Restricted Negation in Orthomodular Logic

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Orthomodular logic has long been studied as a fundamental logical framework within quantum logic. In addition to its lattice-theoretic foundations, various formulations have been proposed that incorporate specialized implication connectives and more complex logical operators. In this study, we reconstruct orthomodular logic by introducing an operator referred to as restricted negation, which has received comparatively little attention in the existing literature. We demonstrate that this operator provides a natural and expressive basis for formulating the core principles of orthomodular logic. Furthermore, as a main result, we construct new proof system for orthomodular logic founded on this operator.

1 Introduction

Quantum logic (QL) is a logical framework that studies the distinctive properties of propositions arising in quantum mechanics and quantum information science. In contrast to classical propositions, quantum-mechanical statements exhibit non-classical features such as incompatibility and the failure of distributive laws, which make classical logic insufficient as a foundational framework for quantum theory. This motivates the development of alternative logical systems tailored to the structure of quantum phenomena. Research in quantum logic has proceeded in various directions, including the development of proof systems, the analysis of algebraic and lattice-theoretic models, applications to quantum information science, and investigations into their semantic and structural properties [1, 2, 3, 5].

Among the systems studied in this context, *orthomodular logic* (OML) plays a fundamental role. Quantum states are represented as vectors in a *Hilbert space*, while observational propositions are associated with closed subspaces of it. OML employs *orthomodular lattice*, an abstraction of the lattice of closed subspaces, as its model. OML has been extensively studied for several decades and forms a core component of the logical foundations of quantum theory. A notable difficulty of OML is that it employs the same logical symbols as classical logic, such as conjunction, disjunction, and negation, while assigning them meanings that differ in subtle but essential ways. This similarity in notation, combined with a divergence in behavior, often complicates both the interpretation and the formal analysis of the logic [2, 6, 8].

In this study, we investigate a particular logical connective that is motivated by quantum-mechanical considerations. We show that OML can be reconstructed using this connective as the sole primitive, with all other logical operations defined in terms of it. This reformulation provides a conceptually transparent perspective on OML and highlights its intrinsically quantum-logical structure.

Here, we consider operators constructed by combining the basic operators *negation* $\sim$ and *conjunction* $\wedge$. Among such operators, *Sasaki hook* $A \rightarrow_S B = \sim(A \wedge \sim(A \wedge B))$, *material implications* other than Sasaki hook, and *Sasaki projection* $A \wedge \sim(A \wedge \sim B)$ have been studied extensively [8, 9, 14, 17, 18, 21]. These operators are associated with physically important meanings. For example, Sasaki hook can be

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interpreted as “B holds after projection onto the subspace in which A is true.” In this study, we analyze *restricted negation* $A \wedge \sim (A \wedge B)$. This operator is similar to the two operators above, yet its properties have not been well studied. This operator has an important meaning associated with subspaces of a Hilbert space, which is discussed in section 3. Furthermore, it is shown in this study that this operator has the following advantages.

In ordinary OML, at least two logical connectives, namely conjunction and negation, are required. However, since both of these connectives can be expressed in terms of restricted negation, all propositions can be handled using a single connective.

Orthomodular law has a rather complicated form when formulated using conjunction and negation, whereas it can be represented in a relatively simple form when expressed in terms of restricted negation.

We can construct a new proof system based on restricted negation. Together with the properties described above, this system allows the concept of orthomodular law to be naturally incorporated into the inference rules. Proof systems for OML have been extensively studied, but no single one has received disproportionate attention; each has its own advantages and disadvantages [4, 6, 7, 11, 12, 13, 15, 16, 20]. Since the *decidability* of orthomodular logic has not yet been established, the development of proof systems from diverse perspectives is an important task.

In Section 2, we review the basic notions of orthomodular logic. In Section 3, we analyze the fundamental properties of restricted negation. In Section 4, we construct a new proof system using restricted negation.

2 Orthomodular logic

Definition 1. *Language L_1 and formulas of OML*

The language L_1 of OML consists of a countable infinite set of propositional variables $p, q, r, \ldots$, and the logical connectives $\sim, \wedge$. Formulas of OML (for L_1) are defined as follows:

$$A ::= p \mid \sim A \mid A \wedge A$$

For the sake of simplicity, disjunction and implication (Sasaki hook) are defined as abbreviations: $A \vee B = \sim (\sim A \wedge \sim B)$, $A \rightarrow_S B = \sim (A \wedge \sim (A \wedge B))$.

Definition 2. *Ortho lattices and orthomodular lattices*

An *ortho lattice* is a complemented lattice $(\mathscr{L}, \sqcap, \sqcup, ', \mathbf{1}, \mathbf{0})$ that satisfies the following conditions for all $a, b \in \mathscr{L}$:

1. $\exists! a'$,
2. $a'' = a$,
3. $a \sqsubseteq b \Rightarrow b' \sqsubseteq a'$,
4. $a \sqcup b = (a' \sqcap b')'$,
5. $a \sqcap a' = \mathbf{0}$,
6. $a \sqcup a' = \mathbf{1}$.

where $a \sqsubseteq b$ is defined as $a \sqcap b = a$ (or, equivalently, $a \sqcup b = b$).

An *orthomodular lattice* is defined as an ortho lattice satisfying the following *orthomodular law*:

$\forall a,b \in \mathscr{L}$, $a \sqcap (a' \sqcup (a \sqcap b)) \sqsubseteq b$ (or, equivalently, $a \sqcap (a \sqcap (a \sqcap b)')' \sqsubseteq b$)

The orthomodular law admits the following equivalent formulations.

