A Complete and Natural Rule Set for Multi-Qudit Clifford Circuits in All Odd Prime Dimensions

Xiaoning Bian [*] Sarah Meng Li[†] Neil J. Ross [‡] John van de Wetering[†] Yuming Zhao [§]

Many physical platforms for qubits naturally possess higher energy levels that can encode qudits, enabling greater information density at the cost of increased control complexity [15, 21, 36, 37, 40]. In particular, when the qudit dimension d is an odd prime, qudit systems exhibit rich algebraic structure that result in unique error correction capabilities [13, 39], lower overhead magic state distillation [14, 35], stronger quantum correlations [12, 17], and improvements to various quantum algorithms [9, 33]. Ultimately, there is a trade-off between the engineering cost of accessing higher dimensions and the computational advantages they provide. By characterising qudit circuits in a systematic manner, we can better understand this balance and exploit the potentials of higher-dimensional quantum information processing.

Recently, algebraic approaches to quantum circuits have gained significant attention. An important problem in this line of work is to develop sound and complete equational theories for circuit families, meaning any two circuits representing the same linear map can be transformed into each other using a finite rule set. While there has been substantial progress for qubits [2, 3, 6, 7, 8, 24, 30, 31, 38, 16], characterisation of qudit circuits in terms of generators and relations remains much less explored.

Among quantum circuit fragments, the Clifford group plays a prominent role. Clifford unitaries underpin the stabiliser formalism [1, 19], fault-tolerant quantum computing [22, 23, 25], efficient classical simulation [26, 32], and practical circuit optimisation techniques [10, 11]. As a result, obtaining a complete and finite equational theory for qudit Clifford circuits is a natural and foundational problem.

Building on the characterisation of the qutrit Clifford group [29], where a qutrit is a three-dimensional ($d = 3$) qudit, we present a complete set of rewrite rules for multi-qudit Clifford circuits in all odd prime dimensions. There are 19 *Clifford relations* in total, each involving at most three qudits and admitting an intuitive interpretation.

Theorem. *The rewrite rules in Figure 1 are* complete *for n-qudit Clifford circuits over any qudit dimension d that is an odd prime. That is, given two n-qudit unitary Clifford circuits C_1 and C_2 implementing the same linear map, there is a sequence of rewrites from Figure 1 that proves C_1 and C_2 are equal.*

To prove this, we leverage the isomorphism between the symplectic group $\mathrm{Sp}(2n, \mathbb{Z}_d)$ and the quotient of the Clifford group by the Pauli group. We first derive a complete set of *symplectic relations* for $\mathrm{Sp}(2n, \mathbb{Z}_d)$, and then lift them to Clifford relations by incorporating Pauli corrections. To do this, we introduce a *symplectic normal form* that captures the stabiliser tableau of a Clifford operator. This normal form is then unique for a Clifford unitary up to a Pauli correction. This simplification enables a streamlined derivation of a complete set of 66 parametrised relations, which we further compress to 18 symplectic relations. They suffice to reduce any Clifford circuit to this normal form. Finally, we show how the lifted symplectic relations alongside the Pauli relations can be further reduced to a small set of 19 Clifford relations in Figure 1. In addition to the Clifford generators H, S and CZ, we also use "derived generators" listed in Figure 2, such as SWAP, CX, and the *multiplier* M_g which is defined by its action on the computational basis state, $M_g|x\rangle = |gx\rangle$, where multiplication is over $\mathbb{Z}_d$. These derived generators allow us to present the rewrite rules as intuitive (quasi-)commutation gate relations.

All computations in $\mathrm{Sp}(2n, \mathbb{Z}_d)$ are formalised in the Agda proof assistant, providing a machine-verified proof of correctness. The corresponding code is available in the GitHub repository [5]. Our results not only bridge algebraic group theory with syntactic quantum circuit reasoning, but also lay the foundation for formal verification of quantum protocols in qudit architectures, automated circuit optimisation and synthesis, and the mathematical study of symplectic groups.

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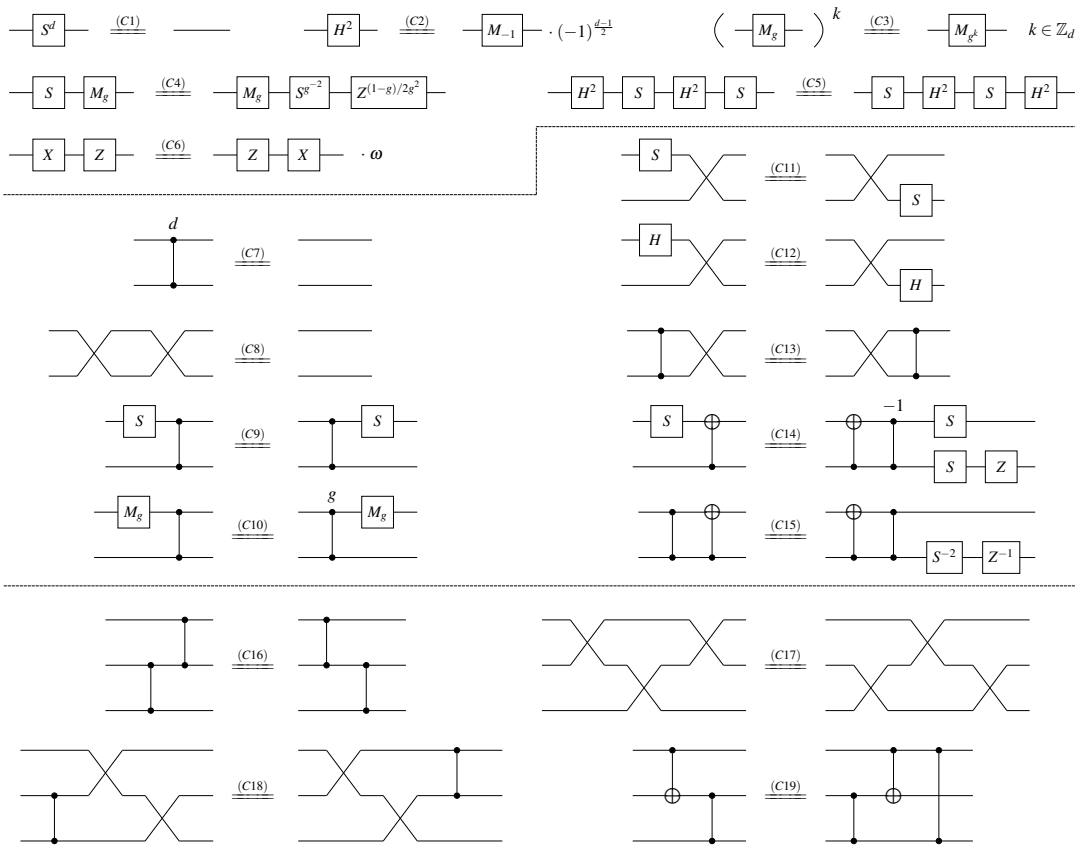

Figure 1: A complete set of rewrite rules for n-qudit Clifford circuits where d is an odd prime and $n \in \mathbb{N}$. Derived generators such as M_g, X, Z, SWAP, and CX are defined in Figure 2.

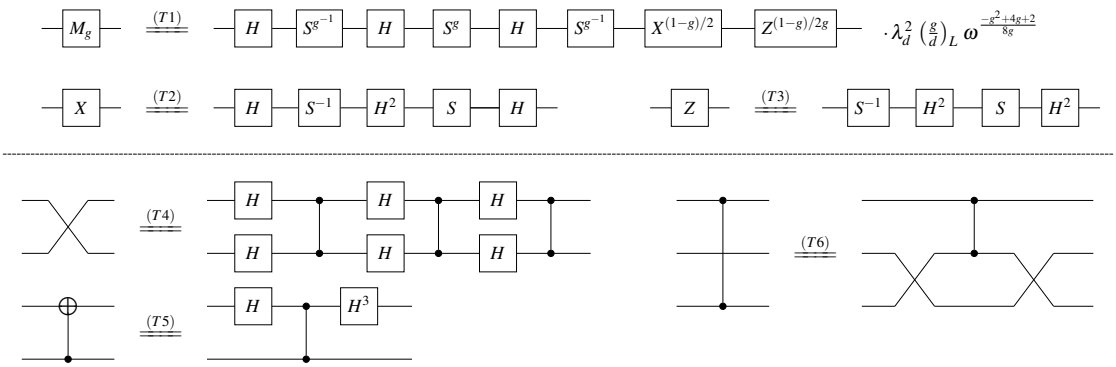

Figure 2: A set of derived Clifford generators, expressed as circuits over the generating set $\{H, S, \mathrm{CZ}\}$. Here, g generates the multiplicative group $\mathbb{Z}_d^*$, and $\left(\frac{g}{d}\right)_L$ denotes the Legendre symbol in number thoery [34, Appendix A].

Preliminaries Let d be an odd prime and let a qudit denote a d-dimensional quantum system. In what follows, we define unitary operators by their actions on the computational basis states. The single-qudit Pauli operators are defined as $X|j\rangle = |j+1\rangle$, $Z|j\rangle = \omega^j|j\rangle$, where $\omega = e^{2\pi i/d}$ and addition is over $\mathbb{Z}_d$. The n-qudit Pauli group $\mathcal{P}_n$ is generated by tensor products of X and Z, together with phases ω^t for $t \in \mathbb{Z}_d$. The qudit H, S, and CZ gates [18, 20, 22, 27, 28, 34] are defined below, where $i = e^{2\pi i/4}$ is the imaginary unit. The choice of global phases ensures that each matrix determinant is equal to 1.

$$H : |j\rangle \mapsto \frac{1}{\lambda_d \sqrt{d}} \sum_{\ell=0}^{d-1} \omega^{j\ell} |\ell\rangle, \;\; \lambda_d = (-i)^{\frac{d-1}{2}}, \qquad S : |j\rangle \mapsto \omega^{\frac{j(j-1)}{2}} |j\rangle, \qquad CZ : |j\rangle|\ell\rangle \mapsto \omega^{j\ell} |j\rangle|\ell\rangle.$$

The n-qudit Clifford group $\mathcal{C}_n$ is the normaliser of $\mathcal{P}_n$ in $\mathcal{U}(d^n)$. It is generated by H, S, and CZ gates via matrix multiplication and tensor product up to global phases that we will ignore. It is well known that the qudit Clifford group modulo Paulis is isomorphic to the symplectic group $\mathrm{Sp}(2n,\mathbb{Z}_d)$ [4]. This correspondence allows Clifford generators to be represented by symplectic matrices over the finite field $\mathbb{Z}_d$, forming the basis of stabiliser tableau representation of Clifford operators [1, 22]. Building on this correspondence, we generalise the prior Clifford completeness results [29, 31, 38] for qubits and qutrits to all odd primes.

A Unique Normal Form for Multi-Qudit Clifford Circuits To find a complete set of rewrite rules, we first define a unique normal form for Clifford circuits. To simplify the construction, we introduce a normal form with two components: a *symplectic normal form* characterising the stabiliser tableau of the Clifford circuit, followed by a *Pauli normal form* giving the necessary Pauli correction. This decomposition reflects the semidirect product structure of the projective Clifford group $\widehat{\mathcal{C}_n}$, where $\widehat{\mathcal{C}_n} \cong \mathrm{Sp}(2n,\mathbb{Z}_d) \ltimes (\mathbb{Z}_d)^{2n}$ [41]. We show that symplectic normal forms $N^{(n)}$ are in one-to-one correspondence with symplectic matrices $M \in \mathrm{Sp}(2n,\mathbb{Z}_d)$. The normal form $N^{(n)}$ is defined inductively using structured layers of circuit fragments, which we call *normal boxes*. These are labelled A, B, D, and E, each implementing a specific action on Pauli generators. An illustration of the multi-qudit Clifford normal form is given in Figure 3.

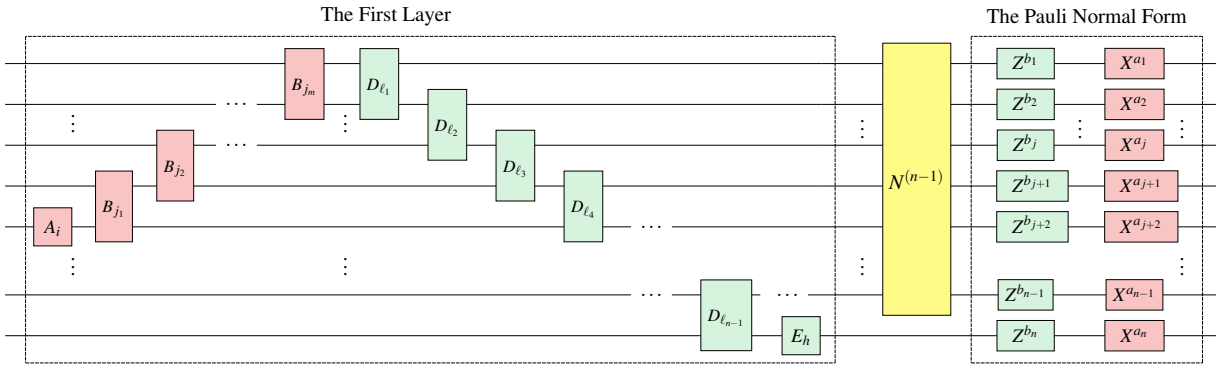

Figure 3: The inductive construction of the normal form for any $C \in \mathcal{C}_n$, up to a global phase. Each layer of the normal form alternates the Z-normal (red) and X-normal (green) circuits, progressively eliminating nontrivial stabiliser tableau entries. The first layer trivialises the conjugation action on the n-th qudit. By induction, the normal form $N^{(n-1)}$ faithfully implements the action of C on the first $n-1$ qudits. Finally, the Pauli normal form carries out the Pauli correction.

Establishing Qudit Clifford Completeness To obtain a complete set of Clifford relations, it suffices to find a set of rewrite rules that transforms any Clifford circuit into its unique normal form. We proceed in three steps. First, we show that if a Clifford generator (H, S, or CZ) is applied before a normal form, the resulting circuit can be rewritten into a new normal form. This is achieved by showing how each generator can be "pushed" through each distinct normal box, yielding equations of the form illustrated below. We refer to them as the *box relations*.

(1)

Here, dir denotes the *residual dirty gates*, which consist of Clifford generators and whose precise form depends on the parameters a, b, and the dimension d. These dirty gates are then pushed further into the remaining part of the normal form. By grouping these relations together in a systematic way, we obtain 66 box relations, each parametrised by the indices of the corresponding normal boxes.

We then reduce these relations to a smaller set of 18 gate relations, forming a compact and natural presentation of the multi-qudit symplectic Clifford group. Finally, we update these relations with Pauli corrections, and show that they imply all the relations governing Paulis and their interaction with Cliffords. This gives Figure 1, a complete set of multi-qudit Clifford relations.

