAG puts certain constraints on probability distributions that are compatible with it, some of which are in the form of conditional independences. As explained in Section 3.3 in cases where a probability distribution *cannot* be factorized based on the conditional independences coming from the Local Markov condition imposed by the DAG, it is not considered classically compatible with the DAG. To test the compatibility of a probability distribution with respect to a DAG, we can either verify the Markov condition or resort to graphical conditions, the d-separation conditions, that can be used to obtain all of the conditional independence constraints imposed by a DAG on the probability distributions that are compatible with it.[4] We describe these conditions now.

Definition 5 (d-separation)**.** If G is a DAG with nodes A and $X,Y,Z \subseteq A$ are sets of nodes in G, the d-separation algorithm says whether X and Y are "d-separated" by Z. X and Y are "d-separated" by Z iff all the undirected paths (paths that ignore the direction of the arrows) from X to Y are *blocked* by Z. A path is blocked iff one or more of the following is true:

1. There is a chain of nodes along the path: $i \to m \to j$ such that, $i \in X, j \in Y$ and $m \in Z$.

2. There is a fork along the path: $i \leftarrow m \to j$ such that, $i \in X, j \in Y$ and $m \in Z$.

[3]From here on, for simplicity, we will occasionally denote probabilities by the concise notation $P(XY) \equiv P_{XY}(X=x, Y=y)$.

[4]Note that till now all the nodes in graph are such that their corresponding probability distributions are known.

3. There is a collider along the path: $i \to m \leftarrow j$ such that, $i \in X, j \in Y$ and $m \notin Z$ and $d \notin Z$ for all the descendants d of m.

If X is d-separated of Y given Z in the DAG under consideration, we denote it as $X \perp Y|Z$. If Z, is the empty set we simply denote that X is d-separated of Y by $X \perp Y$.

For example, for the DAG of Fig. 3 the d-separation criterion says that $A \perp C|B$ and $B \perp E|A,C$. This means that the event $A\,(B)$ should become independent of the event $C\,(E)$ upon knowledge of $B\,(A,C)$. So if for example a distribution P *does not* satisfy the constraint $P(AC|B) = P(A|B)P(C|B)$, then it is not compatible with the DAG of Fig. 3 and thus Fig. 3 is *not* a valid causal explanation for it.

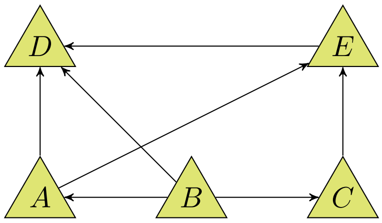


Figure 3: According to the d-separation criterion, this DAG imposes $A \perp\!\!\!\perp C|B$ on the probability distributions classically compatible with it.

The following theorem, proven in [17], makes explicit that the d-separation criterion is indeed complete to find all the conditional independence constraints that a DAG imposes on the classically compatible probability distributions when there are no latent variables. This is because in the absence of latent variables d- separation conditions are equivalent to the (Local) Markov conditions.

Theorem 1 (d-separation and conditional independence)**.** *Let G be a DAG with nodes A, and let $X = \{x_1, \ldots\ldots, x_n\}$, $Y = \{y_1, \ldots\ldots, y_n\}$ and $Z = \{z_1, \ldots\ldots, z_n\}$ be three disjoint subsets of A. Then:*

1. *If P is a probability distribution over the variables A that is Markov with respect to G, then $X \perp Y|Z$ implies that $X \perp\!\!\!\perp Y|Z$ in P.*

2. *If $X \perp\!\!\!\perp Y|Z$ for* every *probability distribution that is Markov with respect to G, then $X \perp Y|Z$.*

Note that from Theorem 1 every d-separation relation implies a conditional independence on the classically compatible probability distributions, but *not all* conditional independence constraints presented by a specific classically compatible probability distribution P have to be associated with corresponding d-separation relations of the DAG.

Remark 1. In a DAG without any latent variables, it is possible to verify the Markov condition without relying on the d-separation conditions. This is because the probability distribution across all the nodes in the graph is known. Thus by marginalizing the joint probability distribution, the probabilities for each node can be obtained and the Markov condition can be confirmed. In other words, in the absence of latent variables, the Markov condition implies the observed[5] d-separation relations and vice versa. However, when latent variables are present, satisfying the observed[6] conditional independences may not be sufficient to satisfy the Markov condition. This is because the observed d-separation relations do not necessarily imply the Markov condition when latent variables are present.

[5] By observed d-separation relations we mean d-separation relations involving only the observed nodes.

[6] Observed conditional independences are those which involve only the observed nodes.

3.5 Latent Variables and Marginal Models

Sometimes there might be variables in an experiment which we cannot measure but we expect them to influence the experiment. In order to provide a causal explanation of such an experiment which could be dependent on certain events which we do not observe, we need to consider *latent* or *hidden* variables. These "latent variables" model the unobserved events as "latent" nodes in a DAG. "Latent variables" are different from observed variables, which are the ones that appear in the empirical probability distribution in the sense that we do not know the probability distributions of these "latent variables". The term "latent" or "hidden" is used to emphasize that these variables that are not directly measurable and we don't know their corresponding probability distributions.

Here, in a DAG, we will represent the nodes associated with latent variables by circles, and the nodes associated with observed variables by triangles. Accordingly, we will call them "latent nodes" and "observed nodes". A first example of a DAG that has these two types of nodes is depicted in Figure 4. This causal structure is called "Bell DAG".

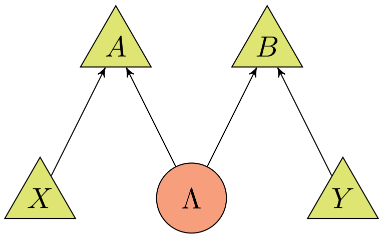


Figure 4: The Bell DAG. It encompasses the assumptions of Bell's theorem for a Bell Scenario where X and Y are the measurement settings of Alice and Bob, A and B are their outcomes and Λ is a classical hidden variable. The probability distributions that are classically compatible with this DAG are those that decompose as in Equation (3).

Definition 6 (Marginal Models)**.** Let G be a DAG that has the set of nodes $A = V \cup L$, V and L disjoint, where V are the observed nodes and L are the latent nodes. Borrowing the terminology used for the case without latent variables, we will say that a probability distribution $P(V)$ over the observed variables V is *classically compatible with* G if there exists some probability distribution $Q(V,L)$ over all nodes such that:

- $Q(V,L)$ is Markov with respect to G.
- The marginal of $Q(V,L)$ over V is the original probability distribution that we were interested in, i.e. $\sum_L Q(V,L) = P(V)$.

As an example, a probability distribution P_{ABXY} over the random variables A, B, X and Y is classically compatible with the DAG of Figure 4 if it can be decomposed as:

$$P_{ABXY}(a,b,x,y) = \sum_\lambda Q_{ABXY\Lambda}(a,b,x,y,\lambda) = \sum_\lambda Q_{A|X,\Lambda}(a|x,\lambda) Q_{B|Y,\Lambda}(b|y,\lambda) Q_X(x) Q_Y(y) P_\Lambda(\lambda) \tag{3}$$

When a directed acyclic graph (DAG) only includes observed variables V, the only constraints that it imposes on probability distributions that are compatible with it are the conditional independence constraints, as described in Theorem 1. Therefore, in this case, the d-separation criterion alone is enough to identify all of the constraints that the DAG imposes on the probability distribution $P(V)$.

If a directed acyclic graph (DAG) G includes latent variables L, using the d-separation criterion over *all* the nodes can identify *all* the constraints that the DAG imposes on the joint probability distribution over *all* the nodes, $Q(V,L)$. However, since our empirical probability distribution is the marginal $P(V) = \sum_L Q(V,L)$, we are only concerned with the distribution over the observed variables V. In this case, the conditional independences between the observed variables V are not the only constraints on $P(V)$. The conditional independence constraints that involve the latent variables L can lead to additional complex constraints on the probability distribution over just the observed variables V, such as Bell's inequalities, as shown in Section 5. A probability distribution $P(V)$ is classically compatible with G if it satisfies both the conditional independences between the observed variables V and the extra constraints on those observed variables resulting from the conditional independences involving the latent variables L.

Remark 2. In summary, it is possible to obtain all the d-separation relations that a DAG with both observed nodes V and latent nodes L implies, which provides conditional independence constraints on $Q(V,L)$. From there, one can infer the constraints on the probability distribution over the observed variables, $P(V)$. However, the process of inferring inequality constraints on $P(V)$ from the conditional independences on $Q(V,L)$ is not always straightforward, as we will illustrate in the case of the Bell DAG in Section 5.

Throughout this text, when we refer to the d-separation relations of a DAG that has latent nodes we will in general be referring to the d-separation relations that *only involve observed nodes*, unless otherwise stated.

3.6 Faithfulness and Minimality

When we analyse the IC^* algorithm in Section 6, we will see that it returns "faithful" [2] causal scenarios corresponding to the input observed conditional independences. So it is important to understand what the assumption of "faithfulness" (also called no fine-tuning) means.

From Theorem 1 every d-separation relation implies a conditional independence on the classically compatible probability distributions, but *not all* conditional independence constraints presented by a specific classically compatible probability distribution P have to be associated with corresponding d-separation relations of the DAG. There might not be any d-separation relation corresponding to a conditional independence. This leads us to the following definition.

Definition 7 (Faithfulness or no fine-tuning)**.** If the only conditional independences present in a probability distribution P, which is compatible with a DAG G, are those that follow from the d-separation relations in the DAG G, then we say that the DAG G explains it faithfully or with no fine-tuning. On the other hand if the probability distribution has some conditional independence corresponding to which there is no d-separation relation in the DAG G,[7] then we say that the DAG G is a fine-tuned explanation of P.