$\forall a,b \in \mathscr{L}$, $a \sqcap (a \sqcap (a \sqcap b)')' = a \sqcap b$

Intuitively, an orthomodular lattice can be regarded as an abstraction of a Hilbert space. Elements of $\mathscr{L}$ represent closed subspaces of a Hilbert space, $\mathbf{1}$ represents the entire space, $\mathbf{0}$ represents the space consisting only of the origin, $'$ represents the orthogonal complement, $\sqcap$ represents the intersection of two subspaces, and $\sqcup$ represents the subspace spanned by two subspaces. One of the main features of orthomodular lattice is that the *distributive law* $a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c)$ does not hold. For example, when we consider a two-dimensional complex Hilbert space, the set of all one-dimensional closed subspaces, $\mathbf{0}$, and $\mathbf{1}$ forms an orthomodular lattice; however, because of $C_1(|0\rangle + |1\rangle) \sqcap (C_2|0\rangle \sqcup C_3|1\rangle) = C_1(|0\rangle + |1\rangle) \sqcap \mathbf{1} = C_1(|0\rangle + |1\rangle)$ and $(C_1(|0\rangle + |1\rangle) \sqcap C_2|0\rangle) \sqcup (C_1(|0\rangle + |1\rangle) \sqcap C_3|1\rangle) = \mathbf{0} \sqcup \mathbf{0} = \mathbf{0}$, the distributive law does not hold.

Definition 3. *Valuations and validity*

A *valuation* v is a function from the set of propositional variables to $\mathscr{L}$ of $(\mathscr{L}, \sqcap, \sqcup, ', \mathbf{1}, \mathbf{0})$. Then, we extend the definition of v inductively as follows:

$v(\sim A) = v(A)'$

$v(A \wedge B) = v(A) \sqcap v(B)$

We say that A is *valid* in orthomodular lattices if $v(A) = \mathbf{1}$ for all v and all orthomodular lattices.

Definition 4. *O-frame*

An *O-frame* is a tuple $(W, \perp)$ where,

W: non empty set.

$\perp$: binary relation on W which is irreflexive and symmetric.

Given $X \subseteq W$, $X^{\perp} \stackrel{\text{def}}{=} \{w \in W | \text{for all } x \text{ in } X, x \perp w\}$. We say that X is $\perp$*-closed* or *testable* if $X^{\perp\perp} = X$.

Intuitively, each element of W represents one-dimensional closed subspace, and a $\perp$-closed set denotes a closed subspace.

Definition 5. *OM-model and validity*

An *OM-model* is a tuple $(W, \perp, \sigma, V)$ where,

$(W, \perp)$ is an O-frame.

σ is a set of $\perp$-closed set that satisfies the following conditions:

$W \in \sigma, \emptyset \in \sigma$,
closed under $\cap$ and $^{\perp}$,
for all $X, Y \in \sigma$, $X \cap (X \cap (X \cap Y)^{\perp})^{\perp} = X \cap Y$ (orthomodular law).

V is a function from the set of propositional variables to σ. Then, we extend the definition of V inductively as follows:

$$V(\sim A) = V(A)^{\perp}$$
$$V(A \wedge B) = V(A) \cap V(B)$$

We say that A is *true* at $x \in W$ if $x \in V(A)$ and write it as $x \models A$. We say that A is *false* at $x \in W$ if A is not true at x. We say that A is *valid* in OM-models if $V(A) = W$ for all OM-models. We write $A = B$ for $V(A) = V(B)$.

Theorem 1 ([3]). *For all formulas A, $V(A)$ is $\perp$-closed.*

Theorem 2 ([3]). *For all formulas A, A is valid in orthomodular lattices iff A is valid in OM-models.*

These theorems allows us to formulate OML using either orthomodular lattices or OM-models. Moreover, as in the case of algebraic and Kripke models for other logics, the respective operators can also be straightforwardly associated ($\sqcap \leftrightarrow \cap$ and $' \leftrightarrow ^{\perp}$) [3]. Therefore, in what follows, we adopt whichever is more appropriate for the purpose of exposition.

Next, we review a proof system for OML. Since OML is well suited to *sequent calculus*, we adopt this formulation. Among the various sequent calculi, **GOM**[15] is the most semantically transparent; therefore, we adopt it as the basis of this study. A *sequent* is defined by $\Gamma \Rightarrow \Delta$ where Γ and Δ are finite sets of formulas. We define that sequent $\Gamma \Rightarrow \Delta$ is *false* at $x \in W$ if all $A \in \Gamma$ is true at x and all $A \in \Delta$ is false at x. If $\Gamma \Rightarrow \Delta$ is not false at x, then it is *true* at x. If $\Gamma \Rightarrow \Delta$ is true at all $x \in W$ in all OM-models, then $\Gamma \Rightarrow \Delta$ is *valid* in OM-models. In the following definition of **GOM**, minor details have been altered from the original for convenience, but the essence remains unchanged.

GOM [15]
Axiom:

$$A \Rightarrow A$$

Rules:

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\ (\text{cut})$$

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ (\text{wL}) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\ (\text{wR})$$

$$\frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ (\wedge\text{L}) \qquad \frac{B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ (\wedge\text{L}) \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ (\wedge\text{R})$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\sim A, \Gamma \Rightarrow \Delta}\ (\sim\text{L}) \qquad \frac{A \Rightarrow \Delta}{\sim \Delta \Rightarrow \sim A}\ (\sim\text{R})$$

$$\frac{A, \Gamma \Rightarrow \Delta}{\sim\sim A, \Gamma \Rightarrow \Delta}\ (\sim\sim\text{L}) \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \sim\sim A}\ (\sim\sim\text{R})$$

$$\frac{\sim B \Rightarrow \sim A \qquad \sim A, B \Rightarrow}{\sim A \Rightarrow \sim B}\ (\text{OM})$$

The features of **GOM** are ($\sim$R) and (OM); otherwise, it is largely the same as classical logic. When $\Gamma \Rightarrow \Delta$ is derivable in the system **S**, we write $\mathbf{S} \vdash \Gamma \Rightarrow \Delta$.