All computations in $\mathrm{Sp}(2n,\mathbb{Z}_d)$ were verified in the proof assistant Agda [5], giving a fully formalised proof of the completeness of the rewrite system. This framework provides a generic method for constructing circuit presentations and enables future verified reasoning about higher-dimensional quantum circuits.

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We present a complete set of rewrite rules for multi-qudit Clifford circuits, where *qudit* denotes a d-level quantum system with d an odd prime. Completeness means that any two Clifford circuits representing the same linear map can be transformed into each other using these rules. In total, there are 19 *Clifford relations*, each involving at most three qudits and admitting an intuitive interpretation.

Our approach leverages the isomorphism between the symplectic group $\mathrm{Sp}(2n,\mathbb{Z}_d)$ and the quotient of the Clifford group by the Pauli group. We first derive a complete set of *symplectic relations* for $\mathrm{Sp}(2n,\mathbb{Z}_d)$, and then lift them to Clifford relations by incorporating Pauli corrections. To do this, we introduce a *symplectic normal form* that captures the stabiliser tableau of a Clifford operator and is unique up to Pauli correction. This simplification enables a streamlined derivation of a complete set of 66 relations, which we further compress to 18 symplectic relations. Our computations in $\mathrm{Sp}(2n,\mathbb{Z}_d)$ are formalised in the Agda proof assistant, providing a machine-verified proof of correctness.

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1 Introduction

Many physical platforms for qubits naturally possess higher energy levels that can encode qudits, enabling greater information density at the cost of increased control complexity [17, 24, 46, 47, 51]. In particular, when d is an odd prime, qudit systems exhibit rich algebraic structure that result in unique error correction capabilities [15, 49], lower overhead magic state distillation [16, 45], stronger quantum correlations [14, 20], and improvements to various quantum algorithms [10, 42]. Ultimately, there is a trade-off between the engineering cost of accessing higher dimensions and the computational advantages they provide. By characterising qudit circuits in a systematic manner, we can better understand this balance and exploit the potentials of higher-dimensional quantum information processing.

Recently, algebraic approaches to quantum circuits have gained significant attention. An important problem in this line of work is to develop sound and complete equational theories for circuit families, meaning any two circuits representing the same linear map can be transformed into each other using a finite rule set. Such presentations enable syntactic reasoning about circuits and underpin applications ranging from circuit optimisation to formal verification. While there has been substantial progress in the qubit setting [2, 3, 7, 8, 9, 27, 36, 37, 48], analogous results for qudit circuits remain comparatively underdeveloped.

Among quantum circuit fragments, the Clifford group plays a particularly prominent role. Clifford unitaries underpin the stabiliser formalism [1, 22], fault-tolerant quantum computing [25, 26, 28], efficient classical simulation [29, 39], and practical circuit optimisation techniques [12, 13]. As a result, obtaining a complete and finite equational theory for qudit Clifford circuits is a natural and foundational problem.

One might hope to derive such a theory from existing graphical calculi. Indeed, a complete ZX-calculus for qudit stabiliser linear maps is known [11, 43, 50]. However, ZX-diagrams represent general linear maps, not just unitaries. Hence, this completeness result does not restrict to the unitary Clifford group in any straightforward way. In general, a completeness result through a graphical calculus for all linear maps of a certain type does not seem to easily imply a method of deriving a calculus for just the unitaries. Consider for instance that a complete ZX-calculus for universal qubit linear maps was found in 2017 [31, 40], but it was not until 2022 that a complete calculus of universal quantum circuits appeared and it used very different methods [18]. Moreover, there is no complete calculus for, for instance, Toffoli-Hadamard circuits, even though there is one for the corresponding set of linear maps [5].

1.1 Main Results

Building on the characterisation of the qutrit Clifford group [35], where a qutrit is a three-dimensional ($d = 3$) qudit, we present a complete set of rewrite rules for multi-qudit Clifford circuits in all odd prime dimensions. There are 19 *Clifford relations* in total, each involving at most three qudits and admitting an intuitive interpretation.

Theorem 4.10. *The rewrite rules in Figure 1 are* complete *for n-qudit Clifford circuits over any qudit dimension d that is an odd prime. That is, given two n-qudit unitary Clifford circuits C_1 and C_2 implementing the same linear map, there is a sequence of rewrites from Figure 1 that proves C_1 and C_2 are equal.*

To prove Theorem 4.10, we introduce a symplectic normal form that corresponds to the stabiliser tableau of a Clifford operator up to Pauli corrections, which simplifies arguments derived from the more involved normal form for qutrits. We then find a set of 66 parametrised relations which suffice to reduce any Clifford circuit to this normal form. Finally, we show how these relations can be further reduced to a small set of 19 Clifford relations in Figure 1. In addition to the Clifford generators H, S and CZ, we also use "derived generators" listed in Figure 2, such as SWAP, CX, and the *multiplier* M_g. M_g is defined by its action on the computational basis state,

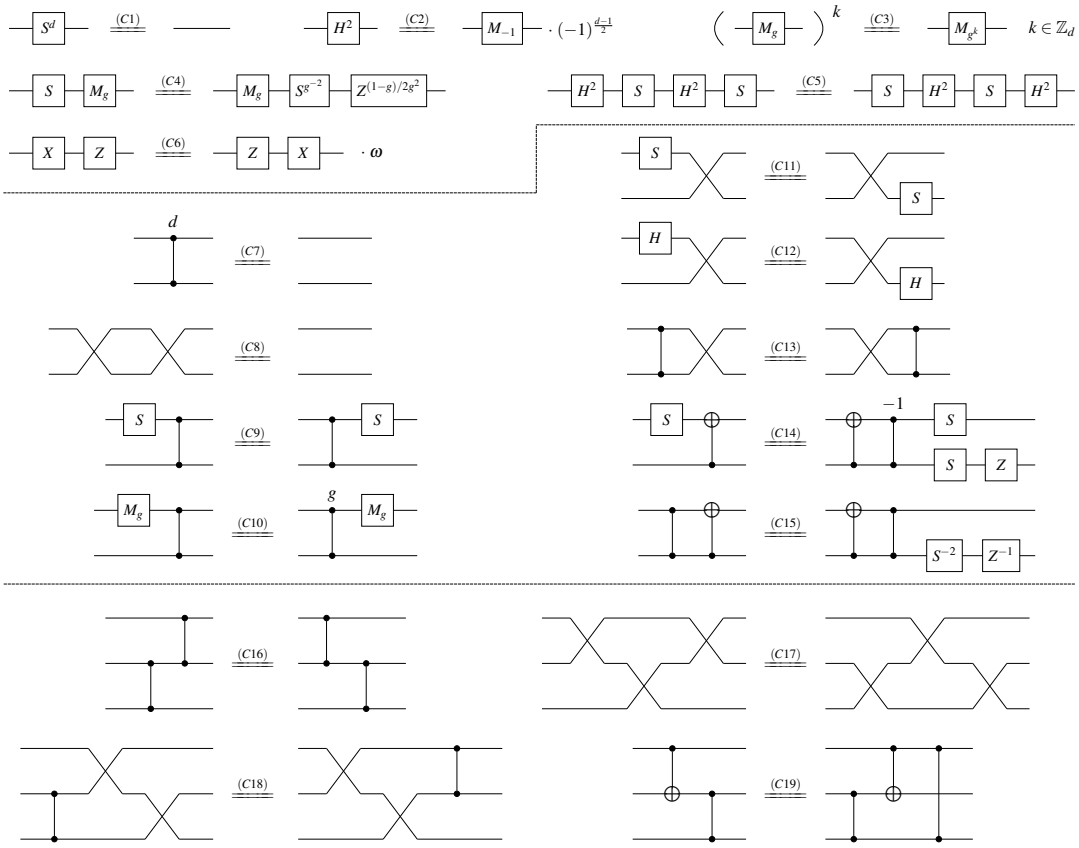

Figure 1: A complete set of rewrite rules for n-qudit Clifford circuits where d is an odd prime and $n \in \mathbb{N}$. Derived generators such as M_g, X, Z, SWAP, and CX are defined in Figure 2.

$M_g|x\rangle = |gx\rangle$, where multiplication is over $\mathbb{Z}_d$. These derived generators allow us to present the rewrite rules as intuitive (quasi-)commutation gate relations.

All computations in $\mathrm{Sp}(2n, \mathbb{Z}_d)$ are formalised in the Agda proof assistant, providing a machine-verified proof of correctness. The corresponding code is available in the GitHub repository [6]. Our results not only bridge algebraic group theory with syntactic quantum circuit reasoning, but also lay the foundation for formal verification of quantum protocols in qudit architectures, automated circuit optimisation and synthesis, and the mathematical study of symplectic groups.

1.2 Proof Overview

Let d be an odd prime and let a qudit denote a d-dimensional quantum system. In what follows, we define unitary operators by their actions on the computational basis states. The single-qudit Pauli

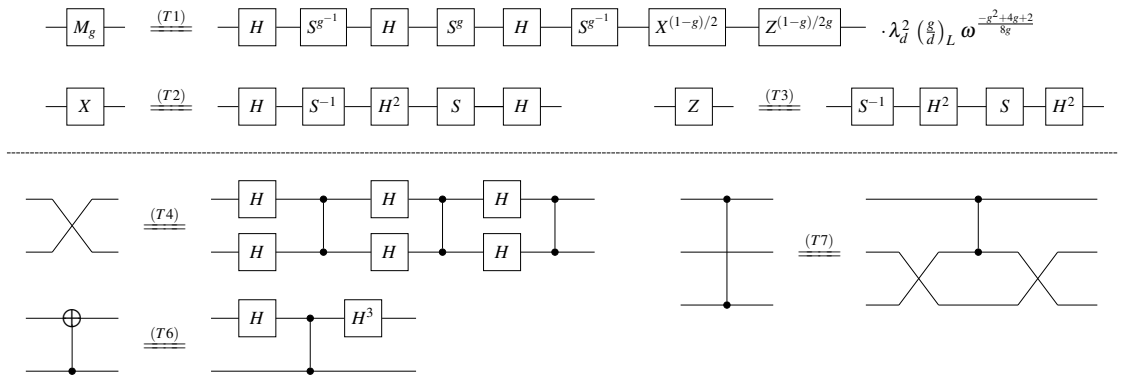

Figure 2: A subset of the derived Clifford generators, expressed as circuits over the generating set $\{H, S, \mathrm{CZ}\}$. These generators are formally defined in Figure 4. Here, g generates the multiplicative group $\mathbb{Z}_d^*$, and $\left(\frac{g}{d}\right)_L$ denotes the Legendre symbol in number thoery [44, Appendix A].

operators are defined as $X|j\rangle = |j+1\rangle$, $Z|j\rangle = \omega^j|j\rangle$, where $\omega = e^{2\pi i/d}$ and addition is over $\mathbb{Z}_d$. The n-qudit Pauli group $\mathcal{P}_n$ is generated by tensor products of X and Z, together with phases ω^t for $t \in \mathbb{Z}_d$. The qudit *H, S, and CZ gates* [21, 23, 25, 33, 34, 44] are defined below, where $i = e^{2\pi i/4}$ is the imaginary unit. The choice of global phases ensures that each matrix determinant is equal to 1.

$$H : |j\rangle \mapsto \frac{1}{\lambda_d\sqrt{d}} \sum_{\ell=0}^{d-1} \omega^{j\ell}|\ell\rangle, \;\; \lambda_d = (-i)^{\frac{d-1}{2}}, \qquad S : |j\rangle \mapsto \omega^{\frac{j(j-1)}{2}}|j\rangle, \qquad CZ : |j\rangle|\ell\rangle \mapsto \omega^{j\ell}|j\rangle|\ell\rangle.$$

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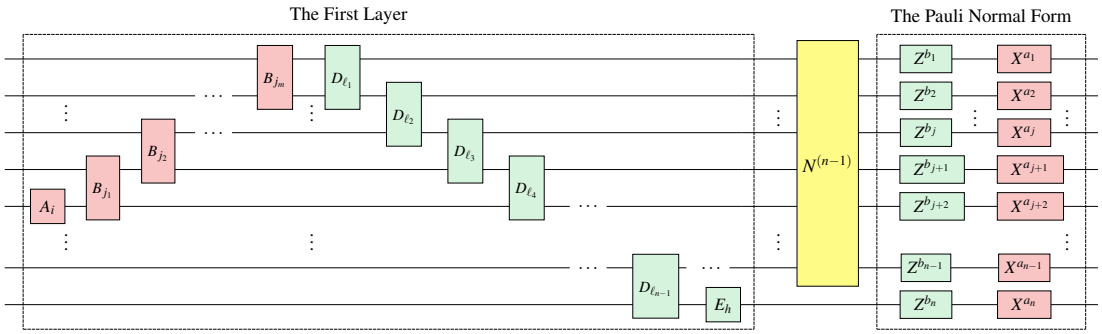

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Establishing Qudit Clifford Completeness To obtain a complete set of Clifford relations, it suffices to find a set of rewrite rules that transforms any Clifford circuit into its unique normal form. We proceed in three steps. First, we show that if a Clifford generator (H, S, or CZ) is applied before a normal form, the resulting circuit can be rewritten into a new normal form. This is achieved by showing how each generator can be "pushed" through each distinct normal box, yielding equations of the form illustrated below. We refer to them as the *box relations*.

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We then reduce these relations to a smaller set of 18 gate relations, forming a compact and natural presentation of the multi-qudit symplectic Clifford group. Finally, we update these relations with Pauli corrections, and show that they imply all the relations governing Paulis and their interaction with Cliffords. This gives Figure 1, a complete set of multi-qudit Clifford relations.

1.3 Paper Organisation

The remainder of this paper is organised as follows. In Section 2, we introduce the necessary algebraic and circuit-theoretic preliminaries, including qudit Pauli and Clifford operators, circuit

interpretations, and the rewriting framework used throughout the paper. Section 3 designs a unique normal form for multi-qudit Clifford circuits, based on a symplectic normal form together with a Pauli correction. In Section 4, we derive a complete set of rewrite rules for multi-qudit Clifford circuits: we first establish completeness at the symplectic level and then lift these results to unitary Clifford circuits by accounting for Pauli corrections. Section 5 concludes with a brief summary of our results and a discussion of future directions. For readability, most technical proofs are deferred to the appendix.

2 Foundations

Here, we introduce the algebraic background and conventions used throughout the paper. In Section 2.1, we introduce the rings and roots of unity underlying our odd-prime qudit formalism. In Section 2.2, we define the Pauli and Clifford operators, review basic identities, and recall the symplectic structure of the Clifford group. In Section 2.3, we move from operators to circuits by specifying Clifford circuit families, their interpretation maps (unitary, projective, and symplectic), as well as the notions of soundness and completeness for rewriting systems that will be used in later sections. Most of the technical proofs can be found in Section A.