In other words all the conditional independences present in the probability distribution should be a consequence of the causal scenario alone. There should not be any extra conditional independences that are not explained by the d-separation relations in the DAG for it to be a faithful explanation of the probability distribution. For example, suppose that there is some conditional independence $X \perp\!\!\!\perp Y|Z$ that is observed in a distribution P compatible with a DAG G that *does not* correspond to a d-separation relation $X \perp Y|Z$ in G. If we adopt G as the causal explanation for P, this implies that there would be some correlations that for some conspiratorial reason do not appear in nature (because $X \perp\!\!\!\perp Y|Z$) , even if they are allowed

[7] Note that such a probability distribution will still satisfy the Markov condition because we are only considering distributions that are compatible with DAG G.

by the underlying causal structure, G (as $X \perp Y|Z$ does not hold in G). Then G could be considered as a bad causal explanation [16] of this probability distribution, and there could be other DAGs that explain all the conditional independences in this probability distribution through their corresponding d-separation relations.

As an explicit example, taken from [18], consider the DAG in Figure 5. In the DAG, there is a direct arrow from A to C, hence $A \perp C$ does not hold. *Any* probability distribution P is compatible with the DAG because by the definition of conditional probabilities *any* P can be written as $P(ABC) = P(A)P(B|A)P(C|AB)$. Therefore, the Markov condition is satisfied for any P and there are no d-separation relations in the DAG to impose any additional constraints on P. However, consider the distribution over all the nodes where A and C are independent, i.e $\sum_B P(ABC) = P(A)P(C)$. Such a distribution is compatible with the DAG because we saw that *any distribution* satisfies the Markov constraint. Further such a distribution could arise because the indirect causal influence from A to C via B could correlate A and C negatively whereas the direct influence from A to C could correlate them equally positively thereby rendering C independent of A. Here the DAG is a fine-tuned explanation of such a probability distribution.

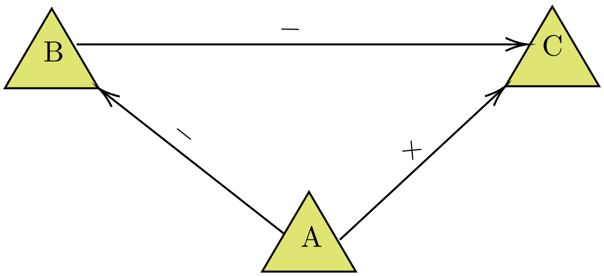


Figure 5: Direct causaul influence from A to C can lead to positive correlation between the two nodes (shown with $+$ over the causal arrow). Whereas indirect casual influence from A to C via B can lead to equivalent negative correlation between the two nodes (shown with $-$ over the causal arrows). The net result could be that A and C are independent of each other even though $A \perp C$ does not hold in the DAG. This is an example of fine-tuning.

Further conditional independence-based algorithms follow one more assumption, which is of minimality, which we explain now. To understand the assumption of minimality, it is important to understand when a causal scenario A *can simulate* another causal scenario B.

Definition 8 (Simulate)**.** In the context of causal modelling, a causal scenario A is said to simulate another causal scenario B, both on the same set of variables V, if for any possible causal-statistical parameters chosen for the latter scenario, there exists a corresponding set of causal-statistical parameters for the former scenario that produces the same probability distribution on the set of variables V.

Definition 9 (Minimality)**.** If two causal scenarios A and B produce the same probability distribution on a set of observed variables V, and B can simulate A on V but A cannot simulate B on V, then A is considered to be a better causal explanation for the probability distribution over V than B.

Choosing A over B even when B explains more might seem a bit counter-intuitive but the fact that A explains less than B makes A more falsifiable than B. So when choosing a causal scenario to explain the observed distribution we will choose the scenario which can be proven to be a wrong explanation more easily. This is in-fact what causal discovery algorithms do.

3.7 Classically causal faithful explanation of correlations

Now we have all the mathematical definitions needed to understand what a classically causal faithful explanation of any observed correlations is.

Definition 10 (Classically causal faithful explanation of any correlations)**.** Let P be any observed probability distribution over a set of observed nodes, V. Let there exist a DAG, G, over V observed nodes and L latent nodes. If P is both classically compatible and not fine-tuned with respect to G, then we say that G is a *classical causal faithful explanation* of P or the observed correlations. If P is just classically compatible with respect to G, then we say that G is a *classically causal unfaithful explanation* of P.

Notice that given a DAG G the only difference between a "classically compatible" probability distribution P with respect to G and a probability distribution P' whose "classically causal faithful explanation" is the DAG G, is just that P could be fine-tuned with respect to G, whereas P' has to be non fine-tuned with respect to G. This is because of the "Faithfulness" assumption introduced in Section 3.6.

For example, consider the Bell DAG of Figure 4 and consider any probability distribution P' which obeys Equation 3 and hence all the Bell inequalities. Further, consider that all the observed conditional independences in P' are the ones that are implied by the observed d-separtion relations in the Bell DAG. Then we say that the Bell DAG (Figure 4) is a classically causal faithful explanation of such a probability distribution P'. On the contrary, consider a probability distribution P which obeys Equation 3 and (thus) the Bell inequalities but has more conditional independences among the observed nodes than those implied by the observed d-separation relations in the Bell DAG. Then we say that the Bell DAG is a classically causal unfaithful explanation of P. An example of such a probability distribution is the probability distribution which violates the CHSH [19] inequality maximally.

3.8 Quantum correlations

We now present an intuitive definition for quantum correlations. For a more precise definition we refer the reader to the definition by Henson, Lal and Pusey in [9].

Definition 11 (Quantum correlations)**.** Consider a DAG, G, with nodes $A=V\cup L$, V and L being disjoint, and where V are the observed nodes and L are the latent nodes. Suppose, each latent node is associated with a quantum system. Then the correlations obtained in such a DAG are termed as quantum correlations.

An example will help illustrate the definition more clearly. Consider the Bell DAG of Figure 4. If the latent common cause, Λ, corresponds to a quantum state ρ_Λ, such that $\rho_\Lambda \in \mathcal{S}(\mathcal{H}_\Lambda) = \mathcal{S}(\mathcal{H}_{\Lambda_A} \otimes \mathcal{H}_{\Lambda_B})$ and not a classical random variable then any probability distribution over the observed nodes, $P(ABXY)$ that is quantum mechanically feasible and compatible with the Bell DAG must satisfy the following factorization:

$$P_{ABXY}(a,b,x,y) = \mathrm{Tr}((E_X^A \otimes E_Y^B)\rho_\Lambda) P_X(x) P_Y(y). \tag{4}$$

Here, the observed nodes A, B correspond to $POVMs$, $\{E_X^A\}$ and $\{E_Y^B\}$ that act on subsystems $\mathcal{H}_{\Lambda_A}$ and $\mathcal{H}_{\Lambda_B}$ and depend on the inputs X and Y respectively.

Throughout this paper we assume that the only conditional independences exhibited by the observed quantum correlations are the ones that follow from the observed d-separation relations of the corresponding DAG G.

4 Non-Classical Correlations

Correlation does not imply causation, but under the assumption of no fine-tuning causation does necessarily imply correlation. As shown in Figure 5, if A causes C, then it is not necessary that A and C will be correlated. Only if we assume no fine-tuning then a correlation between A and C is guaranteed if A causes C. On the other hand, any correlations we observe in a set of data[8] between two random variables A and B implies-:

1. Either A is the cause of B
2. Or B is the cause of A
3. Or A and B both share a common cause in the past.

Implication three is Reichenbach's common cause principle which says that if A and B are correlated and none is the cause of the other, then the correlation between the two is due to some unobserved common cause which affects both A and B. Any correlation observed between two variables must be explainable via the above three possibilities. This is the basic assumption of classical causality theory.

Explaining quantum correlations causally has been a program since Einstein, Podolski and Rosen [20] who showed that quantum mechanics on its own does not provide a local explanation of certain quantum correlations (EPR correlations). Further, it has been shown that these EPR correlations can be explained locally by adding hidden variables to quantum mechanics. The notion of locality here is that any cause of any effect should be in the past light cone of the effect i.e., causal influence must propagate at a speed less than or equal to the speed of light. However, soon Bell [21] was able to show that any theory reproducing the results of quantum theory would be necessarily non-local. He showed this by showing that certain quantum mechanical correlations (Bell correlations) cannot be explained locally, or that the causes of certain two observed correlated events, space-like separated, do not all necessarily lie in their past light cones. A pedagogical review of these results can be found in [22, 23, 24, 25, 26, 27]. The non-local nature of these quantum correlations (Bell correlations) is often interpreted to be the reason behind them being non-classical. Using the language of DAGs and Bayesian networks introduced in the last section we now explain what are non-classical correlations.

The causal scenario Bell considered is shown in Figure 6. Λ is the complete description (including any hidden variables) of all the real things that the theory under consideration posit to be "beables" [28] and which could influence the experiment under consideration. Λ may or may not include the quantum state depending on whether our theory posits the quantum state to be real or not. The important thing is that Λ is a complete description of all the things in the past light cone of the experiment.[9] A maximally entangled state is understood to be distributed to two labs, space like separated. The measurement settings for the labs are denoted by X and Y, respectively, and the corresponding outputs of the labs are A and B. Since the labs are space-like separated, Bell formulated the condition of locality we have described above in a mathematically rigorous way by the following factorization of the joint probability distribution over the outcomes:

$$P(A,B|X,Y,\Lambda) = P(A|X,\Lambda)P(B|Y,\Lambda) \tag{5}$$

Equation 5 just means that the probability of the outcome A is independent of the outcome B and the setting of B, i.e Y, once the setting of A, i.e X, and the complete description of

[8] Obtaining how the two random variables A and B are correlated is a different question and here we assume that if correlation is observed between A and B then it is not coincidental because coincidental correlations in a data set can be removed by performing the experiment a large number of times.

[9] Except the measurement settings which are assumed to be chosen randomly.

anything in the past light cone of A, i.e Λ is known. Similar condition holds for the outcome B. Under the assumption of measurement independence

$$P(X,Y,\Lambda)=P(X,Y|\Lambda)P(\Lambda)=P(X,Y)P(\Lambda) \tag{6}$$

and Equation 5, Bell derived an inequality which he showed was violated by certain correlations produced by using a maximally entangled bi-partite quantum state; thus showing that certain quantum correlations do not obey the above (Equation 5) notion of locality and are hence non-local.