Theorem 3 ([15]). **GOM** $\vdash \Gamma \Rightarrow \Delta$ *iff* $\Gamma \Rightarrow \Delta$ *is valid in OM-models.*

3 Restricted negation

Orthomodularity is a property whose conceptual essence is not easily captured, owing to the complexity of its formal definition. This property has been analyzed from a variety of physical and algebraic viewpoints. In this study, we examine this law from the perspective of combinations of operators. For example, we can regard the orthomodularity as *modus ponens* of Sasaki hook. $A \wedge A \to_S B \Rightarrow B$. We focus on an interpretation via *restricted negation* that has received little attention so far. Restricted negation of B by A is defined as A/B. We define this operator to be semantically equivalent to $A \wedge \sim (A \wedge B)$. In the lattice representation, we also introduce b^a as $a \sqcap (a \sqcap b)'$. The term "restriction" refers to taking negation in the subspace of the Hilbert space in which A is satisfied. Let H_1 be a closed subspace of a Hilbert space H. (H_1 itself is also a Hilbert space.) Let $^{\perp(H)}$ and $^{\perp(H_1)}$ be the operations that take orthogonal complements in their respective spaces. Then, by a well-known result concerning Hilbert spaces, for a closed subspace $H_2 \subseteq H_1$, we have $H_2^{\perp(H_1)} = H_1 \cap H_2^{\perp(H)} = H_1 \cap (H_1 \cap H_2)^{\perp(H)}$. From this viewpoint, $A \wedge \sim (A \wedge B)$ can be regarded as the orthogonal complement relative to A of the closed subspace B which is included in A. In terms of restricted negation, orthomodularity may be interpreted as a property of closed subspaces $H_2^{\perp(H_1)\perp(H_1)} = H_2$ in H_1.

$$
\begin{aligned}
A \wedge B = &(\text{orthomodularity}) = A \wedge \sim (A \wedge \sim (A \wedge B)) \\
&= A \wedge \sim (A \wedge A \wedge \sim (A \wedge B)) = A/A/B
\end{aligned}
$$

Therefore, by using this operator, orthomodularity can be expressed in a simple form, denoted by $A/A/B \Rightarrow B$ or $A/A/B \Leftrightarrow A \wedge B$. In the lattice representation, $b^{aa} \sqsubseteq b$ or $b^{aa} = a \sqcap b$. Based on the above analysis, this may be viewed as a generalization of the property $\sim\sim B = B$ concerning orthogonal complements in the entire space. When $V(A)$ coincides with the entire space $V(\top)$, this is nothing but the usual property of closed subspaces:

$$B = \top \wedge B = \top \wedge \sim (\top \wedge \top \wedge \sim (\top \wedge B)) = \sim\sim B$$

Let $\mathscr{L}_a = \{b \in \mathscr{L} | b \sqsubseteq a\}$ and $b \sqcup_a c = (b^a \sqcap c^a)^a$. Also in orthomodular lattices, as in the Hilbert space case, the following theorem provides evidence that restricted negation represents the orthogonal complement-taking operation in abstract subspace $\mathscr{L}_a$.

Theorem 4. *Let* $(\mathscr{L}, \sqcap, \sqcup, ', \mathbf{1}, \mathbf{0})$ *be an orthomodular lattice and* $a \in \mathscr{L}$. *Then,* $(\mathscr{L}_a, \sqcap, \sqcup_a, ^a, a, \mathbf{0})$ *is an orthomodular lattice.*

Proof.
1. This follows directly from the definition.
2. $b^{aa} = a \sqcap (a \sqcap (a \sqcap (a \sqcap b)'))' = a \sqcap (a \sqcap (a \sqcap b)')' = a \sqcap b = b$
3. $b \sqsubseteq c \Rightarrow a \sqcap b \sqsubseteq a \sqcap c \Rightarrow (a \sqcap c)' \sqsubseteq (a \sqcap b)' \Rightarrow a \sqcap (a \sqcap c)' \sqsubseteq a \sqcap (a \sqcap b)' \Rightarrow c^a \sqsubseteq b^a$
4. From the definition, $b \sqcup_a c = (b^a \sqcap c^a)^a$.
5. $b \sqcap b^a = b \sqcap (a \sqcap (a \sqcap b)') = (b \sqcap a) \sqcap (a \sqcap b)' = \mathbf{0}$
6. $b \sqcup_a b^a = a \sqcap (a \sqcap (a \sqcap (a \sqcap b)' \sqcap b^{aa}))' = a \sqcap (a \sqcap (a \sqcap (a \sqcap b)' \sqcap b))' = a \sqcap (a \sqcap \mathbf{0})' = a \sqcap \mathbf{1} = a$
7. (Orthomodularity) $b \sqcap (b \sqcap (b \sqcap c)^a)^a = b \sqcap a \sqcap (a \sqcap b \sqcap a \sqcap (a \sqcap b \sqcap c)')' = (b \sqcap a) \sqcap ((b \sqcap a) \sqcap ((b \sqcap a) \sqcap c)')' = b \sqcap (b \sqcap (b \sqcap c)')' = b \sqcap c$

□

Another advantage of restricted negation is that, when combined with $\mathbf{1}$, it allows all existing operators $'$, $\sqcap$, and $\sqcup$ to be expressed in terms of this single binary operator, as shown below:

$b' = \mathbf{1} \sqcap (\mathbf{1} \sqcap b)' = b^{\mathbf{1}}$

$a \sqcap b = b^{aa}$

From the properties of orthomodular lattices, $\sqcup$ is already expressible in terms of $\sqcap$ and $'$.