2.1 Rings

We write $\mathbb{N}$ for the collection of non-negative integers, $\mathbb{Z}$ for the ring of integers, $\mathbb{Z}_m$ for the ring of integers modulo $m \in \mathbb{N}$, and $\mathbb{Z}_m^*$ for the collection of non-zero elements of $\mathbb{Z}_m$. If a and b are two integers, we write $a \equiv_m b$ if a and b are equivalent modulo m (i.e., equal in Z_m). If $a \in \mathbb{Z}_m$ is invertible, we sometimes denote its multiplicative inverse by $1/a$. Throughout the paper, d denotes an arbitrary odd prime of $\mathbb{Z}$.

Recall that a complex number $c \in \mathbb{C}$ is a *root of unity* if $c^n = 1$ for some positive $n \in \mathbb{N}$. We will be especially interested in d-th roots of unity.

Definition 2.1. *We write ω_d for the* primitive d-th root of unity $\omega_d = e^{2\pi i/d}$.

In what follows, we drop the subscript and write ω for ω_d, leaving d implicit. Note that $\omega^d = 1$, $\omega^\dagger = \omega^{-1} = \omega^{d-1}$, and $\sum_{j=0}^{d-1} \omega^j = 0$.

Definition 2.2. *The ring $\mathbb{Z}[\omega]$ of* cyclotomic integers of degree d *is the smallest subring of $\mathbb{C}$ containing ω.*

Since $\omega^\dagger = \omega^{d-1}$, the ring $\mathbb{Z}[\omega]$ is closed under complex conjugation. The elements of $\mathbb{Z}[\omega]$ can be written as a linear combinations of integers in $\mathbb{Z}$, so that

$$\mathbb{Z}[\omega] = \left\{ \sum_{j=0}^{d-1} a_j \omega^j \mid a_j \in \mathbb{Z} \right\}. \tag{2}$$

Definition 2.3. *The ring $\mathbb{Z}[1/d, \omega]$ is the smallest subring of $\mathbb{C}$ containing $1/d$ and ω.*

The ring $\mathbb{Z}[1/d,\omega]$ is the localisation of $\mathbb{Z}[\omega]$ by the integer powers of d. We have

$$\mathbb{Z}[1/d,\omega] = \left\{ \frac{u}{d^\ell} \mid u \in \mathbb{Z}[\omega], \ell \in \mathbb{Z} \right\}. \tag{3}$$

Definition 2.4. *Let $i = e^{2\pi i/4}$ be the imaginary unit. Then $\lambda_d = (-i)^{\frac{d-1}{2}}$.*

The ring $\mathbb{Z}[1/d,\omega]$ has two properties that will be useful later.

Proposition 2.5. $\frac{1}{\lambda_d\sqrt{d}} \in \mathbb{Z}[1/d,\omega]$.

Proposition 2.6. *The roots of unity in $\mathbb{Z}[1/d,\omega]$ are $\pm\omega^t$, $t \in \mathbb{Z}_d$.*

2.2 Operators

We now introduce the operators of primary interest in this work, namely the *Pauli* operators [25] and the *Clifford* operators [21, 23, 25, 33, 34, 44].

In what follows, we write I_m for the $m \times m$ identity matrix, and we omit the subscript m when it is not relevant or can be inferred from the context. We say that an operator U is a *scalar*, if it is a scalar multiple of the identity operator, i.e., if $U = \lambda I$, for some $\lambda \in \mathbb{C}$. For simplicity, we denote U by λ in such a case.

We write $\mathrm{Sp}(2n,\mathbb{Z}_d)$ for the *symplectic group* and $\mathcal{U}_m$ for the *unitary group*. The collection $\{\alpha I_m \mid \alpha \in \mathcal{U}_1\}$ of scalar multiples of the identity matrix forms a subgroup of $\mathcal{U}_m$ called the *group of global phases*. By a slight abuse of notation, we denote it by $\mathcal{U}_1$.

Definition 2.7. *Let $m \in \mathbb{N}$. $\mathcal{U}_m(\mathbb{Z}[1/d,\omega])$ is the group of $m \times m$ unitaries with entries in $\mathbb{Z}[1/d,\omega]$.*

Throughout the paper, we define unitary operators by specifying their action on the computational basis. This uniquely determines their action on all states. Unless stated otherwise, juxtaposition AB denotes matrix multiplication of the unitaries A and B.

2.2.1 Pauli Operators

Definition 2.8. *The single-qudit* Pauli X and Z gates *are defined as follows.*

$$X : |j\rangle \mapsto |j+1\rangle \qquad \textit{and} \qquad Z : |j\rangle \mapsto \omega^j |j\rangle, \tag{4}$$

where $0 \le j \le d-1$ and addition is performed modulo d.

We record several properties of the Pauli X and Z gates. The corresponding derivations can be found in Section A. We have

$$X^d = Z^d = 1. \tag{5}$$

It follows from Equation (5) that $X^\dagger = X^{d-1}$ and $Z^\dagger = Z^{d-1}$. In addition, we have $ZXZ^\dagger X^\dagger = \omega$, which implies that $\omega \in \mathcal{P}_1$ and that

$$ZX = \omega XZ, \qquad ZXZ^\dagger = \omega X, \qquad \text{and} \qquad XZX^\dagger = \omega^{-1} Z. \tag{6}$$

Definition 2.9. *For $n \in \mathbb{N}$, the n-qudit* Pauli group $\mathcal{P}_n$ *is the group of n-qudit unitary operators generated by X and Z under composition and tensor product.*

Note that the group $\mathcal{P}_n$ is not closed under tensor product. Definition 2.9 states that among all of the operators that can be constructed using X and Z through composition and tensor product, $\mathcal{P}_n$ contains exactly those that act on n qudits. Using Equations (5) and (6) and the usual properties of the tensor product, one can show that

$$\mathcal{P}_n = \left\{ \omega^c \left(X^{a_1} Z^{b_1} \otimes \cdots \otimes X^{a_n} Z^{b_n} \right) \mid c, a_j, b_j \in \mathbb{Z}_d \text{ for all } 1 \leq j \leq n \right\}.$$

As a consequence, we have $|\mathcal{P}_n| = d^{2n+1}$.

2.2.2 Clifford Operators

Definition 2.10. *The single-qudit H and S gates and the two-qudit CZ gate are defined as follows. λ_d is given in Definition 2.4.*

$$H : |j\rangle \mapsto \frac{1}{\lambda_d \sqrt{d}} \sum_{\ell=0}^{d-1} \omega^{j\ell} |\ell\rangle, \qquad S : |j\rangle \mapsto \omega^{\frac{j(j-1)}{2}} |j\rangle, \qquad \text{and} \qquad CZ : |j\rangle|\ell\rangle \mapsto \omega^{j\ell} |j\rangle|\ell\rangle,$$

where $0 \leq j \leq d-1$ and multiplication is performed modulo d.

The gates H, S, and CZ are called the *Hadamard* gate, the *phase* gate, and the *controlled-Z* gate, respectively. We use these gates to define the *Clifford* group, similarly to how the X and Z gates were used to define the Pauli group in Definition 2.9.

Definition 2.11. *For $n \in \mathbb{N}$, the n-qudit* Clifford group $\mathcal{C}_n$ *is the group of n-qudit unitary operators generated by H, S, and CZ under composition and tensor product.*

We record a few properties of Clifford operators. In particular, we show that the Clifford phases are exactly the integer powers of ω.

Proposition 2.12. *For $0 \leq j \leq d-1$, $H^2|j\rangle = \lambda_d^{-2}|-j\rangle$, subtraction is performed modulo d.*

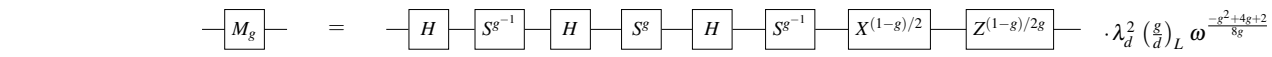
Note that H^2 is a special case of a *multiplier* since up to a global phase, $H^2|j\rangle = |-j\rangle$. Multipliers admit simple commutation rules with the Clifford generators, which helps streamline the relation reduction in [6].

Definition 2.13. *For every $g \in \mathbb{Z}_d^*$, we define the* multiplier by g *as the Clifford unitary M_g that acts on computational basis states as $M_g|x\rangle = |gx\rangle$.*

By direct computation, we can verify that $M_g^{-1} = M_{g^{-1}}$ where g^{-1} is the inverse of g modulo d. Since it sends computational basis states to computational basis states, it must be unitary so that $M_g^\dagger = M_g^{-1}$. It is Clifford since $M_g Z M_{g^{-1}} = Z^{g^{-1}}$ and $M_g X M_{g^{-1}} = X^g$.

In Lemma 2.14, we show how to construct a multiplier using H, S, X, and Z gates. Here, $\left(\frac{g}{d}\right)_L$ denotes the Legendre symbol in number theory [44, Appendix A].

Lemma 2.14. *For every $g \in \mathbb{Z}_d^*$, M_g defined below satisfies $M_g|x\rangle = |gx\rangle$ for all $x \in \mathbb{Z}_d$.*



Lastly, we record some important properties of Clifford operators and Clifford scalars.

Corollary 2.15. $H^4 = 1$.

Lemma 2.16. $S^d = 1$.

Lemma 2.17. $\mathcal{C}_n \subseteq \mathcal{U}_{d^n}(\mathbb{Z}[1/d, \omega])$.

Proof. This follows from the fact that the matrices representing H, S, and CZ in the computational basis have entries in $\mathbb{Z}[1/d, \omega]$. In the case of H, this follows from Proposition 2.5. □

Lemma 2.18. $\det(S) = \det(H) = \det(CZ) = 1$.

Proposition 2.19. *The Clifford phases are exactly the integer powers of ω. That is, for all $n \in \mathbb{N}$, we have $\mathcal{C}_n \cap U(1) = \{\omega^t \mid t \in \mathbb{Z}_d\}$.*

Proof. All zero-qudit Clifford operators are of the form ω^t for some $t \in \mathbb{Z}_d$, so that the statement holds when $n = 0$. When $n \geq 1$, the scalar $\omega = \omega I_{d^n}$ has determinant 1. It then follows from properties of the determinant and Proposition 2.19, that every element of $\mathcal{C}_n$ has determinant 1. Now consider a phase $cI_{d^n} \in \mathcal{C}_n$. Then $c \in \mathbb{Z}[1/d, \omega]$ and

$$c^{d^n} = \det(cI) = 1. \tag{7}$$

Hence, c is a root of unity in $\mathbb{Z}[1/d, \omega]$ and therefore, by Proposition 2.6, $c = \pm\omega^t$ for some $t \in \mathbb{Z}_d$. Now suppose that $c = -\omega^t$, for some $t \in \mathbb{Z}_d$. Then $(-\omega^t)^{d^n} = -\left(\omega^d\right)^{t \cdot d^{n-1}} = -1$, which contradicts Equation (7). Hence, c must be of the form $c = \omega^t$, for some $t \in \mathbb{Z}_d$. □

2.2.3 Symplectic Structure of the Clifford Group

Definition 2.20. *For $n \in \mathbb{N}$, the* normaliser of the Pauli group $\mathcal{P}_n$ *is denoted by $\widetilde{\mathcal{C}_n}$,*

$$\widetilde{\mathcal{C}_n} \coloneqq \{U \in \mathcal{U}_{d^n} \mid UPU^\dagger \in \mathcal{P}_n \text{ for all } P \in \mathcal{P}_n\}.$$

Note that Definition 2.20 includes all possible global phases $e^{i\alpha}$, making $\widetilde{\mathcal{C}_n}$ uncountably infinite for all $n \in \mathbb{N}$.

As the notation indicates, the Clifford group $\mathcal{C}_n$ defined in Definition 2.11 is related to the normaliser $\widetilde{\mathcal{C}_n}$ of the Pauli group defined in Definition 2.20. As is well known, any Clifford $C \in \mathcal{C}_n$ maps any Pauli $P \in \mathcal{P}_n$ to another Pauli $CPC^\dagger \in \mathcal{P}_n$ under conjugation, so $\mathcal{C}_n$ is a subgroup of $\widetilde{\mathcal{C}_n}$.

The global phases in $\mathcal{C}_n$ are exactly powers of ω. That is, $\mathcal{C}_n \cap \mathcal{U}_1 = \langle\omega\rangle$. The groups $\widetilde{\mathcal{C}_n}$ and $\mathcal{C}_n$ are related as follows. Global phases commute with everything, hence $\mathcal{U}_1 \trianglelefteq \widetilde{\mathcal{C}_n}$ and $\langle\omega\rangle \trianglelefteq \mathcal{C}_n$. The quotients $\widetilde{\mathcal{C}_n}/\mathcal{U}_1$ and $\mathcal{C}n/\langle\omega\rangle$ are therefore well-defined. It is a fact that

$$\widetilde{\mathcal{C}_n}/\mathcal{U}_1 \cong \mathcal{C}_n/\langle\omega\rangle.$$

In other words, the Clifford group (up to global phases) is the normaliser of the Pauli group (up to global phases). Because $\mathcal{C}_n \leq \widetilde{\mathcal{C}_n}$, we have $\mathcal{P}_n \trianglelefteq \mathcal{C}_n$, and the quotient $\mathcal{C}_n/\mathcal{P}_n$ is also well-defined. It is well-known that $\mathcal{C}_n/\mathcal{P}_n \cong \mathrm{Sp}(2n, \mathbb{Z}_d)$.

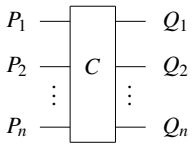
Definition 2.21. *Let $\widehat{\mathcal{P}}_n = \mathcal{P}_n/\langle\omega\rangle$ be the n-qudit* projective Pauli group *and $\widehat{\mathcal{C}}_n = \mathcal{C}_n/\langle\omega\rangle$ be the n-qudit* projctive Clifford group.

We have, $\widehat{\mathcal{P}}_n \trianglelefteq \widehat{\mathcal{C}}_n$ and $\widehat{\mathcal{C}}_n/\widehat{\mathcal{P}}_n \cong Sp(2n, \mathbb{Z}_d)$.

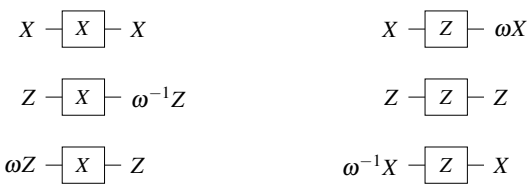
2.2.4 Clifford Conjugation of Pauli Generators

We will denote the action of the Clifford group $\mathcal{C}_n$ by conjugation on the Pauli group $\mathcal{P}_n$ as $C \bullet P := CPC^\dagger$. Note that if we write $Q := C \bullet P$, we then also have $CP = QC$. Then we can interpret this as "pushing P through C gives us Q". It will be useful to introduce a circuit notation for this.