But note that Bell's inequality would still be violated by these quantum correlations even if the labs are not space-like separated. This is because even if the labs are not space-like separated but we carry out the same experiment using the same apparatus as compared to when the labs were space-like separated, we can obtain the same probability distribution $P(A,B,X,Y)$ that we obtained when the separation between the labs and the settings was space-like. We can obtain the same probability distribution $P(A,B,X,Y)$ as before for two simple reasons:

1. The outcomes A and B do not change with the distance between the labs.
2. If the labs were not communicating when they were space-like separated, we expect them to not communicate even when they are time-like separated. Just because the labs are made time-like separated does not imply that they should necessarily communicate. Especially, if we use the same labs and the same experimental setups in both the experiments, we expect the same no-communication condition between the labs.

Hence Equation 5 which along with measurement independence leads to Bell's inequalities would still hold true and thus $P(A,B,X,Y)$ would still violate the Bell inequalities even when the labs are time-like separated. Such a causal set up is shown in Figure 7[10]. Nothing non-local is going on here and yet the Bell inequalities are violated. If in this setup we call the Bell inequality violating correlations as non-classical, it is evident that these non-classical correlations have characteristic properties beyond just non-locality.

On the other hand if a priori we see such correlations when the labs are time-like separated we can explain them locally by positing for example that the labs signalled to each other because now they could. But the important thing is that they would still violate the Bell inequality. Wood and Spekkens [16] have showed that all the explanations of Bell inequality violating non-local quantum correlations are fine-tuned. But the same is true for these locally explainable Bell inequality violating quantum correlations; they are fine-tuned as well (that is why they will violate the Bell inequality). Figure 8 shows a fine-tuned explanation of these local Bell inequality violating quantum correlations (which are non-classical nonetheless) Together these local and non-local Bell inequality violating quantum correlations form a set of Bell inequality violating correlations that we call non-classical because they cannot be explained using the framework of Classical Causality theory (that is both will require some kind of fine tuning for their causal explanation).

The important takeaway is that non-locality is not the only characteristic property of these non-classical correlations. That they are fine-tuned is another important property that these non-classical correlation posses even if they are not non-local. We show that the same is the case for other scenarios apart from the Bell scenario in Section 7. They possess non-classical correlations which cannot be explained classically without resorting to fine-tuning, irrespective of whether measurements are done over space-like separated or time-like separated labs (and thus irrespective of whether the correlations are local or non-local in nature).

[10] In the Figures 6 and 7, it is assumed that Λ does not influence X and Y and that the measurement settings are chosen randomly

Just like the non-locality of these non-classical correlations creates a conflict between Relativity and Quantum Mechanics, similarly the fact that their causal explanation requires fine-tuning (even if they are local and hence do not create a conflict between Relativity and Quantum Mechanics) creates a tension between Quantum Mechanics and Classical Causality theory.

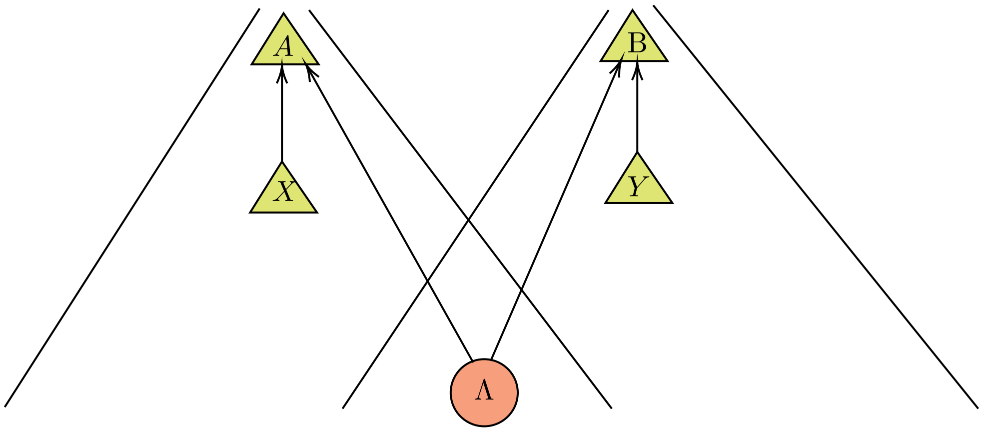


Figure 6: Bell DAG embedded in space-time where a non-classical $P(A,B,X,Y)$ is non-local and fine-tuned. $P(A,B,X,Y)$ is non-classical because it can violate Bell's inequality. It is non-local because two space-like separated nodes are correlated and the correlation is not coming completely from their common past light cone but from somewhere outside their past light cones. It is fine-tuned because any causal model that can explain it will require some kind of fine-tuning.

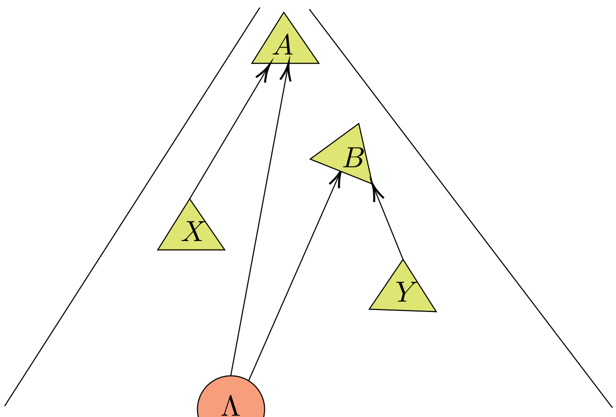


Figure 7: Bell DAG embedded in space-time where a non-classical $P(A,B,X,Y)$ is local but fine-tuned. $P(A,B,X,Y)$ is non-classical because it can again violate a Bell-inequality. It is local because both the outcomes and settings of the opposite parties are time-like separated and so all causal influences can propagate subluminally. It is fine-tuned because any causal model that can explain it will require some kind of fine-tuning.

5 What is an Interesting causal scenario

As discussed in Section 3.5, the d-separation criterion gives us all the classical compatibility constraints imposed on the probability distribution over *all the nodes*, $P(V,L)$. In the case of the Bell DAG, we have $P(V,L) = P(ABXY\Lambda)$, while $P(V) = P(ABXY)$.
Let G' be the DAG depicted in Figure 9, where all the nodes (including Λ) are observed. Note that if a certain $P(ABXY\Lambda)$ is classically compatible with G', then its marginal $P(ABXY) = \sum_\Lambda P(ABXY\Lambda) = \sum_\Lambda P(ABXY|\Lambda)P(\Lambda)$ is classically compatible with the Bell DAG.

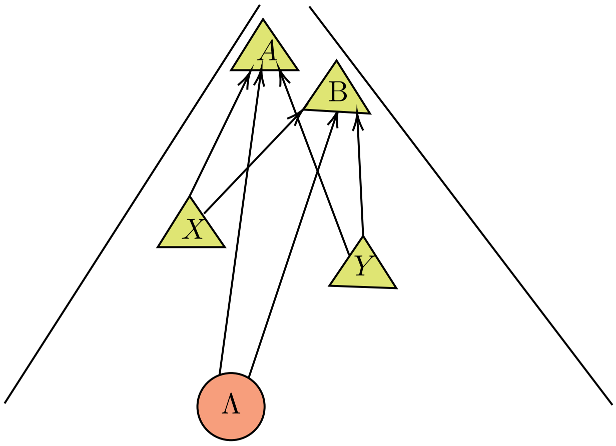


Figure 8: A fine-tuned and local explanation of non-classical $P(A,B,X,Y)$ Bell-inequality violating quantum correlations. $P(A,B,X,Y)$ can again be non-classical (in the Bell scenario) because it can violate Bell's inequalities. Though it is local because all the nodes are time-like separated and thus support subluminal signals between them. It is fine-tuned since corresponding to the observed conditional independence $B \perp\!\!\!\perp X|Y$ there is no corresponding d-separation relation $B \perp X|Y$. Via this unfaithful scenario, de-Broglie-Bohmian mechanics can explain Bell inequality violating local or non-localquantum correlations. Note that $P(A,B,X,Y)$ is not non-classical in the DAG of Figure 8, it is just fine-tuned.

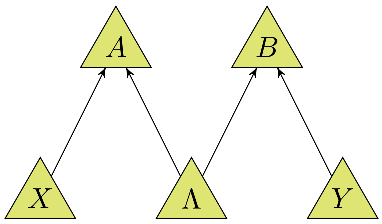


Figure 9: A causal structure with five observed nodes and no latent nodes, here called G'.

The conditional independence relations of $P(ABXY\Lambda)$ that come from the d-separation relations of G' are:

$$P(\Lambda|XY) = P(\Lambda) \tag{7}$$

$$P(AB|XY\Lambda) = P(A|X\Lambda)P(B|Y\Lambda) \tag{8}$$

$$\begin{cases} P(A|XY) = P(A|X) \\ P(B|XY) = P(B|Y) \\ P(XY) = P(X)P(Y) \end{cases} \tag{9}$$

All of the constraints imposed by DAG G' on the compatible probability distributions are given by the conditional independence relations that result from d-separation. According to the result in [17], these conditional independence relations are the only constraints that are imposed on the compatible probability distributions by DAGs that do not have latent variables.

Equations (7)-(9) provide the constraints imposed by the Bell DAG on $P(ABXY)$ by marginalizing Λ. Equation (9), which does not involve Λ, is automatically translated as observed-conditional independence constraints imposed by the Bell DAG on $P(ABXY)$. Equations (7) and (8), which involve the latent variable Λ, give rise to another constraint on $P(ABXY)$, namely Bell's inequality. Equation (7) encodes the no-superdeterminism assumption (measurement independence), and Equation (8) encodes the local causality assumption, which are used to derive Bell's inequality.

Therefore, to fully study the Bell DAG, one needs to examine not only the conditional independence constraints on the compatible distributions $P(ABXY)$ (Equations (9)), but also Bell's inequality, because the set of probability distributions $P(ABXY)$ that satisfy only the conditional independence relations of Equations (9) is larger than the set of probability distributions that satisfy these conditional independence relations and Bell's inequality.