On the basis of these considerations, we define a new language for OML.

Definition 6. *Language L_2 and formulas of OML*

The language L_2 of OML consists of a countable infinite set of propositional variables $p, q, r, \ldots$, constant symbol $\top$, and the logical connective $/$. Then, formulas of OML (for L_2) are defined as follows:

$A ::= \top \mid p \mid A/A$

The set of all formulas for L_2 is denoted by Ω_2. Hereafter, for the purposes of the proof of completeness theorem of sequent calculi, we develop the argument based on OM-models rather than the lattices. As in the previous section, we define a valuation V as a function from the set of propositional variables to σ. Then, we extend the definition of V inductively as follows:

$V(\top) = W$

$V(A/B) = V(A) \cap (V(A) \cap V(B))^{\perp}$

The notions of truth and validity are defined in the same way as in the previous section. This binary operator $/$ is assumed to be right-associative. Since there is only one binary operator, we abbreviate A/B as AB. For example, $pqrs = p/(q/(r/s))$. In terms of the framework of OM-models, the correspondences mentioned above are given as follows:

Lemma 1. *For all OM-models, $V(B)^{\perp} = V(\top B)$, and $V(A) \cap V(B) = V(AAB)$*

Proof. This is immediate from the definition of V together with the orthomodularity of σ. □

We include $\top$ in L_2 because the following theorem holds.

Theorem 5. *There exists no valid formula consisting solely of propositional variables and $/$.*

Proof. This is for the same reason that valid formulas cannot be constructed from propositional variables and $\wedge$ in classical logic. That is, if A/B is valid, then A must be valid because $A/B = A \wedge \sim (A \wedge B)$. Applying this to A and iterating the same argument, we would eventually obtain that a single propositional variable is valid. However, this is not the case. □

4 Sequent calculi with restricted negation

We now construct sequent calculi of OML based on L_2. We begin by introducing a basic system $\mathbf{OMR}^{-}$. While $\mathbf{OMR}^{-}$ is sound, its completeness is currently unknown. Subsequently, we introduce two extensions of $\mathbf{OMR}^{-}$, namely $\mathbf{OMR}^{S}$ and $\mathbf{OMR}^{\top}$. Both satisfy completeness and exhibit distinct characteristics.

$\mathbf{OMR}^{-}$
Axioms:

$$A \Rightarrow A \qquad\qquad \Rightarrow \top$$

Rules:

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma} \text{ (cut)}$$

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \text{ (wL)} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A} \text{ (wR)}$$

$$\frac{A, \Gamma \Rightarrow \Delta}{AAB, \Gamma \Rightarrow \Delta} \text{ ($\wedge$L1)} \qquad \frac{B, \Gamma \Rightarrow \Delta}{AAB, \Gamma \Rightarrow \Delta} \text{ ($\wedge$L2)}$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, AAB} \text{ ($\wedge$R)}$$

$$\frac{B \Rightarrow C}{AC, B \Rightarrow} \text{ ($A \sim$L)} \qquad \frac{B \Rightarrow C}{AC \Rightarrow AB} \text{ ($A \sim$R)}$$

One may also regard ($A \sim$L) and ($A \sim$R) as rules derived from the rules for the ordinary negation $\sim$, where $\sim$ is replaced by the restricted negation $A/$.

Theorem 6 (Soundness theorem for **OMR**$^-$)**.** *If $\Gamma \Rightarrow \Delta$ is provable in* **OMR**$^-$*, then $\Gamma \Rightarrow \Delta$ is valid.*

Proof. The proof proceeds by induction on the inference rule used in the last step. The proof for (cut), (wL), and (wR) are analogous to that in the classical case. The proof for ($\wedge$L1), ($\wedge$L2), and ($\wedge$R) are also analogous to that in the classical case because $AAB = A \wedge B$.

($A \sim$L) Suppose that $B \Rightarrow C$ is valid. In all OM-models, $V(B) \subseteq V(C)$. Then, $(V(A) \cap V(B) \subseteq V(A) \cap V(C)$, $(V(A) \cap V(B)) \cap (V(A) \cap V(C))^\perp = \emptyset$, and $(V(A) \cap (V(A) \cap V(C))^\perp) \cap V(B) = V(AC) \cap V(B) = \emptyset$. Therefore, $AC, B \Rightarrow$ is valid.

($A \sim$R) Suppose that $B \Rightarrow C$ is valid. In all OM-models, $V(B) \subseteq V(C)$. Then, $V(A) \cap V(B) \subseteq V(A) \cap V(C)$, $(V(A) \cap V(C))^\perp \subseteq (V(A) \cap V(B))^\perp$, and $V(A) \cap (V(A) \cap V(C))^\perp \subseteq V(A) \cap (V(A) \cap V(B))^\perp$. Therefore, $AC \Rightarrow AB$ is valid.

□

We define **OMR**^{S} as the system obtained by adding the following axioms (S1) and (S2) to **OMR**$^-$. (Here, S stands for "semantic equivalence axiom".)

$$AB \Rightarrow AA\top AAB \quad \text{(S1)}$$

$$AA\top AAB \Rightarrow AB \quad \text{(S2)}$$

Intuitively, (S1) and (S2) correspond to expressing the meaning of AB in terms of $V(AAB) = V(A) \cap V(B)$ and $V(\top B) = V(B)^\perp$. In other words, this can be seen as a "translation" of AB in L_2 into L_1 using $\wedge$ and $\sim$. Although some rules of **GOM** and those of **OMR**^{S} are semantically similar, there are essential differences between them. By removing the orthomodularity rule (OM) from **GOM**, we obtain a system for logic on ortho lattices, called *ortho logic* (OL).