Definition 2.22. *Let $C \in \mathcal{C}_n$ and $P, Q \in \mathcal{P}_n$ with $Q := C \bullet P$. Write $P := P_1 \otimes \ldots \otimes P_n$ and $Q := Q_1 \otimes \ldots \otimes Q_n$ for $P_j, Q_j \in \mathcal{P}_1$. In the circuit diagram, this conjugation relation is denoted below.*



Example 2.23. *By Equation (6), $X, Z \in \mathcal{C}_1$. Diagrammatically, we have*



(8)

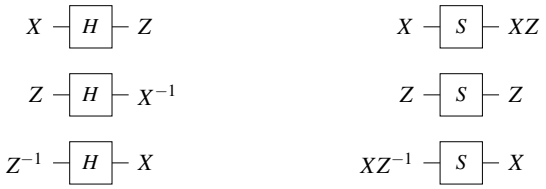
A Clifford unitary is fully determined by its action of conjugating Pauli operators. Since this conjugation is a group homomorphism, it suffices to know its action on a set of generators of $\mathcal{P}_n$. A useful set of generators are those with only one Z or X gate acting on a single qudit, and identities everywhere else.

Definition 2.24. *Let Z_j denote the n-qudit Pauli $P := P_1 \otimes \cdots \otimes P_n$ with $P_\ell = I$ if $\ell \neq j$, and $P_j = Z$ otherwise. We write X_j for the analogous n-qudit Pauli by substituting X with Z in the above definition. Let $\mathcal{B}_n := \{X_j, Z_j \mid 1 \leq j \leq n\}$ be the n-qudit* Pauli basis *for $\mathcal{P}_n$. An element in $\mathcal{B}_n$ is called a* Pauli generator.

Note that $\mathcal{B}_n$ generates $\mathcal{P}_n$. Given any n-qudit Clifford operator C, for $1 \leq j \leq n$, $P_j = C \bullet Z_j$ and $Q_j = C \bullet X_j$ fully determine C (up to a global phase). Moreover, P_j and Q_j inherit the commuting and non-commuting relations from Z_j and X_j. By Equation (6), $P_j Q_j = \omega Q_j P_j$. When $j \neq \ell$, $P_j Q_\ell = Q_\ell P_j$.

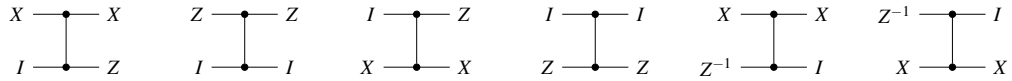
Lemmas 2.25 and 2.26 identify the Clifford generators H, S, and CZ by specifying their action on the Pauli generators under conjugation. Full proofs are given in Section A.

Lemma 2.25. *$H, S \in \mathcal{C}_1$. Specifically, $HXH^\dagger = Z$, $HZH^\dagger = X^{-1}$, $SXS^\dagger = XZ$, $SZS^\dagger = Z$.*



(9)

Lemma 2.26. *$CZ \in \mathcal{C}_2$. Specifically,*



(10)

Finally, we introduce a property of Clifford operators that will be used to establish the existence and uniqueness of the symplectic normal form defined in Section 3.1.

Proposition 2.27. *Let $C \in \mathcal{C}_n$. If $C \bullet P = P$ for all $P \in \mathcal{P}_n$, then C is a scalar (i.e. proportional to the identity).*

Proof. Any complex $d \times d$-matrix can be written in the form $\sum_{i=1}^{d^2} c_i P_i$, $c_i \in \mathbb{C}$, $P_i \in \{X^a Z^b \mid a, b \in \mathbb{Z}_d\}$. It follows that $\mathcal{P}_n$ spans the vector space formed by $d^n \times d^n$ complex matrices. Since $C \bullet P = P$ for all $P \in \mathcal{P}_n$, $C \bullet M = CMC^\dagger = M$ for all operators M. Hence $CM = MC$ so that C must commute with all matrices. The only matrices with this property are the scalars. □

Proposition 2.27 justifies that two Clifford circuits implementing the same automorphism of the Pauli group differ only by a global phase, so constructing a circuit that matches the desired Pauli action suffices to prove existence (and uniqueness up to phase) of the normal form.

2.2.5 Derived Generators

For convenience, Figure 4 defines a set of derived Clifford operators. They will be used to compactly present the qudit Clifford relations. The soundness of the phases, the Pauli X and Z gates, and well as the SWAP gate is established in Lemmas B.8 to B.11.

2.3 Circuits

We assume familiarity with the standard graphical notation for quantum circuits. For background, the reader may consult any standard textbook on quantum computation [41].

In Section 2.2, the Clifford generators H, S, and CZ, as well as the derived generators listed in Figure 4 were treated as operators. In this section, we consider them as generators for circuits. The difference between these two perspectives is that when X is viewed as an operator, we have

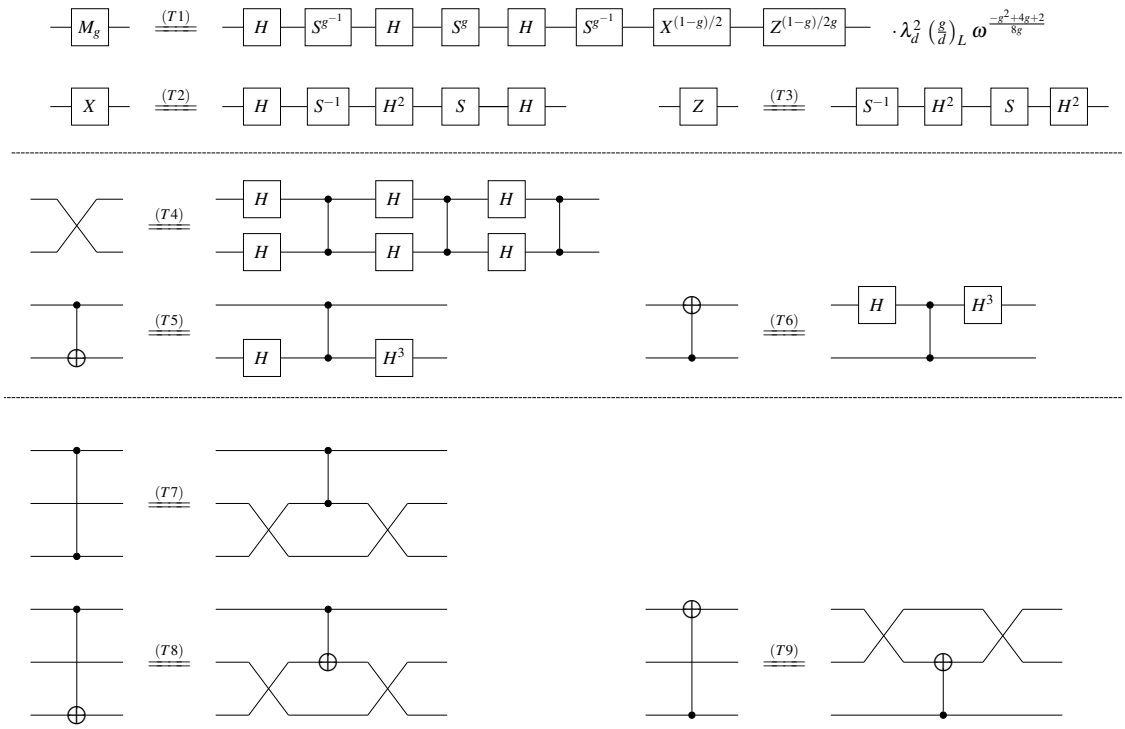


Figure 4: A set of derived Clifford generators, expressed as circuits over the generating set $\{H,S,\mathrm{CZ}\}$. Here, g generates the multiplicative group $\mathbb{Z}_d^*$, and $\left(\frac{g}{d}\right)_L$ denotes the Legendre symbol in number theory [44, Appendix A].

for instance, $X^d = 1$. However, the circuit X^d is not equal to the circuit of an identity matrix (the former contains d gates, while the latter contains none).

We write $\mathfrak{C}$ for the collection of all circuits over the gate set $\{\omega,X,Z,H,S,\mathrm{CZ}\}$ and $\mathfrak{C}_n$ for the subset of $\mathfrak{C}$ containing only the circuits on exactly n qudit wires. We similarly define $\mathfrak{P}$ and $\mathfrak{P}_n$ as the collection of circuits over the gate set $\{\omega,X,Z\}$ and the collection of circuits in $\mathfrak{P}$ on exactly n qudit wires. Note that $\mathfrak{P} \subseteq \mathfrak{C}$ and $\mathfrak{P}_n \subseteq \mathfrak{C}_n$.

2.3.1 Interpretations

An *interpretation map* for a family of circuits is a function which assigns an operator to every circuit. The standard interpretation is the *unitary interpretation*, in which a circuit is interpreted as the unitary operator it represents.

Definition 2.28. *Let $C \in \mathfrak{C}_n$. We define the* unitary interpretation $[\![C]\!]_u \in \mathcal{C}_n$ *of C to be the unitary operator obtained by interpreting the gates in $\{\omega,X,Z,H,S,CZ\}$ as the corresponding operators,*

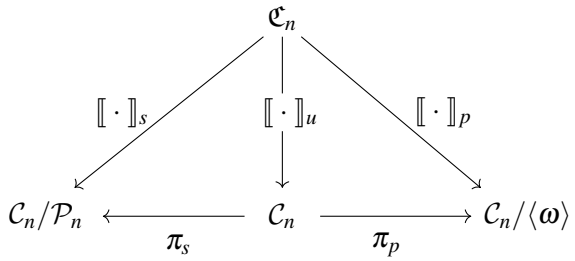
the parallel composition of circuits as the tensor product of operators, and the sequential composition of circuits as the matrix multiplication of operators.

Now recall that the quotients $\mathcal{C}_n/\mathcal{P}_n$ and $\mathcal{C}_n/\langle\omega\rangle$ are well-defined and let $\pi_s : \mathcal{C}_n \to \mathcal{C}_n/\mathcal{P}_n$ and $\pi_p : \mathcal{C}_n \to \mathcal{C}_n/\langle\omega\rangle$ be the corresponding canonical projections. We use these projections to define two more interpretation maps.

Definition 2.29. *Let $C \in \mathfrak{C}_n$. We define:*

- *the* interpretation of C up to Paulis $[\![C]\!]_s \in \mathcal{C}_n/\mathcal{P}_n$ *as* $[\![C]\!]_s = \pi_s([\![C]\!]_u)$ *and*
- *the* interpretation of C up to global phases $[\![C]\!]_p \in \mathcal{C}_n/\langle\omega\rangle$ *of* C *as* $[\![C]\!]_p = \pi_p([\![C]\!]_u)$.

We sometimes refer to $[\![\cdot]\!]_s$ as the *symplectic interpretation* (since $[\![C]\!]_s$ belongs to the symplectic group) and to $[\![\cdot]\!]_p$ as the *projective interpretation* (since $[\![C]\!]_s$ belongs to the projective group). The three interpretations introduced so far organise themselves into the diagram below.



The interpretation maps $[\![\cdot]\!]_u$, $[\![\cdot]\!]_s$, and $[\![\cdot]\!]_p$ correspond to three ways to interpret Clifford circuits, and each interpretation comes with a level of granularity. We now leverage these interpretation maps to define relations among the elements of $\mathfrak{C}$.

Definition 2.30. *Let $C_1, C_2 \in \mathfrak{C}_n$. We write*

- $C_1 \equiv C_2$ *if* C_1 *and* C_2 *are the same circuit.*
- $C_1 \equiv_u C_2$ *if* $[\![C_1]\!]_u = [\![C_2]\!]_u$.
- $C_1 \equiv_p C_2$ *if* $[\![C_1]\!]_p = [\![C_2]\!]_p$.
- $C_1 \equiv_s C_2$ *if* $[\![C_1]\!]_s = [\![C_2]\!]_s$.

As Definition 2.30 states: $C_1 \equiv C_2$ when C_1 and C_2 are identical circuits; $C_1 \equiv_u C_2$ when C_1 and C_2 represent the same unitary; $C_1 \equiv_p C_2$ when C_1 and C_2 represent the same unitary up to a global phase; $C_1 \equiv_s C_2$ when C_1 and C_2 represent the same unitary up to a Pauli. Note that we have the following implications

$$C_1 \equiv C_2 \Rightarrow C_1 \equiv_u C_2 \Rightarrow C_1 \equiv_p C_2 \Rightarrow C_1 \equiv_s C_2,$$

None of the above implications can be reversed in general. However, we do know, for example, that if $C_1 \equiv_s C_2$, then there exists a unique $P \in \mathcal{P}_n$ such that $C_1 \equiv_u C_2 P$.

Since $\mathfrak{P} \subseteq \mathfrak{C}$, all of the above interpretations restrict to $\mathfrak{P}$, giving well-defined interpretations of Pauli circuits.

2.3.2 Relations and Rewriting

A *relation* on $\mathfrak{C}$ (or $\mathfrak{P}$) is a pair (C, C') of elements of $\mathfrak{C}$ (or $\mathfrak{P}$). If $\mathcal{R}$ is a set of relations on $\mathfrak{C}$, we write $\sim_{\mathcal{R}}$ for the smallest *congruence* on $\mathfrak{C}$ that contains all of the relations in $\mathcal{R}$. By congurence, we mean an equivalence relation on $\mathfrak{C}$ that is compatible with composition and tensor product. For two circuits D and D', we have $D \sim_{\mathcal{R}} D'$ if and only if D can be rewritten into D' by applying the relations from $\mathcal{R}$. That is, $D \sim_{\mathcal{R}} D'$ if and only if there is a sequence of rewrites

$$D \to D_1 \to \cdots \to D_k \to D'$$

such that each rewrite is an application of a relation in $\mathcal{R}$.

Let $[\![\cdot]\!]_x$ be one of the three interpretations defined above (i.e., one of $[\![\cdot]\!]_u$, $[\![\cdot]\!]_p$, and $[\![\cdot]\!]_s$) and let $\equiv_x$ be the corresponding equivalence of circuits. We say that a collection $\mathcal{R}$ of relations is

- *sound with respect to* $[\![\cdot]\!]_x$ if $C \sim_{\mathcal{R}} D$ implies $C \equiv_x D$ for all $C, D \in \mathfrak{C}$, and
- *complete with respect to* $[\![\cdot]\!]_x$ if $C \equiv_x D$ implies $C \sim_{\mathcal{R}} D$ for all $C, D \in \mathfrak{C}$.