In contrast, for DAG G' (where Λ is observed), all the constraints of Equations (7)-(9) are conditional independence relations between observed variables. Thus, the set of probability distributions that satisfy the conditional independence relations resulting from d-separation relations of G' is the same as the set of probability distributions that are classically compatible with G'. This is the meaning of the concept of interestingness: an Interesting DAG imposes non-trivial inequality constraints on the classically compatible distributions.

A non-trivial inequality constraint is an inequality that is not implied by the conditional independence relations resulting from d-separation. Bell's inequality is non-trivial in the Bell DAG because it is not implied by Equations (9), making the Bell DAG Interesting. In contrast, DAG G' is Non-Interesting, as it also imposes Bell's inequality on the compatible probability distributions, but this inequality is trivial in G' because it follows from conditional independence relations resulting from d-separation between observed variables of G' (Equations (7) and (8)). All DAGs that do not have latent variables only have conditional independence constraints and are therefore necessarily Non-Interesting.

Previously [9, 29] have analyzed and classified Interesting causal structures. For these Interesting causal structures it is know that observed conditional independences alone are not the only constraints on the compatible probability distributions. In other words these are scenarios which could possess non-classical correlations. In Section 7 we use the observable conditional independences corresponding to causal scenarios of 6 nodes from the Appendix E of [9] as input to the IC^* algorithm.

6 IC^* algorithm applied to conditional independences

In Section 3 we saw how a given causal scenario implies conditional independences. Causal discovery algorithms try to answer the inverse problem. Given some observed conditional independences between the observed variables, how can we find a causal scenario that explains them faithfully? This is the problem of causal discovery. Causal discovery in the presence of possible latent variables is significantly more difficult than in the absence of latent variables. IC^* [2, 7] algorithm (inductive causation) which is equivalent to the Causal Inference algorithm (CI)[3, 4] takes as input the observed conditional independences and returns a (graph) pattern in which the random variables obey those conditional independences. It is one of the well-known algorithms to study causal discovery in the presence of latent variables. If there are causal structures which have pairwise common causes and which are faithful to the observed conditional independences then this algorithm returns a minimal set of these causal structures as a pattern.[11]

A pattern is a graph that encodes all the possible causal scenarios consistent with it. It consists of observed nodes and undirected, directed or bi-directed edges which represent the possible causal scenarios (DAGs) consistent with it. We do not review the algorithm here, but just explain how to interpret the pattern it returns subject to some observed conditional independences. We follow this up with a simple example where we show how the algorithm works. For a review of the IC^* algorithm we point the reader towards [2, 6].

[11] The algorithm only searches for pairwise common causes. So if in a scenario a common cause has three children, the algorithm would return that the three pairs of two children each have independent common causes.

The interpretation of the possible edges in the pattern returned by the algorithm in terms of a DAG is shown in Figures 10, 11 and 12. In all of them we show the pattern returned on the left and its interpretation in terms of a DAG on the right.

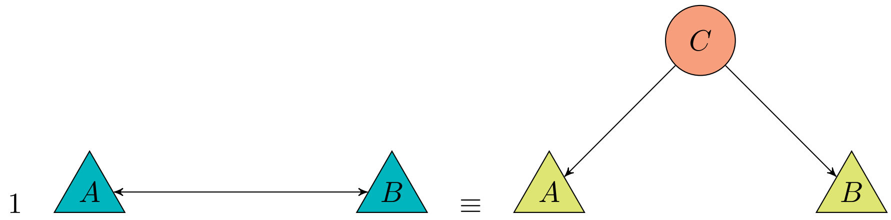


Figure 10: Interpretation of a bi-directed edge in the pattern as a DAG.

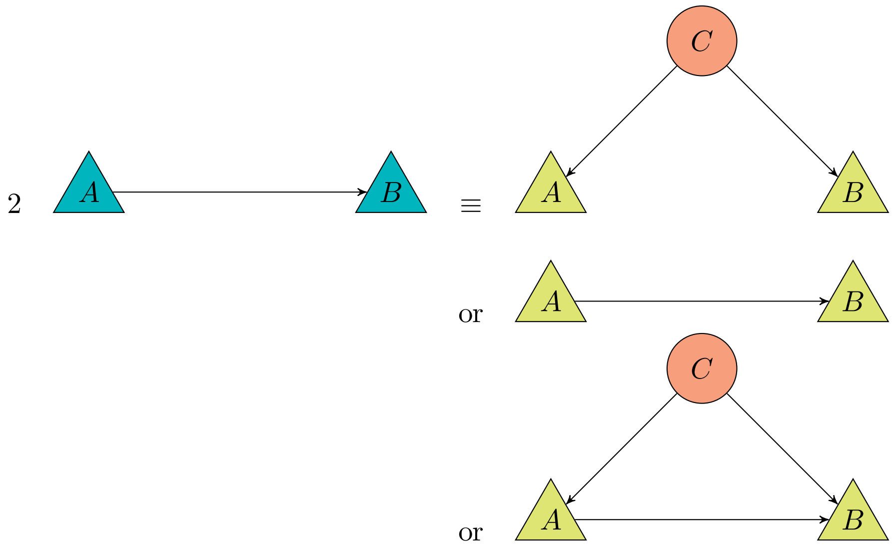


Figure 11: Interpretation of a directed edge in the pattern as a DAG.

The process of selecting the causal scenarios which can be the possible explanations from the returned pattern is done by following the steps below:

1. Assume a particular causal order for the variables.

2. Draw all the possible causal scenarios from the returned pattern which obey the observed conditional independences inputted to the algorithm. These are the possible faithful explanations of the observed conditional independences.

The importance of point 1 can be seen through the examples of Figure 1(a,b). The conditional independences for both the scenarios in Fig 1 (a) and Fig 1 (b) are the same, precisely that there are no conditional independences for either of them. So if no conditional independences are fed into the IC^* algorithm then it will return both the causal scenarios of Figure 1 as possible explanations. It is evident that if we need to select one out of the two cases returned we need to enforce a causal order on the variables considered: Depression will occur before the Recovery.

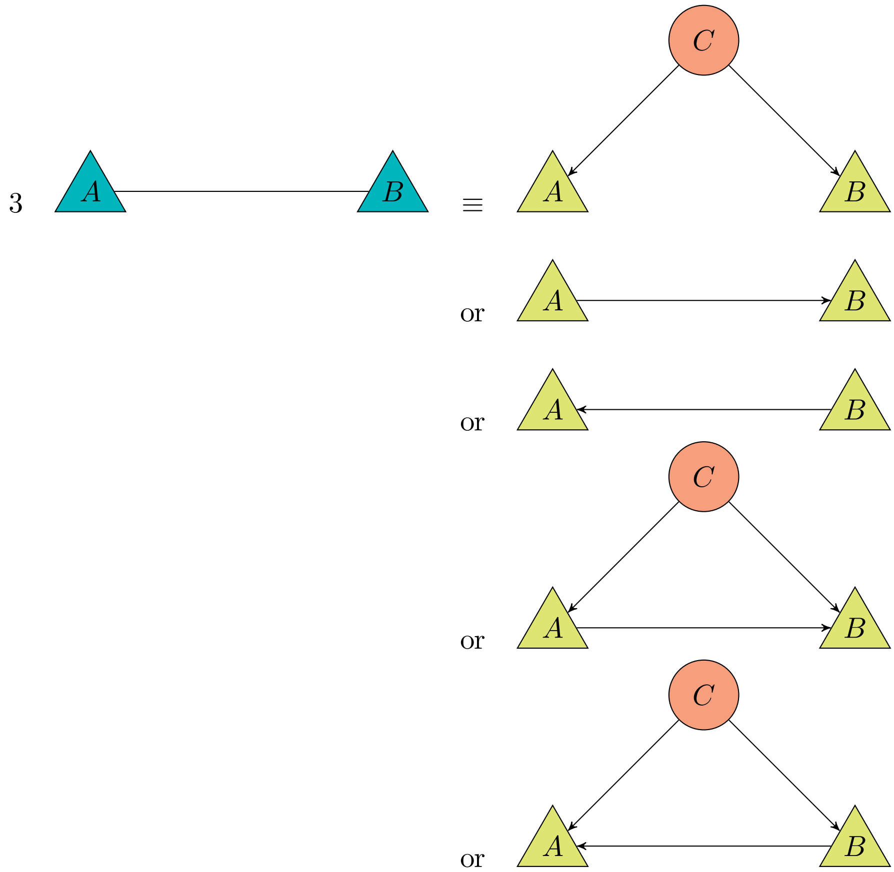


Figure 12: Interpretation of an undirected edge in the pattern as a DAG.

Point 2 above says that we need to select only those causal scenarios from the returned pattern which respect the observed conditional independences. That means that the pattern returned could lead to scenarios which violate some observed conditional independences. This sort of defeats the purpose of the algorithm as we would expect the algorithm to itself only return those causal scenarios which are consistent with the observed conditional independences. Nevertheless, the algorithm and its python implementation [30, 31] that we have used here returns a pattern from which we need to select the possible causal scenarios.

Now we give a simple example to show the working of the algorithm. Suppose we observe some data over three random variables A,B and C. Suppose that the only conditional independences that we observe are: $(A,C \perp\!\!\!\perp B)$. We feed them as inputs to the algorithm and the algorithm works out all the possible scenarios that can support these conditional independences between the three given random variables A,B and C. The pattern that the algorithm returns is shown in Figure 13.

The undirected edge between nodes A and C needs to be interpreted according to the explanation in Figure 12. Thus the causal scenarios shown in Figure 14 are compatible with the returned pattern and with the input observed conditional independences: $(A,C \perp\!\!\!\perp B)$.