In contrast, **OMR**^{S} has no rule corresponding to (OM) and instead internalizes notions concerning the orthomodularity. For example, ($\wedge$L1), ($\wedge$L2), and ($\wedge$R) reflect a property $AAB = A \wedge B$ that holds in OML but not in OL. In other words, it can be regarded that the rule for $\wedge$ and for the orthomodularity are combined. This constitutes one of the major advantages of **OMR**^{S}. Although eliminating (S1) and (S2) does not yield a proof system for OL, this does not pose a problem when considering OML.

It is easy to verify that adding the following rules to **OMR**$^-$ instead of (S1) and (S2) yields a system with exactly the same proof-theoretic strength with **OMR**^{S}.

$$\frac{AB,\Gamma \Rightarrow \Delta}{AA\top AAB,\Gamma \Rightarrow \Delta}\ (\text{*L}) \qquad \frac{\Gamma \Rightarrow \Delta, AB}{\Gamma \Rightarrow \Delta, AA\top AAB}\ (\text{*R})$$

However, from a technical point of view, we use (S1) and (S2) in this study.

Theorem 7 (Soundness theorem for $\mathbf{OMR}^S$)**.** *If $\Gamma \Rightarrow \Delta$ is provable in $\mathbf{OMR}^S$, then $\Gamma \Rightarrow \Delta$ is valid.*

Proof. Validity of (S1) and (S2) is clear from the definition of $V(AB)$, $V(AAB) = V(A) \cap V(B)$, and $V(\top B) = V(B)^{\perp}$. Therefore, from Theorem 6, $\mathbf{OMR}^S$ is also sound. □

A natural approach to proving the completeness of $\mathbf{OMR}^S$ would be to exploit the completeness of **GOM**. Nevertheless, despite the presence of translation-like axioms (S1) and (S2), the differences in language and the significant divergence in the form of the axioms make this approach infeasible at present. (For example, $A \wedge B$ is a combination of two formulas, whereas AAB is a combination of three; therefore, it is difficult to apply the same structural induction to both $A \wedge B$ and AAB.) Moreover, from a proof-theoretic standpoint, establishing the result in a system formulated solely in terms of restricted negation is preferable.In the proof of completeness, a canonical model is constructed. However, this construction is difficult when working with the axioms (S1) and (S2). Therefore, our approach is to develop another system $\mathbf{OMR}^{\top}$, prove its completeness, and through this, establish the completeness of $\mathbf{OMR}^S$. We define $\mathbf{OMR}^{\top}$ as the system obtained by adding the following rules to $\mathbf{OMR}^{-}$.

$$\frac{A, B \Rightarrow C}{A, \top C \Rightarrow AB}\ (\top 1) \qquad \frac{\Gamma \Rightarrow AAB}{\Gamma \Rightarrow \top AB}\ (\top 2) \qquad \frac{A, \Gamma \Rightarrow \Delta}{AB, \Gamma \Rightarrow \Delta}\ (\top 3)$$

These rules are not particularly well-structured in form and are introduced as tools for establishing completeness; however, each of them can also be given a semantic interpretation. ($\top$1) is similar to (A ∼R), making use of the fact that, on the left-hand side of a sequent, $\{A, B\}$ is interpreted as $A \wedge B$. Because $\top AB$ is equivalent to Sasaki hook $\top AB = \sim (A \wedge \sim (A \wedge B)) = \sim A \vee (A \wedge B)$, ($\top$2) can be viewed as analogous to the rule for $\vee$. Because $AB = A \wedge \sim (A \wedge B)$, ($\top$3) can be viewed as a variant of the rule for $\wedge$.

Lemma 2. $\mathbf{OMR}^S \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathbf{OMR}^{\top} \vdash \Gamma \Rightarrow \Delta$.

Proof. The proof proceeds by showing that any axiom occurring only on $\mathbf{OMR}^S$ can be simulated using the rules on $\mathbf{OMR}^{\top}$ and vice versa.

(S1)

$$\dfrac{\dfrac{AB \Rightarrow AB \qquad \dfrac{\Rightarrow \top}{AB \Rightarrow \top}\ (\text{wL})}{AB \Rightarrow \top\top AB}\ (\wedge\text{R}) \qquad \dfrac{\dfrac{AAB \Rightarrow AAB}{AAB \Rightarrow \top AB}\ (\top 2)}{\top\top AB \Rightarrow \top AAB}\ (A\sim\text{R})}{AB \Rightarrow \top AAB}\ (\text{cut})$$

$$\dfrac{\dfrac{A \Rightarrow A}{AB \Rightarrow A}\ (\top 3) \qquad \dfrac{\vdots}{AB \Rightarrow \top AAB}}{AB \Rightarrow AA\top AAB}\ (\wedge R)$$

(S2)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A\Rightarrow A}{A,B\Rightarrow A}\,(\mathrm{wL})\quad \dfrac{B\Rightarrow B}{A,B\Rightarrow B}\,(\mathrm{wL})}{A,B\Rightarrow AAB}\,(\wedge\mathrm{R})}{A,\top AAB\Rightarrow AB}\,(\top 1)}{A,AA\top AAB\Rightarrow AB}\,(\wedge\mathrm{L2})}{AA\top AAB\Rightarrow AB}\,(\wedge\mathrm{L1})$$