3 A Unique Normal Form for Multi-Qudit Clifford Circuits

In this section, we define a normal form in $\mathfrak{C}_n$ for elements in $\mathcal{C}_n$. Similar to the Clifford normal form in the qubit and qutrit lands [35, 37, 48], our normal form for a qudit Clifford operator is based on uniquely representing its stabiliser tableau. However, our construction follows a conceptually different route. To motivate our approach, recall that the symplectic group $\mathrm{Sp}(2n, \mathbb{Z}_d) \cong \mathcal{C}_n/\mathcal{P}_n \cong \widehat{\mathcal{C}}_n/\widehat{\mathcal{P}}_n$, where $\widehat{\mathcal{C}}_n = \mathcal{C}_n/\langle \omega \rangle$ and $\widehat{\mathcal{P}}_n = \mathcal{P}_n/\langle \omega \rangle \cong \mathbb{Z}_d^{2n}$. In Section 3.1, we first define the *symplectic normal form*, as a normal form for elements in $\mathcal{C}_n$ up to Pauli corrections. The unitary implementations of symplectic normal circuits form a complete set of representatives for the cosets of $\mathcal{P}_n$ in $\mathcal{C}_n$. In other words, for every symplectic matrix $M \in \mathrm{Sp}(2n, \mathbb{Z}_d)$, there is a unique circuit $C \in \mathfrak{C}_n$ in the symplectic normal form such that $[\![C]\!]_s = M$. In Section 3.2, we then define a normal form for elements in $\widehat{\mathcal{P}}_n$, and call this the *Pauli normal form*. Since we work with odd prime d, it is well known that:

Theorem 3.1 ([52]). $\widehat{\mathcal{C}}_n \cong \mathrm{Sp}(2n, \mathbb{Z}_d) \ltimes (\mathbb{Z}_d)^{2n}$.

Using this semi-direct product structure, in Section 3.3 we derive the normal form of elements in $\widehat{\mathcal{C}}_n$ by composing the symplectic and Pauli normal forms. The norm form for $\mathcal{C}_n$ follows by taking phases into consideration.

To define the symplectic normal form, our first goal is to map the Pauli generators to the desired outcomes (up to global phases) by using some elementary building blocks in $\mathfrak{C}_1$ and $\mathfrak{C}_2$, which we call *normal boxes*. To obtain a unique normal form, we need to identify some notion of uniqueness when designing these boxes. In total, there are four *types* of normal boxes. They are labelled as A, B, D, and E, as shown in Figure 5. Each type of normal box has different *variants*, and each variant is labelled by the subscript of the box. Note that in Figure 5, Pauli propagation is tracked up to a global phase.

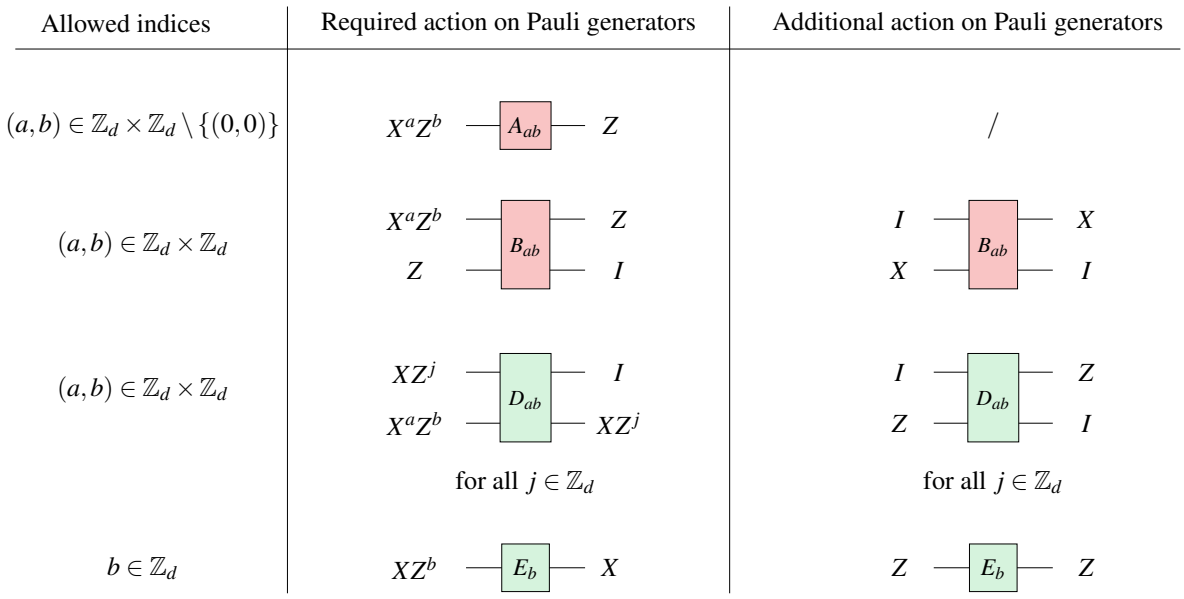


Figure 5: The types of normal boxes, and the specified action they must have on Paulis. The colour of the box (red or green) specifies whether these are part of the Z-part, respectively the X-part, of the normal form.

Based on the Pauli automorphism defined by the second column of Figure 5, Figure 6 defines all the normal boxes for building the symplectic normal form shown in Figure 8. Note that the construction of A, B, and D boxes are given up to Pauli correction. All box constructions are described parametrically, using the index of each box. This feature plays an important role in simplifying the search for box relations. Figures 15, 16 and 18 summarise how A, B, D, and E boxes act on Pauli generators respectively, and all Clifford's actions on X and Z are calculated up to a global phase.

3.1 Symplectic Normal Form for Multi-Qudit Clifford Circuits

Definition 3.2. *For A, B, D, and E normal boxes defined in Figure 6,*

- *an n-qudit Clifford circuit is* Z-normal *if it is of the form in Figure 7a, and*
- *an n-qudit Clifford circuit is* X-normal *if it is of the form in Figure 7b.*

Definition 3.3. *An n-qudit Clifford circuit is* symplectically normal *if it is of the form in Figure 8.*

Remark 3.4. *When $n = 1$, a symplectic normal form is composed of only A and E boxes. A generic construction of $N_S^{(1)}$ is given in Figure 9.*

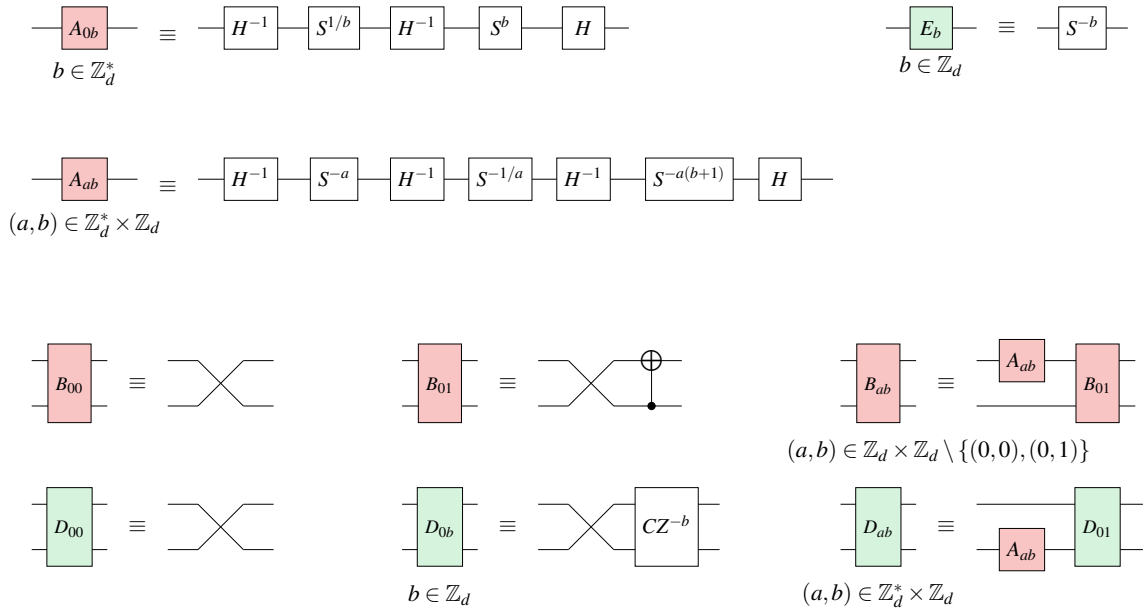
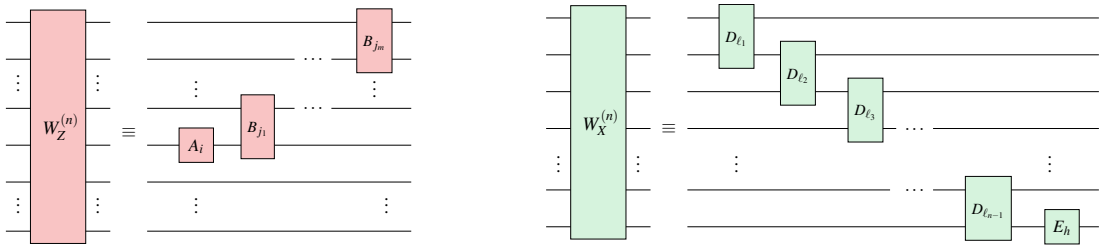


Figure 6: The concrete implementations of the Z and X normal boxes. Here $a, b \in \mathbb{Z}_d$ and $a \neq 0$.



(a) The Z-normal circuit.

(b) The X-normal circuit.

Figure 7: The Z- and X-normal circuits are used to build the symplectic normal form for an n-qudit Clifford circuit. Up to Pauli correction, $W_Z^{(n)}$ gets rid of the C box on the top output wire and $W_X^{(n)}$ gets rid of the F box on the bottom output wire.

Now that we have the symplectic normal form, we need to show that it is *universal*, meaning it can represent an arbitrary symplectic matrix M, and *unique*, meaning that each $M \in \mathrm{Sp}(2n, \mathbb{Z}_d)$ is uniquely represented by one specific instantiation of this normal form. For universality, by Proposition 2.27, it suffices to show that we can represent an arbitrary automorphism of $\mathcal{P}_n$ by a normal form, up to Pauli correction.

Let ϕ be an automorphism of $\mathcal{P}_n$. We claim that one can construct a Z-normal circuit W_Z and an X-normal circuit W_X such that $(W_X W_Z)^{-1}$ acts as ϕ^{-1} on Z_n and X_n. To obtain a normal form for ϕ, it suffices to construct W_Z and W_X and then to recursively iterate this procedure with the updated

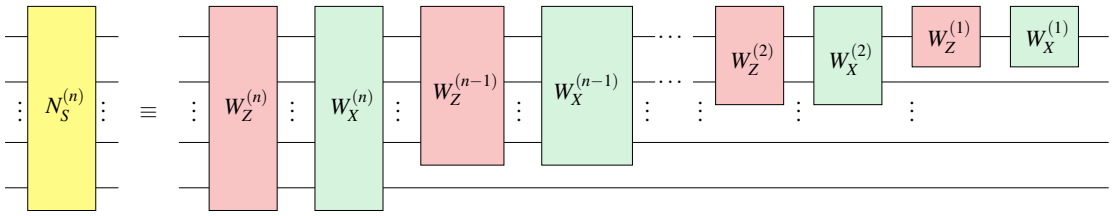


Figure 8: The structure of a symplectic normal form: we alternate layers of Z- and X-normal circuits, with each subsequent layer acting on one fewer qudit. Note that we can view this normal form as being defined inductively, with the n-qudit symplectic normal form adding a layer of Z- and X-normal circuits to the left of an existing $(n-1)$-qudit symplectic normal form.

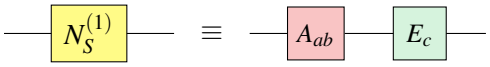


Figure 9: For $(a,b) \in \mathbb{Z}_d \times \mathbb{Z}_d \setminus \{(0,0)\}$ and $c \in \mathbb{Z}_d$, a generic construction of the symplectic normal form of a single-qudit Clifford operator. It is a Clifford normal form up to Pauli correction.

automorphism $\phi' = \phi\,(W_X W_Z)^{-1}$. The proofs of these statements are analogous to those for the qutrit case in [35], so we postpone them to Section E.

Lemma 3.5. *For every $P \in \mathcal{P}_n \setminus \{\omega^t I;\, t \in \mathbb{Z}_d\}$ be a non-scalar n-qudit Pauli operator, there exists a unique Z-normal circuit W_Z such that $[\![W_Z]\!]_u \bullet P \equiv_p Z \otimes I \otimes \cdots \otimes I$.*

Lemma 3.6. *For every $Q \in \mathcal{P}_n$ such that $(Z \otimes I \otimes \cdots \otimes I)Q = \omega Q(Z \otimes I \otimes \cdots \otimes I)$, there exists a unique X-normal circuit W_X such that $[\![W_X]\!]_u \bullet Q \equiv_p I \otimes \cdots \otimes I \otimes X$.*

Lemma 3.7. *Every X-normal circuit W_X satisfies $[\![W_X]\!]_u \bullet (Z \otimes I \otimes \cdots \otimes I \otimes I) \equiv_p I \otimes I \otimes \cdots \otimes I \otimes Z$.*

Lemma 3.8. *Let $P, Q \in \mathcal{P}_n$ such that $PQ = \omega QP$. Then there exist unique normal circuits W_Z and W_X such that $[\![W_X W_Z]\!]_u \bullet P \equiv_p I \otimes \cdots \otimes I \otimes Z$ and $[\![W_X W_Z]\!]_u \bullet Q \equiv_p I \otimes \cdots \otimes I \otimes X$.*

Proposition 3.9. *Let $\phi : \mathcal{P}_n \to \mathcal{P}_n$ be an automorphism of the Pauli group such that ϕ fixes scalars. There exists a unique Clifford circuit $C \in \mathfrak{C}_n$ in symplectic normal form such that for all $P \in \mathcal{P}_n$, $[\![C]\!]_u \bullet P \equiv_p \phi(P)$.*

Since there is a bijection between the symplectic matrices and the normal forms, we can count the number of n-qudit normal forms to compute the cardinality of $\mathrm{Sp}(2n, \mathbb{Z}_d)$. The proof of Lemma 3.10 can be found in Section E.

Lemma 3.10. $|\mathrm{Sp}(2n, \mathbb{Z}_d)| = \prod_{k=1}^{n} \left(d^{2k} - 1\right) \cdot \left(d^{2k-1}\right)$.

3.2 Normal Form for a Multi-Qudit Phase-Free Pauli Operator

Here, we introduce a normal form for multi-qudit Pauli operators, which will serve as a basic building block in Section 3.3. This normal form is unique up to a global phase.

Definition 3.11. *An n-qudit circuit in $\mathfrak{C}_n$ is said to be a* Pauli *normal form if it is of the form in Figure 10.*

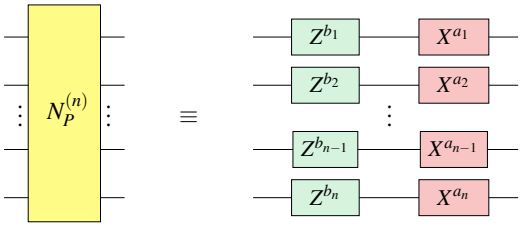


Figure 10: The construction of a Pauli normal form.