Unless we choose a specific causal order for the variables, we cannot narrow down on the choices of the causal scenarios compatible with the conditional independences $(A,C \perp\!\!\!\perp B)$. We may assume that A occurs before C. In that case the only compatible scenarios would be the the first, second and fourth ones in Figure 14. If on the other hand we assume that C

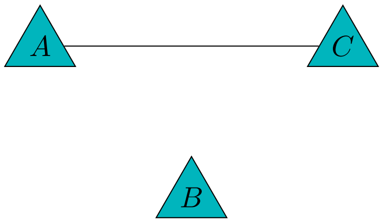


Figure 13: Pattern returned by the IC^* algorithm for conditional independences $(A,C \perp\!\!\!\perp B)$.

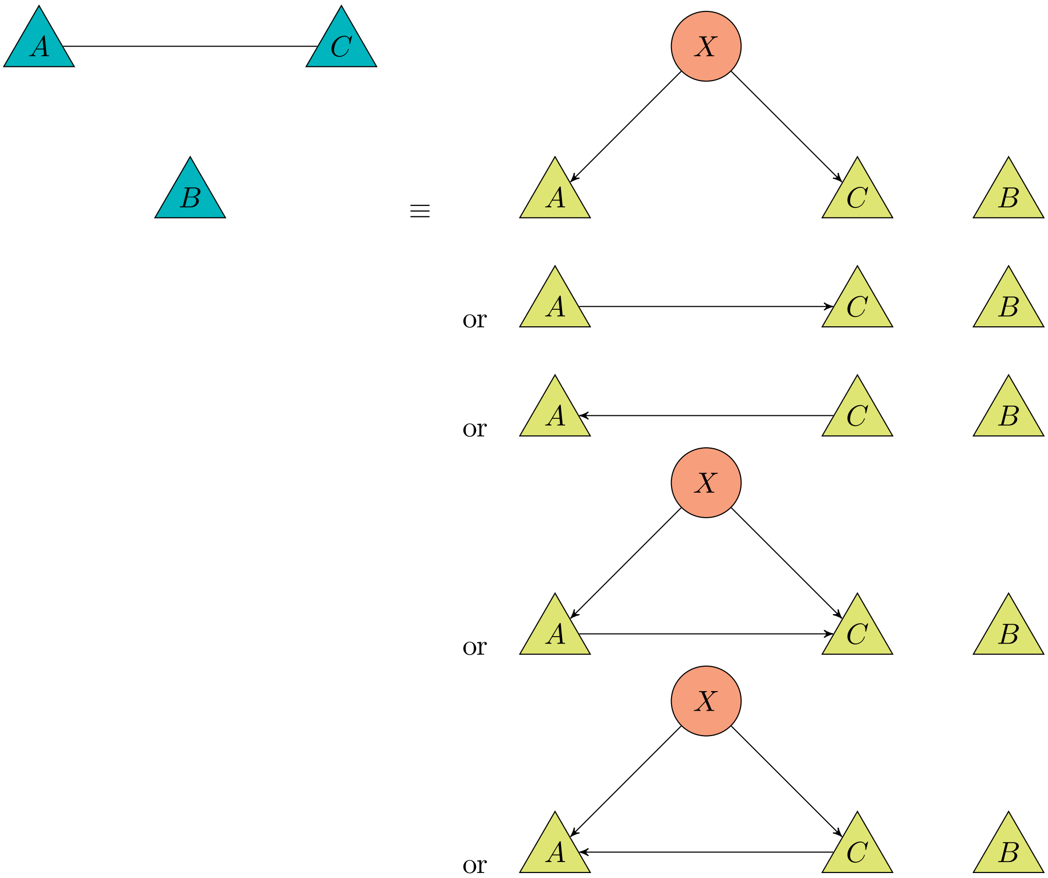


Figure 14: Causal scenarios compatible with conditional independences $(A,C \perp\!\!\!\perp B)$.

occurs before A, then the compatible scenarios would be the first, third and fifth ones in Figure 14. Note that in this simple example the observed conditional independences $(A,C \perp\!\!\!\perp B)$ are satisfied by all the scenarios resulting from the pattern. In a more complicated case this might not necessarily be true and we would need to consider only those scenarios resulting from the returned pattern which obey the given observed conditional independences. Wood and Spekkens [16] have explained this selection through nice pedagogical examples which we do not repeat here but just use the above procedure of selection on the causal scenarios we are interested in, in Section 7.

It is interesting to see what pattern the algorithm returns if we input the conditional independences corresponding to the Bell DAG of Figure 4. The generating set of conditional independences for the Bell DAG of Figure 4 are $(X \perp\!\!\!\perp B,Y)$ and $(Y \perp\!\!\!\perp X,A)$. The pattern returned by the algorithm corresponding to these conditional independences is shown in Figure 15.

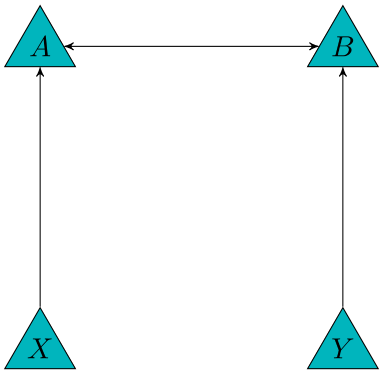


Figure 15: Pattern returned by the IC^* algorithm for conditional independences $(X \perp\!\!\!\perp B,Y)$ and $(Y \perp\!\!\!\perp X,A)$.

Observation shows that the Bell DAG of Figure 4 is the only possible causal scenario compatible with the pattern returned in Figure 15 unless one resorts to fine-tuning. This was first pointed out by Wood and Spekkens [16]. But the Bell DAG cannot explain certain quantum correlations and hence is not an explanation of the set of all possible quantum correlations in this scenario.

It must not be forgotten that the causal discovery algorithms can go wrong because at best they are a heuristic explanation to the observed distribution. As Wood and Spekkens have shown this algorithm cannot distinguish between Bell inequality violating and non-violating correlations and hence cannot return a causal scenario which explains certain (quantum) correlations obtained in a Bell-CHSH test faithfully. All explanations of Bell inequality violating correlations require fine tuning. This shortcoming of the algorithm is not a surprise because the algorithm only takes observed conditional independences as input and consequently does not know anything about the strength of possible correlations the respective conditional dependences can give rise to.

7 Results

We consider some causal scenarios that have already been shown to be capable of possessing non-classical correlations (and are hence called Interesting) They have also been shown to definitely support certain quantum correlations that cannot be explained in any classical way without resorting to fine-tuning in [10, 11]. Now in an actual experiment we will observe some conditional independences and correlations. So, given just these observed conditional independences and correlations that we find in the experiment, we would like to ask what all possible causal scenarios can explain them. In particular, we would like to check if along with an Interesting scenario there also exists a Non-Interesting scenario that can explain those observed conditional independences or not. If there exists such a Non-Interesting scenario as well, then we can very well explain those *experimentally observed* conditional independences classically and the existence of such conditional independences does not imply the existence of non-classical correlations. This is because in such a case, we can always explain any non-classical correlations in the Interesting scenario classically in the Non-Interesting scenario supporting the same observed conditional independences. This is proved in Theorem 2.

On the other hand if we find that an Interesting scenario is the only possible scenario that can explain the corresponding *experimentally observed* conditional independences then the existence of non-classical correlations is *directly implied just by the observation of some* *conditional independences.* Further, if the scenario supports some quantum correlations that are non-classical then there would exist no other scenario that can support those particular quantum correlations in any classical way without resorting to fine-tuning. Hence, no classical model would be able to explain such experimentally realized quantum correlations causally. These scenarios are thus *"genuinely interesting"*. Note that as explained in Section 4, these observed non-classical correlations (including the quantum correlations) need not be necessarily non-local. Irrespective, of whether these observed correlations can or cannot be explained locally, they are non-classical in the sense that classical causal modelling cannot explain them without fine-tuning, an assumption very central to the field.

We find several examples of such scenarios. For example, the Bi-locality scenario [12, 13] that represents the correlations obtained in an entanglement swapping scenario is the only scenario that classical causal modelling says can explain all the corresponding observed conditional independences faithfully. The same is true for the GHZ [14, 15] and multi-partite Bell scenarios with n independent parties. Just like the Bell scenario, they support both non-local and local-non-classical correlations which require fine-tuning to be explained classically. Thus, there are scenarios beyond just the Bell scenario that classical causal theory cannot explain faithfully. In-fact these scenarios highlight the incompleteness of causal modelling.

In Tables 1, 2 and 3 we show the causal scenarios whose corresponding observed conditional independences we input in the IC^* algorithm and the corresponding pattern returned by it. In all the causal scenarios triangular nodes represent observed variables and the circular nodes represent the latent variables. First we prove an important theorem.

Theorem 2. *Let G be an Interesting causal scenario that exhibits some quantum correlations that are non-classical. Let G also exhibit certain observed conditional independences. Let H be another Non-Interesting scenario free of any latent variables that exhibits exactly the same observed conditional independences as G. Then H can support* ***all*** *the quantum correlations that were non-classical in G in a perfectly classical way without resorting to any fine-tuning.*

Proof. Let $\mathcal{C}_\mathcal{G}$, $\mathcal{C}_\mathcal{H}$ be the set of all correlations that are classically explainable in G and H respectively. Similarly, let $\mathcal{Q}_\mathcal{G}$, $\mathcal{Q}_\mathcal{H}$ be the set of all correlations that are explainable using Quantum theory in G and H respectively. Finally, let $\mathcal{I}_\mathcal{G}$, $\mathcal{I}_\mathcal{H}$ be the set of all correlations that respect all the observed conditional independences in G and H respectively. Since G and H have exactly the same observed conditional independences, we have that $\mathcal{I}_\mathcal{G}=\mathcal{I}_\mathcal{H}$. Also from [9] we know the following relations between the above sets of correlations:

$$\mathcal{C}_\mathcal{G}\subseteq\mathcal{Q}_\mathcal{G}\subseteq\mathcal{I}_\mathcal{G} \tag{10}$$

and

$$\mathcal{C}_\mathcal{H}=\mathcal{Q}_\mathcal{H}=\mathcal{I}_\mathcal{H}=\mathcal{I}_\mathcal{G} \tag{11}$$

Using Equations 10 and 11 together we have that

$$\mathcal{Q}_\mathcal{G}\subseteq\mathcal{I}_\mathcal{G}=\mathcal{I}_\mathcal{H}=\mathcal{C}_\mathcal{H} \tag{12}$$

thus showing that any quantum correlations in G are classical in H and that all non-classical correlations in G can be explained classically in H without resorting to fine-tuning. □

Causal scenarios (1), (3) and (6) in Tables 1, 2 can be modelled by the latent free scenario of Figure 16. Scenario (5) in Table 2 can also be modelled by this same latent free scenario with the simple relablelling $D\mapsto F, F\mapsto E, E\mapsto D$. Similarly, scenario (2) in Table 1 can be modelled by the latent free scenario of Figure 17. Thus, by Theorem 2 any non-classical correlations in causal scenarios (1), (2), (3), (5), (6) can be explained classically by using the latent free

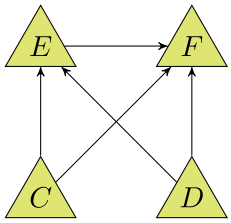


Figure 16: Latent free casual scenario reproducing all the conditional independences corresponding to scenarios (1), (3) and (6) in Tables 1, 2.[12]

scenarios of Figures 16 and 17 respectively. In an upcoming work [10, 11], it is shown that all these scenarios also support quantum correlations that cannot be explained classically using these scenarios. But again by resorting to Theorem 2, we know that such quantum correlations can always be modelled classically by the respective latent free scenarios of Figures 16 and 17.