$(\top 1)$

$$\dfrac{\dfrac{\dfrac{A\Rightarrow A}{A,\top C\Rightarrow A}\,(\mathrm{wL})\quad \dfrac{\dfrac{\dfrac{\dfrac{A,B\Rightarrow C}{A,AAB\Rightarrow C}\,(\wedge\mathrm{L2})}{AAB\Rightarrow C}\,(\wedge\mathrm{L1})}{\top C\Rightarrow \top AAB}\,(A\sim\mathrm{R})}{A,\top C\Rightarrow \top AAB}\,(\mathrm{wL})}{A,\top C\Rightarrow AA\top AAB}\,(\wedge\mathrm{R})\qquad AA\top AAB\Rightarrow AB}{A,\top C\Rightarrow AB}\,(\mathrm{cut})$$

$(\top 2)$

$$\dfrac{\dfrac{AB\Rightarrow AA\top AAB\quad \dfrac{\top AAB\Rightarrow \top AAB}{AA\top AAB\Rightarrow \top AAB}\,(\wedge\mathrm{L2})}{AB\Rightarrow \top AAB}\,(\mathrm{cut})}{\top\top AAB\Rightarrow \top AB}\,(A\sim\mathrm{R})$$

$$\dfrac{\Gamma\Rightarrow AAB\qquad \dfrac{\dfrac{\dfrac{\Rightarrow\top}{AAB\Rightarrow\top}\,(\mathrm{wL})\quad AAB\Rightarrow AAB}{AAB\Rightarrow \top\top AAB}\,(\wedge\mathrm{R})\quad \dfrac{\vdots}{\top\top AAB\Rightarrow \top AB}}{AAB\Rightarrow \top AB}\,(\mathrm{cut})}{\Gamma\Rightarrow \top AB}\,(\mathrm{cut})$$

$(\top 3)$

$$\dfrac{\dfrac{AB\Rightarrow AA\top AAB\quad \dfrac{A\Rightarrow A}{AA\top AAB\Rightarrow A}\,(\wedge\mathrm{L1})}{AB\Rightarrow A}\,(\mathrm{cut})\qquad A,\Gamma\Rightarrow\Delta}{AB,\Gamma\Rightarrow\Delta}\,(\mathrm{cut})$$

□

Theorem 8 (Soundness theorem for $\mathbf{OMR}^\top$)**.** *If $\Gamma\Rightarrow\Delta$ is provable in $\mathbf{OMR}^\top$, then $\Gamma\Rightarrow\Delta$ is valid.*

Proof. A corollary of Theorem 7 and Lemma 2. □

Lemma 3 (Disjunction theorem for $\mathbf{OMR}^-$, $\mathbf{OMR}^S$, and $\mathbf{OMR}^\top$)**.** *If $\Gamma\Rightarrow\Delta$ is provable in $\mathbf{OMR}^-$ ($\mathbf{OMR}^S$, $\mathbf{OMR}^\top$) and $\Delta\neq\emptyset$, then there exists $A\in\Delta$ such that $\Gamma\Rightarrow A$ is provable in $\mathbf{OMR}^-$ ($\mathbf{OMR}^S$, $\mathbf{OMR}^\top$).*

Proof. The proof is almost the same as the one for **GOM** in [15]. □

We now define the tools necessary to prove completeness. An *infinite sequent* is defined as $\Gamma \Rightarrow \Delta$, where Γ and Δ are (could be infinite) set of formulas. (Although we employ the same symbols as for sequent, it should be clear from the context which interpretation is meant.) We define $\Gamma' \Rightarrow \Delta'$ to be a *finite subsequent* of $\Gamma \Rightarrow \Delta$ if $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$, and Γ' and Δ' are finite sets. We say that an infinite sequent is unprovable if all of its finite subsequents are not provable.

Lemma 4. *If $\Gamma \Rightarrow \Delta$ (consisting of L_2) is unprovable in* $\mathbf{OMR}^{\top}$*, there exists unprovable infinite sequent $\Gamma' \Rightarrow \Delta'$ such that $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$, and $\Gamma' \cup \Delta' = \Omega_2$.*

Proof. Because (cut) is included in $\mathbf{OMR}^{\top}$, if $\Gamma \Rightarrow \Delta$ is unprovable, for all $A \in \Omega_2$, at least one of $A, \Gamma \Rightarrow \Delta$ or $\Gamma \Rightarrow \Delta, A$ is unprovable. By adding every $A \in \Omega_2$ to either the left or right side of the sequent in this way, we obtain an infinite unprovable sequent as limit. □

We define that $\top\Gamma = \{\top A | A \in \Gamma\}$. A canonical model $(W_c, \perp_c, \sigma_c, V_c)$ is defined as follows:

1. W_c is defined as the set of all infinite sequents $\Gamma \Rightarrow \Delta$ that satisfy the following conditions:
   $\Gamma \Rightarrow \Delta$ is unprovable in $\mathbf{OMR}^{\top}$.
   $\Gamma \cup \Delta = \Omega_2$.
2. $\perp_c \overset{\text{def}}{=} \{(\Gamma \Rightarrow \Delta, \Gamma' \Rightarrow \Delta') | \exists A \in \Omega_2, A \in \Gamma \text{ and } \top A \in \Gamma'\}$
3. $\sigma_c \overset{\text{def}}{=} \{V(A) | A \in \Omega_2\}$
4. $V_c(p) \overset{\text{def}}{=} \{\Gamma \Rightarrow \Delta \in W_c | p \in \Gamma\}$

Lemma 5. $(W_c, \perp_c, \sigma_c, V_c)$ *is an OM-model.*

Proof. The irreflexivity of $\perp_c$

Because $A, \top A \Rightarrow$ is provable by ($A \sim$L), for all $(\Gamma \Rightarrow \Delta) \in W_c$ and $A \in \Omega_2$, if $A \in \Gamma$, then $\top A \in \Delta$.