Lemma 3.12. *For every element $P \in \mathcal{P}_n$, there exists a unique Pauli normal form $N_P^{(n)}$ such that $[\![N_P^{(n)}]\!]_u \equiv_p P$.*

Proof. This follows from the fact that every $P \in \mathcal{P}_n$ can be uniquely written as $P = \omega^t \bigotimes_{j=1}^n X^{a_j} Z^{b_j}$ for $t, a_j, b_j \in \mathbb{Z}_d$. □

3.3 Compose the Symplectic and Pauli Normal Forms

We now combine the symplectic and Pauli normal forms to obtain a canonical normal form for arbitrary multi-qudit Clifford circuits. Using the semi-direct product structure of the Clifford group, this composition separates the symplectic action from the residual Pauli correction, yielding a unique representation up to global phase. In what follows, $A;B$ denotes the sequential composition of gates in circuit (diagrammatic) order, meaning that A is applied first and then B. This corresponds to the matrix product BA.

Definition 3.13. *An n-qudit Clifford circuit $N^{(n)}$ is a* Clifford norm form *if $N^{(n)} \equiv N_S^{(n)}; N_P^{(n)}$ as shown in Figure 11, where $N_P^{(n)}$ is a Pauli normal form and $N_S^{(n)}$ is a symplectic normal form.*

Proposition 3.14. *For every n-qudit Clifford operator $C \in \mathcal{C}_n$, there exists a unique Clifford normal form circuit $N \in \mathfrak{C}_n$ such that $[\![N]\!]_u \equiv_p C$.*

Proof. We first prove the existence. By Proposition 3.9, there is a unique circuit $N_S \in \mathfrak{C}_n$ in the symmetric normal form such that $[\![N_S]\!]_u \bullet P \equiv_p C \bullet P$. Hence $[\![N_S]\!]_s = \pi_s(C)$. This implies there is a unique Pauli $P \in \mathcal{P}_n$ such that $P \cdot [\![N_S]\!]_u = C$. Then by Lemma 3.12, there is a unique circuit $N_P \in \mathfrak{C}_n$ such that $[\![N_P]\!]_u [\![N_S]\!]_u \equiv_p C$. It follows that $N \equiv N_S; N_P$ is a circuit in Clifford normal form such that $[\![N]\!]_u \equiv_p C$.

Now we prove the uniqueness. Suppose $N' \equiv N'_P N'_S$ is another circuit in $\mathfrak{C}_n$ in Clifford normal form such that $[\![N]\!]_u \equiv_p C$, where N'_S and N'_P are symplectic normal form circuit and Pauli normal form circuit respectively. It follows that $[\![N'_S]\!]_s = [\![N']\!]_s = \pi_s(C) = [\![N_S]\!]_s$. By the uniqueness of symplectic normal form, we must have $N'_S = N_S$. This also implies $[\![N'_S]\!]_u = [\![N_S]\!]_u$. Since $[\![N']\!]_u \equiv_p$

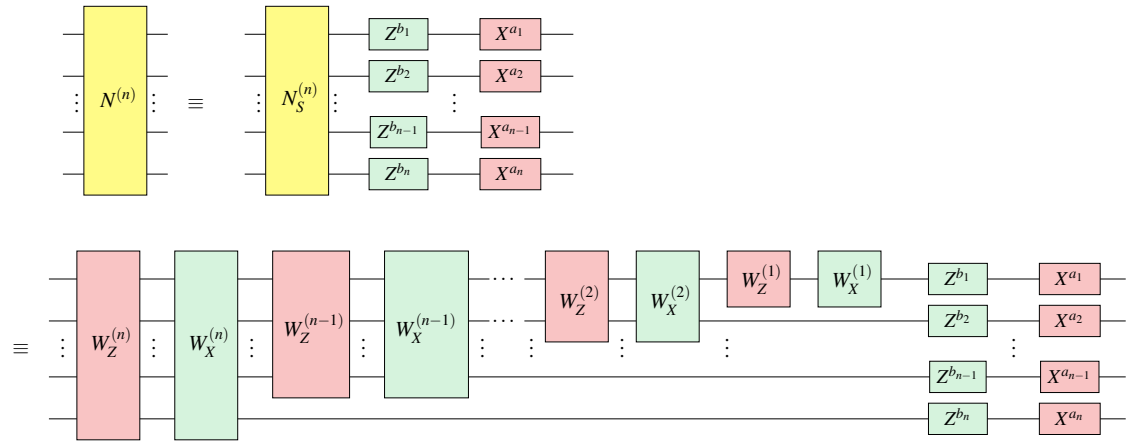


Figure 11: The construction of a Clifford normal form.

$[\![N]\!]_u$, we have $[\![N'_P]\!]_u \equiv_p [\![N_P]\!]_u$ as Pauli operators in $\mathcal{P}_n$. By the uniqueness of Pauli normal form, we must have $N'_P \equiv N_P$. We conclude that $N' \equiv N$. □

Since each element of $\widehat{\mathcal{C}}_n$ is in bijection with a unique Clifford normal form as shown in Figure 11, counting these normal forms yields the cardinality of $\widehat{\mathcal{C}}_n$. Using Proposition 2.12, which characterises the Clifford phases as the integer powers of ω, we then obtain the cardinality of the Clifford group $\mathcal{C}_n$.

Remark 3.15. *By Lemmas 3.10 and 3.12 and proposition 3.14,*

$$|\mathcal{C}_n| = d^{2n+1} \cdot |\mathrm{Sp}(2n, \mathbb{Z}_d)| = d^{2n+1} \prod_{k=1}^{n} \left(d^{2k} - 1\right) \cdot \left(d^{2k-1}\right).$$

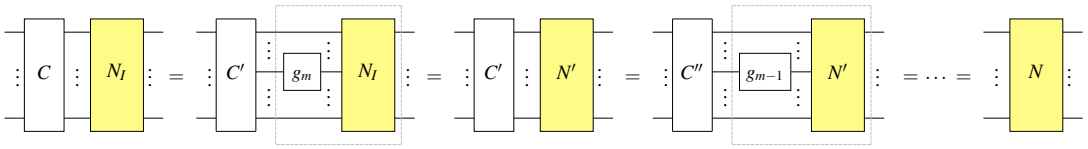
4 A Complete Set of Multi-Qudit Clifford Relations

In this section, we derive a complete set of rewrite rules for multi-qudit Clifford circuits. In Section 4.1, we establish completeness at the symplectic level and then in Section 4.2, we lift these results to unitary Clifford circuits by accounting for Pauli corrections.

4.1 Establish Symplectic Completeness

Here, we show how to derive a complete set of relations that suffices to transform any multi-qudit Clifford circuits into its unique symplectic normal form. We call this the *Clifford normalisation* process, and it proceeds as follows. Let $C \in \mathcal{C}_n$ be a circuit composed of gates $g_1, \ldots, g_m$. Append it on the right with the normal form of the identity circuit (N_I). Starting from the right-most gate g_m, iteratively 'push' each gate of C into the normal form, updating it at each step (e.g., from N_I to N').

This process continues until all gates have been absorbed into the updated normal form N. Below, we sketch a simple example. Since the Pauli component plays no role in the present discussion, we suppress the distinction between the symplectic normal form N_S and the Pauli normal form N_P, and simply write N for the symplectic normal form.



Overview In (11) and (12), we present the symplectic normal form of the identity operator. Since a Clifford circuit is composed of H, S and CZ gates, it suffices to show how a normal form absorbs these gates and is updated to a new normal form. Locally, this reduces to demonstrating how to push a Clifford gate through each of the normal boxes specified in Figure 6. We call each such instance a *box relation*.

Figure 12.(a) gives an abstract illustration of *pushing through a normal box*, which results in an updated normal box M' and *residual dirty gates*, denoted as dir, appearing after M'. M' is uniquely determined by g and M. Lemmas D.2 to D.14 provide examples of finding M' and discuss this process in details. dir denotes a Clifford circuit, and it is determined by g, M, and M' collectively. Figure 12.(b) shows a concrete example of pushing an H gate through a B box.

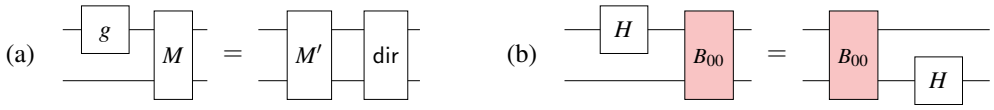


Figure 12: An abstract and concrete example of pushing a gate through a normal box M.

In total, there are 66 parametric box relations, which are summarised in Section F. These relations cover every possible case of pushing through a normal box, making them sufficient to normalise any Clifford operator. Combined with Proposition 3.9, any two Clifford circuits that are equal as symplectic matrices can be rewritten into the same normal form using these box relations. This shows that the box relations are *complete* for multi-qudit Clifford circuits up to Pauli corrections.

Technical details We need to show that the Clifford normalisation terminates, and when this happens, there are no more gates left on the right-hand side of the normal form. Figure 13 provides a schematic summary of what a normal form could look like during the normalisation process. It accounts for all possible cases when pushing through a normal box.

Definition 4.1. *We say that a circuit is in* clean symplectic normal form *if it is of the form in Figure 8. A circuit is in* dirty symplectic normal form *if it is of the form in Figure 13, which has*

the structure of a normal form, but with any number of additional Clifford gates allowed in all the designated spots, subject to the following rules.

- *H gates can be present at any wire labelled* ①*;*
- *S gates can be present at any wire labelled* ①*,* ②*, or* ③*;*
- *CZ gates can be present at any pair of adjacent wires, provided that the top wire is labelled* ①*,* ②*, or* ③*, and the bottom wire is labelled* ①*.*

In the context of a dirty normal form, we will refer to the normal boxes as clean*, and all the other H, S, X, Z, and CZ gates as* dirty*.*

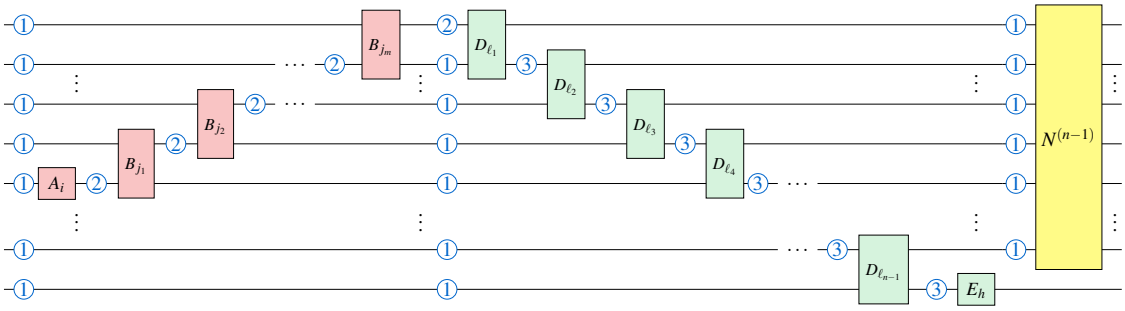


Figure 13: An example of an n-qudit dirty symplectic normal form. Here, $N^{(n-1)}$ is in dirty normal form subject to the same constraints as $N^{(n)}$.

The placement of dirty gates is restricted by which gates can be properly absorbed by each normal box, as this ultimately determines whether Clifford normalisation will terminate. Accordingly, the third column of Figure 5 identifies the additional symmetry each normal box must satisfy to ensure dirty gates always have the expected shape. By manually inspecting the box relations in Section F, we can confirm that all dirty gates follow the pattern described in Definition 4.1.

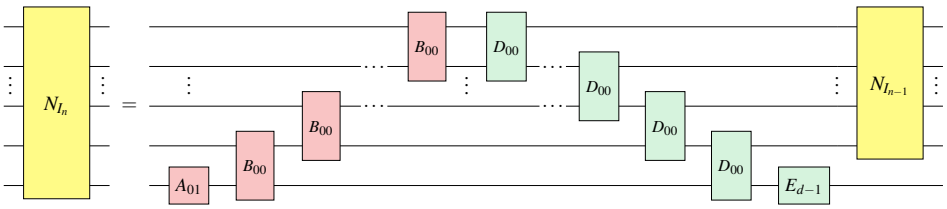
Lemma 4.2. *Any dirty normal form can be converted to its symplectic normal form by applying the box relations in Section F, in the left-to-right direction, a finite number of times.*

The proof is analogous to that of [35, Lemma 4.2].

Proposition 4.3. *Consider a Clifford circuit composed of H, S, and CZ gates. Any such circuit can be converted to its symplectic normal form by using the box relations in Section F, together with the equations:*

(11)

Proof. For $1 \leq k \leq n$, let N_{I_k} be the normal form of the k-qudit identity operator. First note that



(12)

Indeed, an identity circuit can be converted to this normal form by applying (11) a finite number times. Appending the given Clifford circuit on the right with the normal form in (12), we obtain a dirty normal form, which can be converted to its normal form by Lemma 4.2. □

Up to Pauli corrections, Propositions 3.9 and 4.3 show that qudit Clifford operators admit a presentation in terms of the generators H, S, and CZ, together with the derived generators from Figure 4, the normal boxes in Figure 6, and the 66 parametric box relations listed in Section F. This presentation is highly redundant, so we extract a reduced set of relations, shown in Figure 14. The resulting rule set remains symplectically complete for multi-qudit Clifford circuits, while providing a more streamlined and intuitive account of qudit Clifford gate interactions.

Theorem 4.4. *The* 18 *rewrite rules listed in Figure 14 are complete for qudit Clifford circuits up to Pauli corrections.*

Proof. It suffices to show that these 18 relations imply the full set of 66 parametric box relations. All proofs have been formalised, and the corresponding code is available in the GitHub repository [6]. □

4.2 From Symplectic to Unitary Completeness

In Section 4.1, we show that Clifford circuits corresponding to the same symplectic matrix can be rewritten into the same normal form using the relations of Figure 14. To lift this result to unitary completeness, we account for the Pauli corrections that are invisible at the symplectic level. By combining symplectic completeness with the Pauli normal form from Section 3.2, we obtain a complete and sound rule set of Clifford circuits for the unitary interpretation. Again, we use $A;B$ to denote the sequential composition of gates in diagrammatic order, meaning that A is applied first and then B.