Next consider the 4th scenario in Table 2. IC^* algorithm says that the conditional independences corresponding to this scenario can be produced by only one graph which necessarily needs to have two latent variables between $E \longleftrightarrow D$, and $F \longleftrightarrow D$ respectively. This is because any latent common causes between nodes A,E and nodes A,F can be absorbed into node A without any loss of generality. Hence, the 4th scenario in Table 1 is the only possible scenario that produces the corresponding observed conditional independences. A close look at the scenario shows that it is similar to the Bi-locality scenario with the exception that the extreme wings share the same settings.

For the 7th scenario in Table 2 notice that from all the many scenarios resulting from the pattern corresponding to it, only those preserve the respective conditional independences which respect the following:

1. Given the pattern returned, for $C \perp\!\!\!\perp F$ to be true there cannot be a directed edge $E \to F$. So there can be an edge $F \to E$, or $E - F$ can share a common cause or there can be both an edge $F \to E$ and a common cause between $E - F$.

2. But then for $C \perp\!\!\!\perp E|D$ to hold true there can be no collider at D on an undirected path $C - D - E$, unless there is another collider somewhere along the undirected path $C - D - E$ that makes $C \perp E|D$ true. Thus for $C \perp\!\!\!\perp E|D$ to hold true there can be no directed edges $F \to D$ and $F \to E$, but the only possibility is that $D - F$ share a common cause and $E - F$ share another common cause only.

3. For the same reason as the last point, there can be no common cause of $D - E$, else $C \perp\!\!\!\perp E|D$ will not hold true.

4. $C - D$ can share a common cause or a directed edge $C \to D$, or both the edge and the common cause. Without loss of generality we can always absorb the common cause of $C - D$ into C.

A close inspection reveals that there is only one causal scenario resulting from the pattern returned which respects the above conditions and the corresponding observed conditional independences and it is indeed the 7th one in Table 2.

Now consider the Bi-locality scenario which is 8th in Table 2. Inputting its conditional independences in the algorithm returns the corresponding pattern for which edges $G \longleftrightarrow D$ and $D \longleftrightarrow H$ are bi-directed. Remembering the meaning of a bi-directed edge in a pattern from Section 6, we see that the only possible scenario is the Bi-locality one. This is because

[12] Inverting the direction of the edge $E \to F$ also yields a valid scenario.

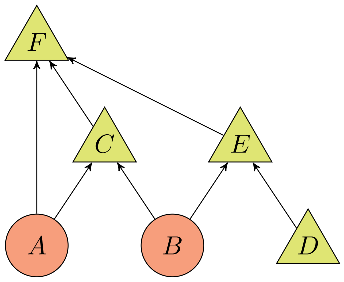
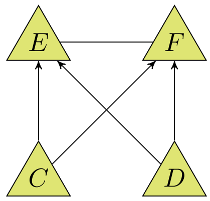
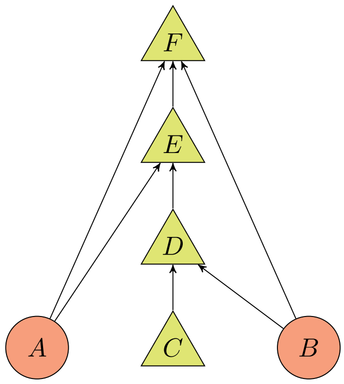
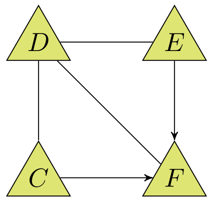
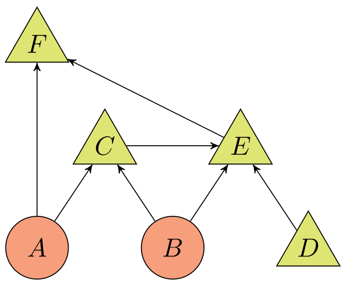
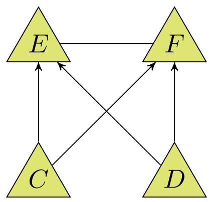
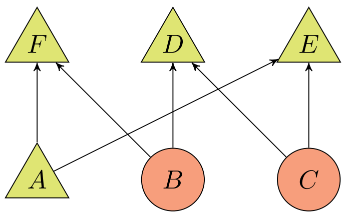
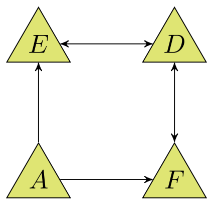
	Causal Scenarios possessing certain observed conditional independences	The pattern obtained by inputting the corresponding observed conditional independences in the IC^* algorithm
(1)	 Condtional Independences = $\{C \perp\!\!\!\perp D\}$	
(2)	 Condtional Independences = $\{C \perp\!\!\!\perp E \,|\, D\}$	
(3)	 Condtional Independences = $\{C \perp\!\!\!\perp D\}$	
(4)	 Condtional Independences = $\{A \perp\!\!\!\perp D,\, E \perp\!\!\!\perp F \,|\, A\}$	

Table 1: Causal scenarios and their corresponding patterns returned by the IC^* algorithm.

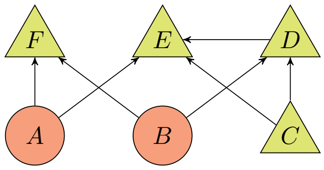
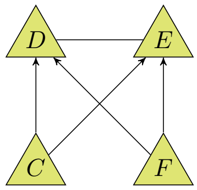
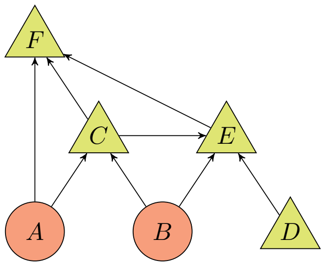
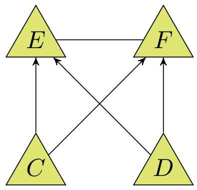
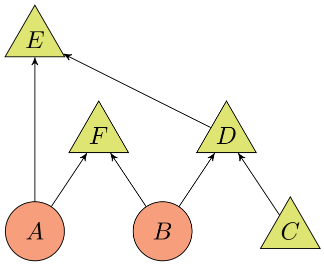
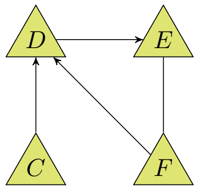
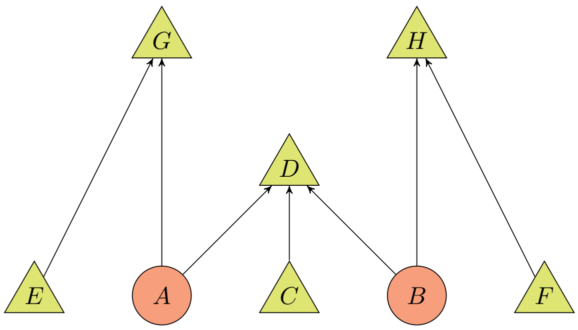
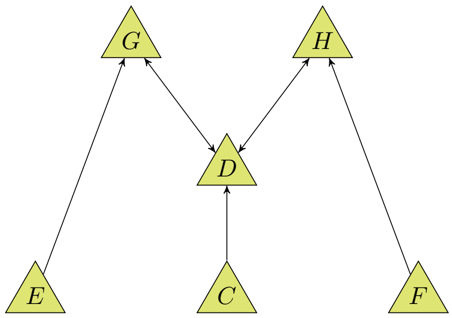
	Causal Scenarios possessing certain observed conditional independences	The pattern obtained by inputting the corresponding observed conditional independences in the IC^* algorithm
(5)	 Condtional Independences = $\{C \perp\!\!\!\perp F\}$	
(6)	 Condtional Independences = $\{C \perp\!\!\!\perp D\}$	
(7)	 Condtional Independences = $\{C \perp\!\!\!\perp F,\ C \perp\!\!\!\perp E \mid D\}$	
(8)	 Condtional Independences = $\{E \perp\!\!\!\perp C,\ F \perp\!\!\!\perp C,\ G \perp\!\!\!\perp C,\ E \perp\!\!\!\perp D,\ F \perp\!\!\!\perp D,\ E \perp\!\!\!\perp F,\ G \perp\!\!\!\perp F,\ E \perp\!\!\!\perp H,\ H \perp\!\!\!\perp C\ \}$	

Table 2: Causal scenarios and their corresponding patterns returned by the IC^* algorithm.