The symmetry of $\perp_c$

Suppose that $(\Gamma \Rightarrow \Delta) \perp_c (\Gamma' \Rightarrow \Delta')$ because of $A \in \Gamma$ and $\top A \in \Gamma'$. Then, $(\Gamma' \Rightarrow \Delta') \perp_c (\Gamma \Rightarrow \Delta)$ because $A \Rightarrow \top\top A$ is provable by (wL) and ($\wedge$R), it follows that $\top\top A \in \Gamma$.

$\perp$-closedness of $V_c(p)$

By the definition of $\perp$-closed set, $X \subseteq X^{\perp\perp}$ holds for all $X \subseteq W$. Therefore, it suffices to prove that $V_c(p)^{\perp\perp} \subseteq V_c(p)$. For the sake of contradiction, suppose that $V_c(p)^{\perp\perp} \not\subseteq V_c(p)$. That is, there exists $(\Gamma \Rightarrow \Delta, p) \in W_c$ such that $(\Gamma \Rightarrow \Delta, p) \perp_c x$ for all $x \in V_c(p)^{\perp}$. If $\top p \Rightarrow \top\Gamma$ is provable, from Lemma 3, there exists $\top C \in \top\Gamma$ such that $\top p \Rightarrow \top C$ is provable. Then, $\Gamma \Rightarrow \Delta, p$ is provable because of ($A \sim$R), (cut), (wL), and the provability of $A \Rightarrow \top\top A$ and $\top\top A \Rightarrow A$. Therefore, $\top p \Rightarrow \top\Gamma$ is not provable. Then, from Lemma 4, there exists unprovable infinite sequent $\Gamma' \Rightarrow \Delta'$ such that $\top p \in \Gamma'$, $\top\Gamma \subseteq \Delta'$, and $\Gamma' \cup \Delta' = \Omega_2$. This is a contradiction because $(\Gamma' \Rightarrow \Delta') \not\perp_c (\Gamma \Rightarrow \Delta, p)$ and $(\Gamma' \Rightarrow \Delta') \in V_c(p)^{\perp}$.

Orthomodularity

The proof is carried out by applying Lemma 6, which does not require the use of this concept. Because $AAB \Rightarrow B$ is provable by ($\wedge$L1), for all $\Gamma \Rightarrow \Delta \in W_c$, if $AAB \in \Gamma$, then $B \in \Gamma$. Therefore, from Lemma 6, for all $A, B \in \Omega_2$, if $(\Gamma \Rightarrow \Delta) \models AAB$, then $(\Gamma \Rightarrow \Delta) \models B$. This implies that $V(AAB)(= V(A) \cap (V(A) \cap (V(A) \cap V(B))^{\perp})^{\perp}) \subseteq V(B)$. Therefore, $V(A) \cap (V(A) \cap (V(A) \cap V(B))^{\perp})^{\perp} \subseteq V(A) \cap V(B)$. An analogous argument applied to $A, B \Rightarrow AAB$ yields $V(A) \cap V(B) \subseteq V(A) \cap (V(A) \cap (V(A) \cap V(B))^{\perp})^{\perp}$.

□

Lemma 6. *In $(W_c, \perp_c, \sigma_c, V_c)$, for all $\Gamma \Rightarrow \Delta \in W_c$ and $A \in \Omega_2$, $(\Gamma \Rightarrow \Delta) \models A$ iff $A \in \Gamma$.*

The proof proceeds by induction on the structure of formulas.

Suppose that $p \in \Gamma$. From the definition of V_c, $(\Gamma \Rightarrow \Delta) \models p$.

Suppose that $p \in \Delta$. From the definition of V_c, $(\Gamma \Rightarrow \Delta) \not\models p$.

Suppose that $AB \in \Gamma$. To show that AB is true at $(\Gamma \Rightarrow \Delta) \in W_c$, it suffices to establish $(\Gamma \Rightarrow \Delta) \models A$ and $(\Gamma \Rightarrow \Delta)$ is orthogonal to all $(\Gamma' \Rightarrow \Delta') \in W_c$ such that $(\Gamma' \Rightarrow \Delta') \models A$ and $(\Gamma' \Rightarrow \Delta') \models B$.

$AB \Rightarrow A$ is provable in $\mathbf{OMR}^\top$ as follows:

$$\dfrac{A \Rightarrow A}{AB \Rightarrow A}\ (\top 3)$$

Therefore, $A \in \Gamma$. From the induction hypothesis, $(\Gamma \Rightarrow \Delta) \models A$.

From the induction hypothesis, for all $(\Gamma' \Rightarrow \Delta') \in W_c$, if $(\Gamma' \Rightarrow \Delta') \models A$ and $(\Gamma' \Rightarrow \Delta') \models B$, then $A, B \in \Gamma'$. $A, B \Rightarrow \top AB$ is provable in $\mathbf{OMR}^\top$ as follows:

$$\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A, B \Rightarrow A}\ (\mathrm{wL}) \quad \dfrac{B \Rightarrow B}{A, B \Rightarrow B}\ (\mathrm{wL})}{A, B \Rightarrow AAB}\ (\wedge\mathrm{R})}{A, B \Rightarrow \top AB}\ (\top 2)$$

Therefore, if $A, B \in \Gamma'$, then $\top AB \in \Gamma'$. From the definition of $\perp_c$, $(\Gamma \Rightarrow \Delta) \perp_c (\Gamma' \Rightarrow \Delta')$. Therefore, $(\Gamma \Rightarrow \Delta)$ is orthogonal to all $(\Gamma' \Rightarrow \Delta') \in W_c$ such that $(\Gamma' \Rightarrow \Delta') \models A$ and $(\Gamma' \Rightarrow \Delta') \models B$.