Definition 4.5. *Let C be an n-qudit Clifford circuit, given as a list of gates. We define its* unitary *interpretation $[\![C]\!]_u$ as the $d^n \times d^n$ unitary matrix the circuit represents, up to global phase. We define its* symplectic *interpretation $[\![C]\!]_s$ as the $2n \times 2n$ symplectic matrix over $\mathbb{Z}_d$ it represents.*

Note that in the symplectic interpretation, the Paulis are all equal to the identity: $[\![P]\!]_s = \mathrm{id}$. We define the unitary interpretation as ignoring global phases. We could define this formally by viewing these matrices as elements of the projective unitary group that mods out global non-zero scalars, but what we really care about is that when we say two circuits have an equal unitary

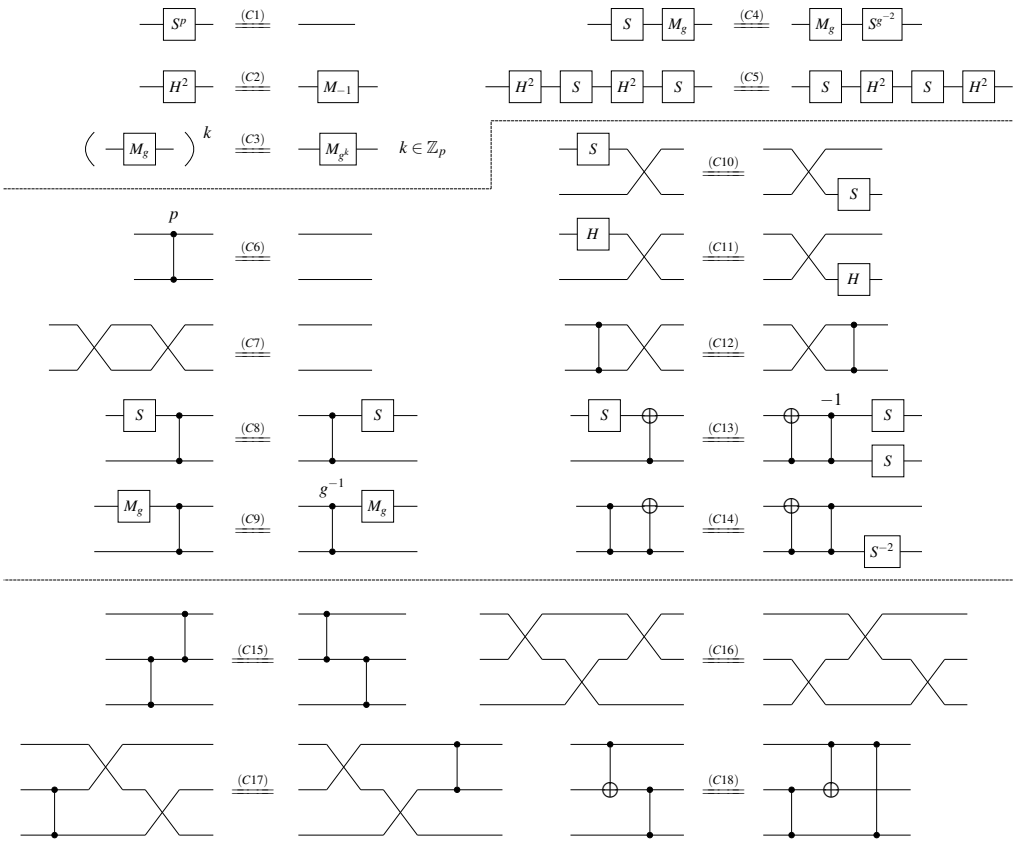


Figure 14: A complete set of symplectic relations for n-qudit Clifford circuits where d is an odd prime and $n \in \mathbb{N}$. Derived generators such as M_g, X, Z, SWAP, and CX are defined in Figure 2.

interpretation that we mean that their linear maps are equal up to global phase. Other than that we can just view the unitary interpretation as being a unitary linear map. Note that if two circuits are equal in the unitary interpretation that they are then also equal in the symplectic interpretation.

Lemma 4.6. *Let C_1 and C_2 be two qudit Clifford circuits with equal symplectic representation: $[\![C_1]\!]_s = [\![C_2]\!]_s$. Then there exists a unique Pauli P such that $[\![C_1]\!]_u = [\![C_2; P]\!]_u$.*

In other words: given two Cliffords implementing the same symplectic map, we can find a unique "Pauli correction" that makes the Clifford unitaries exactly equal up to a global phase.

Definition 4.7. *Let $\mathcal{R}_s := \{C_l^{(i)} = C_r^{(i)}\}$ be a set of rewrite rules of Clifford circuits. We say it is* symplectically complete *when they are* sound *with respect to the symplectic interpretation, $[\![C_l^{(i)}]\!]_s = [\![C_r^{(i)}]\!]_s$ for all i, and* complete *for this interpretation: given circuits C_1 and C_2 such that $[\![C_1]\!]_s = [\![C_2]\!]_s$, we can find a sequence of rewrites from $\mathcal{R}_s$ that rewrites C_1 to C_2.*

Definition 4.8. *Let $\mathcal{R}_p$ be a set of rewrite rules of Clifford circuits. We say it is* Pauli complete *if they are sound for the unitary interpretation, and they suffice to prove the commutation and cancellation identies for Paulis as well as how to push Paulis through all the Clifford generators. That is, $\mathcal{R}_p$ must prove the following equations:*

- $Z;X = X;Z$
- $Z^d = X^d = 1$
- $Z;H = H;X^{-1}$ *and* $X;H = H;Z$
- $Z;S = S;Z$ *and* $X;S = S;Z;X$
- $(Z\otimes I);CZ = CZ;(Z\otimes I)$
- $(X\otimes I);CZ = CZ;(X\otimes Z)$

Finally, our construction follows a generic group-theoretic approach: given a group G with a normal subgroup N and a presentation of both N and the quotient group G/N, one can construct a presentation of G itself [30, Proposition 2.55]. This framework applies directly in our setting, since the Pauli group $\mathcal{P}_n$ is a normal subgroup of the Clifford group $\mathcal{C}_n$, and the quotient group $\mathcal{C}_n/\mathcal{P}_n$ is isomorphic to the symplectic group $\mathrm{Sp}(2n,\mathbb{Z}_d)$. Since $\mathcal{C}_n$ is isomorphic to the semidirect product of $\mathcal{P}_n$ by $\mathrm{Sp}(2n,\mathbb{Z}_d)$, lifting symplectic relations to unitary Clifford relations amounts to accounting for the associated Pauli relations, which leads to a particularly simple form of the general construction.

Proposition 4.9. *Let $\mathcal{R}_s := \{C_l^{(i)} = C_r^{(i)}\}$ be a symplectically complete set of rewrite rules and $\mathcal{R}_p$ a Pauli complete set of rewrite rules. Then we can efficiently find a modified set of rewrite rules $\mathcal{R}'_s := \{C_l^{(i)} = C_r^{(i)};P^{(i)}\}$ where the $P^{(i)}$ are uniquely defined Paulis such that $\mathcal{R}'_s$ is sound in the unitary interpretation and $\mathcal{R}'_S \cup \mathcal{R}_p$ is sound and complete for the unitary interpretation.*

Proof. By Lemma 4.6 we can find unique Paulis to make the equations of $\mathcal{R}_s$ sound in the unitary interpretation by changing them to $\mathcal{R}'_s := \{C_l^{(i)} = C_r^{(i)};P^{(i)}\}$. Hence the equations $\mathcal{R}'_S \cup \mathcal{R}_p$ are indeed all sound for the unitary interpretation. Now, to prove completeness let us assume we have two circuits C_1 and C_2 that are equal in the unitary interpretation: $[\![C_1]\!]_u = [\![C_2]\!]_u$. We need to find a set of rewrites from $\mathcal{R}'_S \cup \mathcal{R}_p$ that rewrites C_1 into C_2.

Note that C_1 and C_2 are also necessarily equal in the symplectic interpretation. Hence, there is a series of rewrite rules from $\mathcal{R}_s$ that rewrites C_1 to C_2 in a symplectically sound way. Let's consider the first two rewrites in this sequence: $C_1 \rightsquigarrow C_1' \rightsquigarrow C_1''$ and suppose the rewrites used were $C_l^{(i)} = C_r^{(i)}$ and $C_l^{(j)} = C_r^{(j)}$. These rewrites were potentially applied locally in some larger context.

Now instead of $C_l^{(i)} = C_r^{(i)}$ we must use the unitarily sound modification $C_l^{(i)} = C_r^{(i)};P^{(i)}$. The LHS is still the same, and hence the rule still matches to C_1, but now we don't end up with C_1', but instead a modified circuit that has $P^{(i)}$ appearing at the end of the application of $C_r^{(i)}$. Using the rules from the Pauli complete $\mathcal{R}_p$ we can now push $P^{(i)}$ further to the right in the modified C_1' past all the generators. This changes $P^{(i)}$ to a potentially different Pauli, but otherwise does not change

any of the structure of C_1'. At the end we hence end up with a circuit $C_1'; Q^{(i)}$ where the $Q^{(i)}$ is uniquely determined by $P^{(i)}$. We hence see that our rewrite rules prove $C_1 \rightsquigarrow C_1'; Q^{(i)}$.

Now, the second rewrite rule $C_l^{(j)} = C_r^{(j)}$ is modified to the unitarily sound $C_l^{(j)} = C_r^{(j)}; P^{(j)}$. The LHS is still the same, and we still have C_1' (the only change is that it is composed with $Q^{(i)}$, but this does not change the fact whether the LHS matches or not). Hence, we can apply this modified rewrite rule to $C_1'; Q^{(i)}$. This then results in a modified C_1'' where there is the Pauli $P^{(j)}$ composed after the matching $C_r^{(j)}$. In the same way as above we can push this to the end of the circuit to end up with $C_1''; Q^{(j)}; Q^{(i)}$.

Repeating this procedure for all the rewrite rules from $\mathcal{R}_s$ that bring us from C_1 to C_2 gives us a set of now unitarily sound rewrites from $\mathcal{R}_S' \cup \mathcal{R}_p$ that rewrites C_1 to $C_2; P$ where P is some big collection of Paulis. Now, because we know that C_1 and C_2 are unitarily equal, we must have $[\![P]\!]_u = \mathrm{id}$, and so $\mathcal{R}_p$ must be able to rewrite P to the identity (empty) circuit. This then gives us a rewrite from C_1 to $C_2; P$ and finally to C_2. □

Theorem 4.10. *The rewrite rules in Figure 1 are* complete *for n-qudit Clifford circuits over any qudit dimension d that is an odd prime. That is, given two n-qudit unitary Clifford circuits C_1 and C_2 implementing the same linear map, there is a sequence of rewrites from Figure 1 that proves C_1 and C_2 are equal.*

Proof. By direct computation, the rewrite rules in Figure 1 imply all Pauli relations listed in Definition 4.8. By Proposition 4.9, the rules in Figure 1 form a complete rewrite system for n-qudit Clifford circuits. □

5 Conclusion and Open Problems

We give a complete and finite equational theory for n-qudit Clifford circuits in every odd prime dimension, by presenting a small set of local rewrite rules that suffices to determine equality of Clifford unitaries. Our approach proceeds by introducing a symplectic normal form for Clifford circuits, deriving relations at the symplectic level, and then lifting these relations to circuit identities by accounting for Pauli corrections.

An immediate open question is to extend this completeness framework beyond odd prime dimensions, in particular to prime-power and even dimensions, where the arithmetic over $\mathbb{Z}_d$ and the behaviour of Clifford phases are more intricate. More broadly, it would be interesting to develop compact circuit-level equational theories for larger circuit fragments, such as Clifford–permutation–dihedral circuits. In the meantime, it will be fruitful to extend our software infrastructure as a verified backend for automated optimisation and synthesis of qudit circuit families.

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A Proofs from Section 2

In this appendix, we provide proofs that were omitted from Section 2 and document algebraic results related to the qudit Clifford group. Recall the summation of a geometric sequence for $a_n = a_1 q^{n-1}$ when $q \neq 1$ is

$$S_n = \frac{a_1(1-q^n)}{1-q}. \tag{13}$$

Definition A.1. *Let $k \in \mathbb{Z}_d$ and $f : \mathbb{Z}_d \mapsto \mathbb{Z}_d$ be an invertible function. A* generalised permutation matrix *is a matrix that maps $|k\rangle$ to $a_k|f(k)\rangle$, for some $a_k \in \mathbb{C}$ and $\|a_k\|^2 = 1$.*

Lemma A.2. *Let ω be the primitive d-th root of unity and $k \in \mathbb{Z}_d$.*

$$\sum_{\ell=0}^{d-1} \omega^{k\ell} = \begin{cases} 0, & k \not\equiv_d 0 \\ p, & k \equiv_d 0. \end{cases} \tag{14}$$

Proof. When $k \equiv_d 0$, $\sum_{\ell=0}^{d-1} \omega^{k\ell} = \sum_{\ell=0}^{d-1} \omega^0 = p$. When $k \not\equiv_d 0$, according to (13),

$$\sum_{\ell=0}^{d-1} \omega^{k\ell} = \frac{1\left(1-(\omega^k)^d\right)}{1-\omega^k} = 0. \qquad \square$$

Lemma A.3. $\det(H) = 1$.

Recall that $H = \frac{1}{\lambda_d \sqrt{d}} F$, where $F = \sum_{j,\ell=0}^{d-1} \omega^{j\ell} |j\rangle\langle\ell|$ is a discrete Fourier transform. The determinant of F is a standard computation (see e.g., [38]), and the phase λ_d is chosen so that $\det(H) = 1$ (see also [44]). For completeness, we give a short elementary proof.

Proof. We first compute

$$\begin{aligned} \det(F) &= \prod_{0 \leq \ell < j \leq d-1} (\omega^j - \omega^\ell) = \prod_{0 \leq \ell < j \leq d-1} \omega^\ell (\omega^{j-\ell} - 1) \\ &= \left(\prod_{t=1}^{d-1} \prod_{\ell=1}^{d-1-t} \omega^\ell \right) \cdot \left(\prod_{t=1}^{d-1} (\omega^t - 1)^{d-t} \right) \\ &= \omega^{(d-1)d(d-2)/6} \left(\prod_{t=1}^{d-1} (\omega^t - 1)^{d-t} \right). \end{aligned}$$

Let $\xi := e^{\pi i/d}$. Since $\omega^t - 1 = 2i\xi^t \sin(\pi t/d)$ for all t, we obtain that

$$\begin{aligned} \prod_{t=1}^{d-1} (\omega^t - 1)^{d-t} &= (2i)^{\sum_{t=1}^{d-1}(d-t)} \cdot \xi^{\sum_{t=1}^{d-1} t(d-t)} \cdot \prod_{t=1}^{d-1} (\sin(\pi t/d))^{d-t} \\ &= (2i)^{d(d-1)/2} \cdot \xi^{(d-1)d(d-2)/6} \cdot \left(\prod_{t=1}^{(d-1)/2} \sin(\pi t/d) \right)^d, \end{aligned}$$

where the last equality pairs $(\sin(\pi t/d))^{d-t}$ with $(\sin(\pi(d-t)/d))^t$. Differentiating the cyclotomic polynomial $x^d-1=\prod_{t=0}^{d-1}(x-\omega^t)$ and evaluating at $x=1$ yields $\prod_{t=1}^{d-1}(1-\omega^t)=p$. Note that $|1-e^{i\theta}|=2\sin(\theta/2)$ for all $\theta\in(0,2\pi)$. This implies $\prod_{t=1}^{d-1}2\sin(\pi t/d)=p$. Since $\sin(\pi t/d)>0$ for all $1\leq t\leq d-1$, by pairing t with $d-t$, we see that $\prod_{t=1}^{(d-1)/2}\sin(\pi t/d)=\left(\prod_{t=1}^{(d-1)}\sin(\pi t/d)\right)^{1/2}=2^{-(d-1)/2}\cdot d^{1/2}$. Hence

$$\prod_{t=1}^{d-1}(\omega^t-1)^{d-t}=i^{d(d-1)/2}\cdot\xi^{(d-1)d(d-2)/6}\cdot d^{d/2}.$$

It follows that

$$\det(F)=d^{d/2}e^{i\pi(d-1)/4}.$$

Since λ_d satisfies $\lambda_d^d=e^{i\pi(d-1)/4}$ for all odd prime d, the lemma follows. □

Lemma A.4. $X^d=Z^d=1$.