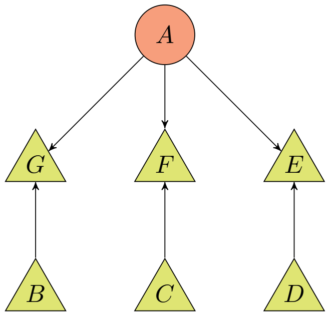
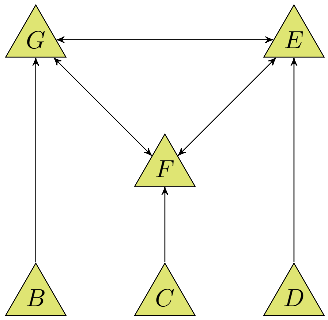
	Causal Scenarios possessing certain observed conditional independences	The pattern obtained by inputting the corresponding observed conditional independences in the IC^* algorithm
(9)	 Conditional Independences = $\{C \perp\!\!\!\perp B,\ D \perp\!\!\!\perp B,\ E \perp\!\!\!\perp B,\ F \perp\!\!\!\perp B,\ D \perp\!\!\!\perp C,\ E \perp\!\!\!\perp C,\ G \perp\!\!\!\perp C,\ F \perp\!\!\!\perp D,\ G \perp\!\!\!\perp D\}$	

Table 3: Causal scenarios and their corresponding patterns returned by the IC^* algorithm.

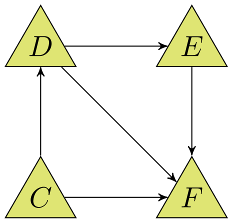


Figure 17: Latent free casual scenario reproducing all the conditional independendces corresponding to the scenario (2) in Table 2.[13]

nodes G and D can share a common cause only without any directed edge between them and the same would be the case for nodes D and H which will share only a common cause as well.[14] So just like the Bell scenario, the Bi-locality scenario is the only faithful scenario which explains the corresponding observed conditional independences.

The conclusion is the same for the GHZ scenario which is 9th in Table 3. It is the only possible scenario that can model the corresponding conditional independences. This is because the bi-directed edges at $G \longleftrightarrow F$, $F \longleftrightarrow E$, and $G \longleftrightarrow E$ imply that the corresponding nodes can share only a latent common cause. This result can be generalized to multi-partite Bell scenarios with arbitrary number of parties with respectively independent settings. We prove this generalised Theorem 3 in the next section.

8 Fine-tuning for multi-partite Bell scenarios

We first introduce the formal definition of a multi-partite Bell scenario.

Definition 12 (Multi-partite Bell scenarios)**.** Consider n parties with m settings and k outputs, all space-like separated to each other, such that $n > 2, m > 1, k > 1$. We call such a scenario the multi-partite Bell scenario.

[14]For the directed edges between two nodes we can absorb the latent variable into the parent node without any loss of generality.

For all scenarios we have dealt with up to now, we have relied on the IC^* algorithm, which worked because the number of conditional independences was not arbitrarily high. But now we consider multi-partite Bell scenarios with an arbitrary large number of parties, with each party's setting being independent of the other. Here we will have arbitrarily many conditional independences and hence a brute-force search is needed to find the possible non fine-tuned scenarios which can support such conditional independences.

Consider a multi-partite Bell scenario with n independent parties. We denote the set of parties as $\mathbb{P} = \{1......n \mid n \in \mathbb{N}\}$. The settings and outputs of the respective parties are denoted as $X_1, X_2, X_3, X_4, ...X_n$ and $A_1, A_2, A_3, A_4, ...A_n$ respectively. Thus, $\mathbb{O} = \{A_i \mid i = 1, ...n\}$ is the set of random variables representing all the outputs and $\mathbb{S} = \{X_i \mid i = 1, ...n\}$ is the set of random variables representing all the settings. Here, A_i and X_i correspond to the outcome and the setting of the ith party. The outputs and settings corresponding to any subset $\mathcal{P} \subseteq \mathbb{P}$ of the parties is denoted as $\mathcal{O}_{\mathcal{P}} = \{A_j \mid j \in \mathcal{P}\}$ and $\mathcal{S}_{\mathcal{P}} = \{X_j \mid j \in \mathcal{P}\}$ respectively.

Now, we begin with the following assumptions for our no-go theorem:

1. **Quantum correlations and observed conditional independences:** On grounds of relativity, we expect that any subset of the parties cannot signal to any other subset of the parties since they are all space-like separated to each other. That is, once the settings corresponding to any subset of the outputs is given that subset of outputs is independent of all the other settings. We also assume that any subset of the settings are independent of any other disjoint set of settings. Hence, we assume that all the observed quantum correlations and observed probability distributions need to respect only the following conditional independences:

$$
\begin{aligned}
(\mathcal{O}_P \perp\!\!\!\perp \mathbb{S} \backslash \mathcal{S}_P \mid \mathcal{S}_P) \ \forall \, \mathcal{P} \subseteq \mathbb{P} \quad &\text{(No-signalling between any subset of parties)} \\
(\mathcal{S}_P \perp\!\!\!\perp \mathbb{S} \backslash \mathcal{S}_P) \ \forall \, \mathcal{P} \subseteq \mathbb{P} \quad &\text{(Any subset of independent parties have independent settings)}
\end{aligned}
\tag{13}
$$

   By the semi-graphoid axioms (in particular the decomposition axiom)[32] we know that the conditional independences in Equation 13 also imply the following conditional independences:

$$
\begin{aligned}
(A_i \perp\!\!\!\perp X_j \mid X_i) \ \forall \, i, j = 1, ...n, \, j \neq i \quad &\text{(Pairwise no-signalling)} \\
(X_i \perp\!\!\!\perp X_j) \ \forall \, i, j = 1, ...n \ j \neq i \quad &\text{(Pairwise independent parties)}
\end{aligned}
\tag{14}
$$

2. **Causal and non fine-tuned explanation:** This assumption states that all probability distributions on observed variables must exhibit a non fine-tuned explanation based on the framework of standard causal modelling. That is there should exist a DAG that is a classically causal faithful explanation of all the observed probability distributions.

Our no-go Theorem 3 is based on the conjunction of the above two assumptions and says that all explanations of Bell inequality violating correlations in any multi-partite Bell scenario can be explained only via some fine-tuning. If we obtain some correlations that violate our no-go Theorem 3, then either of the above two assumptions or both are wrong. Now, we formally state and prove our theorem.

Theorem 3 (Fine-tuning for multi-partite Bell scenarios)**.** *There are no classically causal faithful explanations of Bell inequality violating correlations in any multi-partite Bell scenario with n independent settings. Or alternatively, any classical causal explanation of such correlations requires fine-tuning.*

Proof. We prove this theorem by first showing that the most general DAG that can explain all the conditional independences in Equation 13 is the one shown in Figure 18 which admits the following factorisation:

$$P(A_1,...,A_n,X_1,...,X_n)=\sum_{\Lambda}P(A_1|X_1,\Lambda)...P(A_n|X_n,\Lambda)P(X_1)...P(X_n)P(\Lambda). \tag{15}$$

To prove that the most general DAG explaining all the conditional independences in Equation 13 admits the factorisation in Equation 15 we consider all the possible DAGs that can support the conditional independences in Equation 13 and show that any correlation that can be explained by them classically without any fine-tuning can also be explained classically by the DAG in Figure 18 without resorting to any fine-tuning. Therefore, any correlation that cannot be explained classically without any fine-tuning by the DAG in Figure 18 admits no classically causal faithful explanation. And, it is well known that there exist quantum correlations that violate the factorisation in Equation 15 by violating the corresponding Mermin-Ardehali inequalities [33, 34, 35, 36]. Hence, there do not exist any classically causal faithful explanations of Bell inequality violating correlations in any multi-partite Bell scenario with parties having independent measurement settings.

The first step of the proof is to recognise that under the assumption of no fine-tuning the conditional independences in Equation 14 imply the conditional independences in Equation 13. This follows from the fact that considered as d-separation relations Equations 14 and 13 are equivalent. Hence, if we assume that there is no fine-tuning the two equations are equivalent at the level of conditional independences as well. Thus, we can proceed with the generating set of conditional independeces being Equation 14.

Throughout the proof, we use the mDAG formalism [37, 29] according to which it is sufficient to just consider the DAGs where all the latent variables have no parents. This is because adding any latent variables that are common children of observed or latent variables or lie on a directed path between two observed or latent variables is pointless since adding such variables generates the same probability distributions over the observed variables as DAGs which exclude such possibilities. Hence, we only need to consider cases where latent variables are the common causes of the observed variables.[15]

It follows from the approach taken in the proof of Theorem 2 in [16] that the only sets of observed variables that can have any latent common causes are the following: $\{A_1,A_2,...A_{n-1},A_n\}$, $\{A_1,X_1\},...,\{A_n,X_n\}$. Any other set of observed variables cannot have any latent common causes if all the conditional independences in Equation 14 are to be satisfied without resorting to any fine-tuning. Similarly, the only pairs of observed variables for which there can be a direct causal influence are only $\{A_1,X_1\},...,\{A_n,X_n\}$.. Note that the possibility of a direct causal influence from A_i to A_j, $\forall i,j,i\neq j$ is excluded to respect the conditional independences $A_j \perp\!\!\!\perp X_i|X_j$ $\forall i,j,i\neq j$ without introducing fine-tuning. The possibility of no causal connection (a direct causal influence or a latent common cause) between $\{A_1,A_2,...A_{n-1},A_n\}$, $\{A_1,X_1\},...,\{A_n,X_n\}$, can be excluded; otherwise there would be no way to explain the correlations between the variables in the following sets: $\{A_1,A_2,...A_{n-1},A_n\}$, $\{A_1,X_1\},...,\{A_n,X_n\}$. The direction of direct causal influence between the observed variables in the sets: $\{A_1,X_1\},...,\{A_n,X_n\}$ necessarily has to be $X_i \to A_i$ $\forall i$, because a direct causal influence $A_i \to X_i$ $\forall i$ would violate $A_i \perp\!\!\!\perp X_j|X_i$ $\forall i,j,i\neq j$, unless we resort to fine-tuning.

It follows that the only DAGs which can explain all the conditional independences in Equation 14 without any fine-tuning need to have the following structure:

[15] Considering cases where the latent variables have only one child is again redundant since in such a case the latent variable will not create any correlations between the observed variables and without loss of generality we can absorb such a latent variable inside its observed child.