Suppose that $AB \in \Delta$. If $A \in \Delta$, from the induction hypothesis, $(\Gamma \Rightarrow \Delta) \not\models A$.

Therefore, $(\Gamma \Rightarrow \Delta) \not\models AB$.

Suppose that $A \in \Gamma$. For the sake of contradiction, suppose that $A, B \Rightarrow \top\Gamma$ is provable in $\mathbf{OMR}^\top$. From Lemma 3. there exists $C \in \Gamma$ such that $A, B \Rightarrow \top C$ is provable. Then, $A, C \Rightarrow AB$ is provable as follows:

$$\dfrac{\dfrac{C \Rightarrow \top \quad C \Rightarrow C}{C \Rightarrow \top\top C}\ (\wedge\mathrm{R}) \quad \dfrac{A, B \Rightarrow \top C}{A, \top\top C \Rightarrow AB}\ (\top 1)}{A, C \Rightarrow AB}\ (\mathrm{cut})$$

Because $A, C \in \Gamma$ and $AB \in \Delta$, $\Gamma \Rightarrow \Delta$ is provable. This is a contradiction. Therefore, $A, B \Rightarrow \top\Gamma$ is not provable. Then, from Lemma 4, there exists unprovable infinite sequent $\Gamma' \Rightarrow \Delta'$ such that $A, B \in \Gamma'$, $\top\Gamma \subseteq \Delta'$ and $\Gamma' \cup \Delta' = \Omega_2$. We have $(\Gamma' \Rightarrow \Delta') \not\perp_c (\Gamma \Rightarrow \Delta)$ because $\top\Gamma \subseteq \Delta'$. From the induction hypothesis, $(\Gamma' \Rightarrow \Delta') \models A$ and $(\Gamma' \Rightarrow \Delta') \models B$. This means that one of the above conditions for AB to be true is violated. Therefore, $(\Gamma \Rightarrow \Delta) \not\models AB$.

Theorem 9 (Completeness theorem for $\mathbf{OMR}^\top$)**.** *If $\Gamma \Rightarrow \Delta$ (consisting of L_2) is valid in all OM-models, then $\Gamma \Rightarrow \Delta$ is provable in $\mathbf{OMR}^\top$.*

Proof. We prove the contrapositive. Suppose $\Gamma \Rightarrow \Delta$ is not provable. Then, from Lemma 4, in a canonical model, there exists unprovable infinite sequent $\Gamma' \Rightarrow \Delta'$ such that $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$, and $(\Gamma' \Rightarrow \Delta') \in W_c$. By Lemma 5 and 6, $\Gamma \Rightarrow \Delta$ is false at $(\Gamma' \Rightarrow \Delta') \in W_c$ in the OM-model $(W_c, \perp_c, \sigma_c, V_c)$. $\square$

Theorem 10 (Completeness theorem for $\mathbf{OMR}^S$)**.** *If $\Gamma \Rightarrow \Delta$ (consisting of L_2) is valid in all OM-models, then $\Gamma \Rightarrow \Delta$ is provable in $\mathbf{OMR}^S$.*

Proof. A corollary of Lemma 2 and Theorem 9. □

5 Conclusion and Future Work

In this study, we investigated the restricted negation and developed a proof system of OML based on it. Through a detailed analysis of restricted negation, we clarified several of its fundamental properties and demonstrated how these properties can be exploited to construct a new framework.

One of the main contributions of this work is the introduction of a proof system that enables a novel viewpoint on OML. By explicitly incorporating the characteristics of restricted negation into the inference rules, the proposed system offers a foundation for further logical investigations and serves as a potential tool for analyzing OML in a systematic manner. These results suggest that proof-theoretic approaches based on restricted negation can complement existing methods and contribute to a deeper understanding of OML.

Several directions for future research naturally arise from this study. First, the proposed proof system can be further utilized as a tool for a multifaceted analysis of OML, exploring its behavior from different logical and methodological perspectives.

Second, it is important to clarify the correspondence between restricted negation and existing analytical frameworks for OML, which may reveal deeper connections between our approach and previously established results. For example, when analyzing the relationship between L_1 and L_2, one can also consider the following translation by taking into account the semantics of the formulas:

L_2 to L_1

$$\tau(p) = p$$
$$\tau(AB) = \tau(A) \wedge \sim (\tau(A) \wedge \tau(B))$$

L_1 to L_2

$$\tau'(p) = p$$
$$\tau'(A \wedge B) = \tau'(A)\tau'(A)\tau'(B)$$
$$\tau'(\sim A) = \top \tau'(A)$$

These translations gives rise to formulas similar to those appearing in the present study as follows:

$$\tau'(\tau(pq)) = \tau'(p \wedge \sim (p \wedge q)) = pp\top ppq$$
$$\tau(\tau'(p \wedge q)) = \tau(ppq) = p \wedge \sim (p \wedge \tau(pq)) = p \wedge \sim (p \wedge (p \wedge \sim (p \wedge q)))$$

However, as noted above, an effective method to exploit them has not yet been discovered.

Finally, as in the original proof system, the cut-elimination theorem does not hold in the present framework. Extending the system so that the cut-elimination theorem becomes derivable remains an important and challenging topic for future work. This will likely be achieved by sequents with more complex structures, such as hyper sequent and nested sequent [10, 19].

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