1. The set of observed variables $\{A_1,A_2,...A_{n-1},A_n\}$ can have any number of joint latent common causes.

2. The sets of observed variables: $\{A_1,X_1\},...,\{A_n,X_n\}$ can have latent two party common causes $X_i \leftarrow \mu_i \rightarrow A_i$, or a direct causal influence from $X_i \rightarrow A_i$ $\forall i$, or both.

Now, consider the set of observed variables $\{A_1,A_2,...A_{n-1},A_n\}$. To explain general correlations between A_i,A_j, we should atleast have two party common causes $A_i \leftarrow \lambda_{ij} \rightarrow A_j$ $\forall i,j,i \neq j$. However, there can be further ${}^nC_3$ three party common causes, ${}^nC_4$ four party common causes and so on. Hence, in the most general case the total number of common causes can be ${}^nC_2 + {}^nC_3 + {}^nC_4 + ... + {}^nC_{n-1}$. Now since we wish to explain all types of correlations between the observed nodes and directed edges in a DAG mediate the possibility of a causal influence, we can, without the loss of any generality assume that all the ${}^nC_2 + {}^nC_3 + {}^nC_4 + ... + {}^nC_{n-2}$ multi- party common causes to just be a part of the single n party common cause which we can denote as Λ. In other words this n party common cause can explain all possible classical correlations between the observed variables $A_1,...,A_n$.

Similarly, it is easy to show that any classically achievable correlation the DAGs where A_i and X_i have a latent common cause $X_i \leftarrow \mu_i \rightarrow A_i$ or both a direct causal influence $X_i \rightarrow A_i$ as well as the common cause $X_i \leftarrow \mu_i \rightarrow A_i$ can be classically explained without any fine-tuning by the DAG in which we just have the direct causal influence $X_i \rightarrow A_i$. This follows from:

1. For the case where we have both the direct causal influence $X_i \rightarrow A_i$ as well as the common cause $X_i \leftarrow \mu_i \rightarrow A_i$, we have, $P(A_i,X_i) = \sum_{\mu_i} P(A_i|X_i,\mu_i)P(X_i|\mu_i)P(\mu_i)$. Now we define $X_i' := (X_i,\mu_i)$, with $P(X_i') := P(X_i|\mu_i)P(\mu_i)$, thereby obtaining $P(A_i|X_i,\mu_i) = P(A_i|X_i')$ and consequently $P(A_i,X_i') = P(A_i|X_i)P(X_i)$, thus ending up showing that any classical distribution in such a DAG can be explained without fine-tuning by the DAG where we just have the direct causal influence $X_i \rightarrow A_i$.

2. For the case where we just have the common cause $X_i \leftarrow \mu_i \rightarrow A_i$, we have, $P(A_i,X_i) = \sum_{\mu_i} P(A_i|\mu_i)P(X_i|\mu_i)P(\mu_i) = \sum_{\mu_i} P(A_i|X_i,\mu_i)P(X_i|\mu_i)P(\mu_i)$. And thus again if we define $X_i' := (X_i,\mu_i)$, with $P(X_i') := P(X_i|\mu_i)P(\mu_i)$, we can show that any correlation in such a DAG can be explained classically without fine-tuning by the DAG in which we just have the direct causal influence $X_i \rightarrow A_i$[16].

Since under the assumption of no fine-tuning Equations 14 and 13 are equivalent, we have thus shown by a brute force characterisation, that the most general DAG (of Figure 18) which explains all the conditional independences in Equation 13 without any fine-tuning admits the factorisation:

$$P(A_1,...,A_n,X_1,...,X_n) = \sum_{\Lambda} P(A_1|X_1,\Lambda)...P(A_n|X_n,\Lambda)P(X_1)...P(X_n)P(\Lambda). \tag{16}$$

A distribution is classical with respect to this DAG if and only if it respects Equation 16. However, there exist quantum correlations which violate this factorisation [33, 34, 35, 36] and thus we have shown that there are no classically causal faithful explanations of Bell inequality violating correlations in any multi-partite Bell scenario with n independent measurement settings. □

9 Summary of the results

In the end we have two results. For concreteness we summarize them now:

[16] Actually we can do more and show that all the three possible cases where A_i and X_i have a latent common cause $X_i \leftarrow \mu_i \rightarrow A_i$ or both a direct causal influence $X_i \rightarrow A_i$ as well as the common cause $X_i \leftarrow \mu_i \rightarrow A_i$ or just the direct causal influence $X_i \rightarrow A_i$ are equivalent in terms of supporting classical correlations.

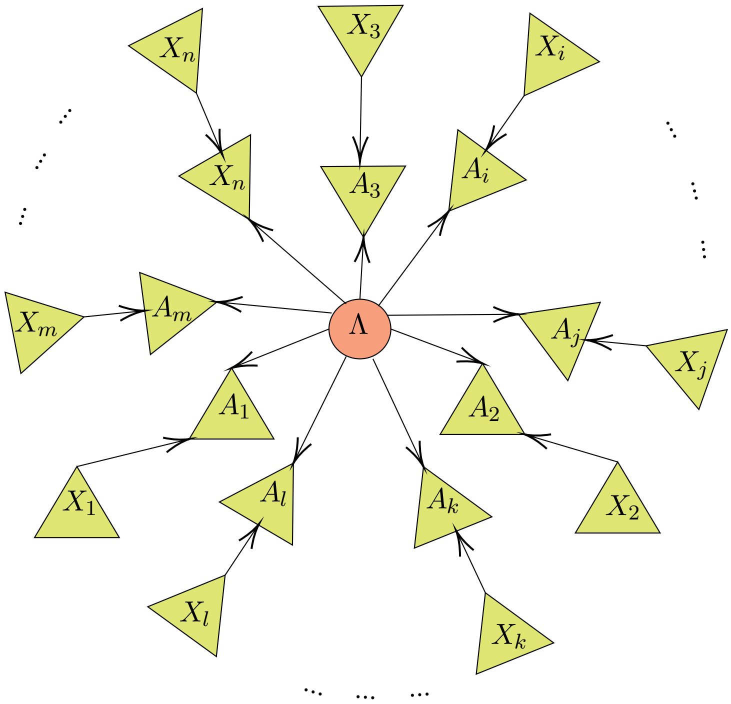


Figure 18: The most general DAG that can explain all the conditional independences in Equation 13.

1. Non-locality is not the only defining property of non-classical correlations. Just like the Bell scenario there are several other scenarios that exhibit local correlations which are non-classical as well. Any causal explanation of these non-classical correlations based on classical causal modelling will require fine-tuning just like the Bell scenario. Hence the result of Wood and Spekkens [16] extends to several other scenarios as well. From the scenarios that we modelled, their result extends to scenarios (4th), (7th), Bi-locality, GHZ scenarios in Tables 1, 2 and 3 and the general multi-partite Bell scenario with n independent parties .

2. Suppose we perform some quantum experiment and observe some conditional independences and quantum correlations. Based on our knowledge of the quantum experiment we try to model it using a causal scenario. We find that there is no way to explain these quantum correlations based on our classical causal model without fine-tuning. This would suggest that no classical theory can reproduce these quantum correlations. But we would still like to search for classical explanations of the obtained quantum correlations. Hence we input the observed conditional independences in the IC^* algorithm and find that it returns that there are number of scenarios which reproduce these quantum correlations. Amongst them is the scenario that we initially unsuccessfully came up with to model the obtained correlations. But importantly, IC^* outpus that there are a number of other possibly latent free causal scenarios that can also model the observed conditional independences. If we find a latent free causal scenario that can model the observed conditional independences then that scenario will also explain all the obtained quantum correlations classically without fine-tuning. This follows from Theorem 2. Thus we will have a valid classical explanation of the experimentally observed quantum correlations which we initially thought to have no possible classical explanation at all. Amongst the scenarios that we studied, examples of such are scenarios (1), (2), (3), (5) and (6) in Tables 1 and 2.

10 Discussion on Modifying the IC^* algorithm for Causal Discovery

In the last two sections, we have reiterated another result; IC^* algorithm is not sufficient for causal discovery. This is because the strength of correlations plays an important role determining the causal mechanism at play. Or put differently conditional independences alone do not characterize an observed probability distribution completely. Now we suggest an improvement to the IC^* algorithm.

It is motivated on the fact that strength of correlations plays an important role in characterizing the classically compatible distributions. Hence we suggest appending the IC^* algorithm with an algorithm that can check the strength of compatible distributions. One such algorithm is provided by Fraser [38]. To get a list of DAGs that obey the observed conditional independences and thus also the observed correlations we can combine the two algorithms in the following way:

1. Input the observed conditional independences in the IC^* algorithm to get extract a list of possibly compatible DAGs.

2. Use Fraser's algorithm to check which of the extracted DAGs support all of the observed correlations.

Only the DAGs which can support all of the observed correlations can be possible explanation of the observed probability distribution.

11 Conclusion

Causal discovery algorithms have become an important field of research in Machine Learning and Artificial Intelligence [39]. In this study, we applied the IC^* algorithm to various causal scenarios with up to 6 total nodes, showing that for most scenarios, the algorithm was able to model the observed conditional independences using both scenarios for which observed d-separation relations are the only constraints on the possible distributions compatible with the scenario and scenarios for which there are constraints beyond observable d-separation that restrict the compatible distributions. However, for 3 scenarios, the algorithm was only able to model the conditional independences using faithful causal scenarios having constraints beyond observed d-separation relations. This suggests that there may not be a classical causal faithful explanation for these scenarios. We also examined the Bi-Locality, the GHZ and the multi-partite Bell scenarios in quantum mechanics and found that the only faithful scenarios that can explain the corresponding observed conditional independences in the above three scenarios are in fact just one in each case; namely, the Bi-Locality, GHZ and the multi-partite Bell scenario themselves. This result highlights the importance of exploring properties of data beyond just observed conditional independences when attempting to reproduce a causal hypothesis for a given set of data.

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