Proof. Note that, for any $0\leq j\leq d-1$, we have $X^d|j\rangle=|j+d\rangle=|j\rangle$ and $Z^d|j\rangle=\omega^d|j\rangle=|j\rangle$. □

Lemma A.5. $ZXZ^\dagger X^\dagger=\omega$.

Proof. Note that, for $0\leq j\leq d-1$, we have

$$ZXZ^\dagger X^\dagger|j\rangle=ZXZ^\dagger|j-1\rangle=ZX\omega^{-(j-1)}|j-1\rangle=Z\omega^{-(j-1)}|j\rangle=\omega^{-(j-1)+j}|j\rangle=\omega|j\rangle.$$

□

Remark A.6. *Next, recall that Proposition 2.5 follows from results in [44].*

Proposition 2.6. *The roots of unity in $\mathbb{Z}[1/d,\omega]$ are $\pm\omega^t$, $t\in\mathbb{Z}_d$.*

Proof. By Theorem 5.3 in [19], the roots of unity in $\mathbb{Q}(\omega)$ are exactly the complex numbers of the form $\pm\omega^t$, $t\in\mathbb{Z}_d$. Since $\mathbb{Z}[1/d,\omega]\subset\mathbb{Q}(\omega)$, this completes the proof. □

Proposition 2.12. *For $0\leq j\leq d-1$, $H^2|j\rangle=\lambda_d^{-2}|-j\rangle$, where subtraction is performed modulo d.*

Proof. We have

$$\begin{aligned} H^2|j\rangle &= H\left(\frac{1}{\lambda_d\sqrt{d}}\sum_{k=0}^{d-1}\omega^{kj}|k\rangle\right) \\ &= \frac{1}{\lambda_d\sqrt{d}}\sum_{k=0}^{d-1}\omega^{kj}H|k\rangle \\ &= \frac{1}{\lambda_d\sqrt{d}}\sum_{k=0}^{d-1}\omega^{kj}\left(\frac{1}{\lambda_d\sqrt{d}}\sum_{\ell=0}^{d-1}\omega^{\ell k}|\ell\rangle\right) \\ &= \frac{1}{\lambda_d^2 d}\sum_{\ell=0}^{d-1}\left(\sum_{k=0}^{d-1}\omega^{k(j+\ell)}\right)|\ell\rangle. \end{aligned}$$

Now consider the sum $\sum_{k=0}^{d-i}\omega^{k(j+\ell)}$. When $\ell \equiv -j \pmod d$, then $w^{k(j+\ell)} = 1$, so that the sum is equal to d. When $\ell \not\equiv -j \pmod d$, then $\omega^{j+\ell}$ is a primitive d-th root of unity, so that the sum is equal to 0. Hence,

$$H^2|j\rangle = \frac{1}{\lambda_d^2 d}\sum_{\ell=0}^{d-1}\left(\sum_{k=0}^{d-1}\omega^{k(j+\ell)}\right)|\ell\rangle = \frac{1}{\lambda_d^2}|-j\rangle,$$

as desired. □

Corollary A.7. *H^2 is a generalised permutation matrix.*

Lemma 2.18. $\det(S) = \det(H) = \det(CZ) = 1$.

Proof. By Lemma A.3, $\det(H) = 1$. Next, we show that $\det(S) = \det(\mathrm{CZ}) = 1$. Using straightforward properties of sums of integers, we have

$$\sum_{j=0}^{d-1} j^2 = \left(\sum_{j=1}^{d} j^2\right) - d^2 = \frac{d(d+1)(2d+1)}{6} - d^2. \tag{15}$$

$$\sum_{j=0}^{d-1} j = \frac{(0+(d-1))\,d}{2} = \frac{d(d-1)}{2}. \tag{16}$$

$$\sum_{j,\ell=0}^{d-1} j\ell = \sum_{j=0}^{d-1} j\left(\sum_{\ell=0}^{d-1}\ell\right) \overset{(16)}{=\!=} \sum_{j=0}^{d-1} j\left(\frac{d(d-1)}{2}\right) \overset{(16)}{=\!=} \left(\frac{d(d-1)}{2}\right)^2 = \frac{d^2(d-1)^2}{4}. \tag{17}$$

Then we have

$$\sum_{j=0}^{d-1} j^2 - \sum_{j=0}^{d-1} j \overset{(15)}{\underset{(16)}{=\!=}} \frac{d(d+1)(2d+1)}{6} - d^2 - \frac{d(d-1)}{2} = \frac{d(d^2-3d+2)}{3}. \tag{18}$$

$$\sum_{j=0}^{d-1} j^2 + \sum_{j=0}^{d-1} j \overset{(15)}{\underset{(16)}{=\!=}} \frac{d(d+1)(2d+1)}{6} - d^2 + \frac{d(d-1)}{2} = \frac{d(d^2-1)}{3}. \tag{19}$$

Since $\omega^d = 1$, it then follows that

$$\det(S) \overset{\text{Definition 2.10}}{=\!=} \prod_{j=0}^{d-1} \omega^{\frac{j(j-1)}{2}} = \omega^{\sum_{j=0}^{d-1} \frac{j(j-1)}{2}} = \omega^{\frac{1}{2}\left(\sum_{j=0}^{d-1} j^2 - \sum_{j=0}^{d-1} j\right)} \overset{(18)}{=\!=} \omega^{\frac{1}{2}\left(\frac{d(d^2-3d+2)}{3}\right)} = \omega^{d(d^2-3d+2)/6} = 1.$$

$$\det(\mathrm{CZ}) = \prod_{j,\ell=0}^{d-1} \omega^{j\ell} = \omega^{\sum_{j,\ell=0}^{d-1} j\ell} \overset{(17)}{=\!=} \omega^{d^2(d-1)^2/4} = 1. \qquad \square$$

Lemma 2.25. *$H, S \in \mathcal{C}_1$. Specifically, $HXH^\dagger = Z$, $HZH^\dagger = X^{-1}$, $SXS^\dagger = XZ$, $SZS^\dagger = Z$.*

Proof. One can check that $\lambda_d \lambda_d^\dagger = 1$. By Definitions 2.8 and 2.10, we have

$$\begin{aligned}
HXH^\dagger|j\rangle &= HX\left(\frac{1}{\lambda_d^\dagger\sqrt{d}}\sum_{k=0}^{d-1}\omega^{-kj}|k\rangle\right) \\
&= H\left(\frac{1}{\lambda_d^\dagger\sqrt{d}}\sum_{k=0}^{d-1}\omega^{-kj}|k+1\rangle\right) \\
&= \frac{1}{\lambda_d^\dagger\sqrt{d}}\sum_{k=0}^{d-1}\omega^{-kj}\left(\frac{1}{\lambda_d\sqrt{d}}\sum_{\ell=0}^{d-1}\omega^{(k+1)\ell}|\ell\rangle\right) \\
&= \frac{1}{d}\sum_{\ell=0}^{d-1}\omega^{\ell}\left(\sum_{k=0}^{d-1}\omega^{k(\ell-j)}\right)|\ell\rangle.
\end{aligned}$$

Now consider the sum $\sum_{k=0}^{d-i}\omega^{k(\ell-j)}$. When $\ell \equiv j \pmod d$, then $w^{k(\ell-j)} = 1$, so that the sum is equal to d. When $\ell \not\equiv j \pmod d$, then $\omega^{\ell-j}$ is a primitive d-th root of unity, so that the sum is equal to 0. Hence,

$$HXH^\dagger|j\rangle = \frac{1}{d}\sum_{\ell=0}^{d-1}\omega^{\ell}\left(\sum_{k=0}^{d-1}\omega^{k(\ell-j)}\right)|\ell\rangle = \omega^j|j\rangle = Z|j\rangle,$$

as desired. Note that, by Corollary 2.15, we have $H^2 = (H^\dagger)^2$. Hence, $HZH^\dagger = H^2X(H^\dagger)^2 = H^2XH^2$. Thus

$$HZH^\dagger|j\rangle = H^2XH^2|j\rangle = H^2X|-j\rangle = H^2|-j+1\rangle = |-(-j+1)\rangle = |j-1\rangle = X^\dagger|j\rangle.$$

Similarly, we have

$$S'XS'^\dagger|j\rangle = S'X\omega^{-\frac{j(j+1)}{2}}|j\rangle = S'\omega^{-\frac{(j+1)(j+2)}{2}}|j+1\rangle = \omega^{-\frac{j(j+1)}{2}+\frac{(j+1)(j+2)}{2}}|j+1\rangle = \omega^{j+1}|j+1\rangle = \omega XZ|j\rangle.$$

Finally, we have

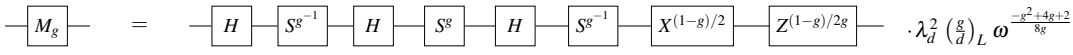
$$SZS^\dagger|j\rangle = SZ\omega-\frac{j(j-1)}{2}|j\rangle = S\omega-\frac{j(j-1)}{2}+1|j\rangle = \omega-\frac{j(j-1)}{2}+1+\frac{j(j-1)}{2}|j\rangle = \omega|j\rangle = Z|j\rangle,$$

and

$$S'ZS'^{\dagger}|j\rangle = SZ\omega - \frac{j(j+1)}{2}|j\rangle = S\omega - \frac{j(j+1)}{2}+1|j\rangle = \omega - \frac{j(j+1)}{2}+1+\frac{j(j+1)}{2}|j\rangle = \omega|j\rangle = Z|j\rangle,$$

□

Lemma 2.14. *For every $g \in \mathbb{Z}_d^*$, M_g defined below satisfies $M_g|x\rangle = |gx\rangle$ for all $x \in \mathbb{Z}_d$.*



Proof. Fix a $g \in \mathbb{Z}_d^*$, we compute

$$M_g|0\rangle = \frac{1}{\lambda_d d^{3/2}} \omega^{-\frac{g^2-4g-2}{8g}} \left(\frac{g}{d}\right)_L \sum_h \omega^{t_h} \mu_h |h + \frac{1-g}{2}\rangle,$$

where

$$t_h = \frac{h(h-1)}{2g} + \frac{1-g}{2g}\left(h + \frac{1-g}{2}\right)$$

and

$$\mu_h = \sum_{k,j} \omega^{\frac{j(j-1)}{2g} + kj + \frac{k(k-1)}{2g} + kh}.$$

On the other hand, one can track the Pauli propagation $M_g X M_g^{\dagger} = X^g$ and $M_g Z M_g^{\dagger} = Z^{g^{-1}}$. This implies there exists $c_g \in \mathbb{C}$ such that $M_g|x\rangle = c_g|x\rangle$ for all $x \in \mathbb{Z}_d$. In particular, $M_g|0\rangle = c_g|0\rangle$. So we must have $\mu_h = 0$ for all $h \neq \frac{g-1}{2}$. It follows that

$$\mu_{\frac{g-1}{2}} = \sum_h \mu_h = \sum_{k,j} \omega^{\frac{j(j-1)}{2g} + kj + \frac{k(k-1)}{2g}} \cdot \sum_h \omega^{kh} = d\left(\sum_j \omega^{\frac{j(j-1)}{2g}}\right) = \lambda_d d^{3/2} \left(\frac{g}{d}\right)_L \omega^{-\frac{1}{8g}}.$$

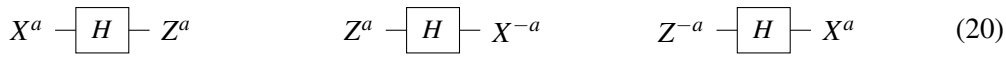
The second last equality uses that $\sum_h \omega^{kh} = \begin{cases} d & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$. The last equality follows from the property of the quadratic Gauss sum $\sum_j \omega^{\frac{j^2}{2g}} = \lambda_d \sqrt{d} \left(\frac{g}{d}\right)_L$, where $\left(\frac{\cdot}{\cdot}\right)_L$ is the Legendre symbol. It follows that

$$c_g = \left(\frac{g}{d}\right)_L \omega^{-\frac{g^2-4g-2}{8g}} \frac{1}{\lambda_d d^{3/2}} \omega^{t_{\frac{g-1}{2}}} \mu_{\frac{g-1}{2}} = 1.$$

We conclude that $M_g|x\rangle = |gx\rangle$ for all $x \in \mathbb{Z}_d$. □

B Clifford Conjugation

Lemma B.1. *For all $a \in \mathbb{Z}_d$,*



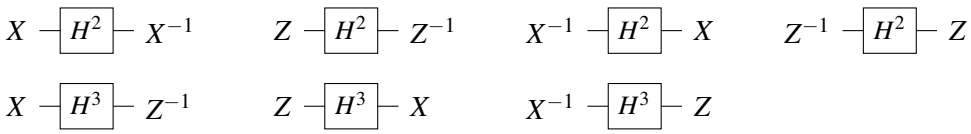
(20)

Proof. Based on Lemma 2.25, we proceed by repeatedly applying the action of H on a copies of Pauli X and Z.

$$HX^aH^\dagger = (HXH^\dagger)^a \overset{(9)}{=\!=} Z^a$$
$$HZ^aH^\dagger = (HZH^\dagger)^a \overset{(9)}{=\!=} \left(X^{-1}\right)^a = X^{-a}$$
$$H^\dagger X^aH = (H^\dagger XH)^a \overset{(9)}{=\!=} \left(Z^{-1}\right)^a = Z^{-a}$$

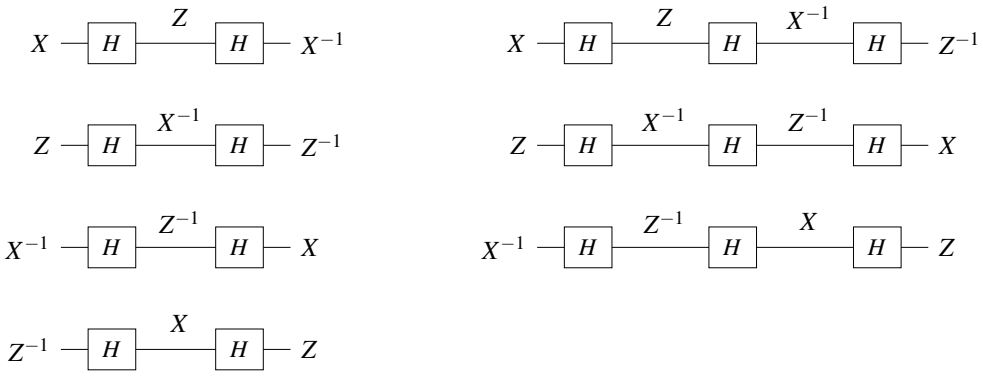
□

Lemma B.2.



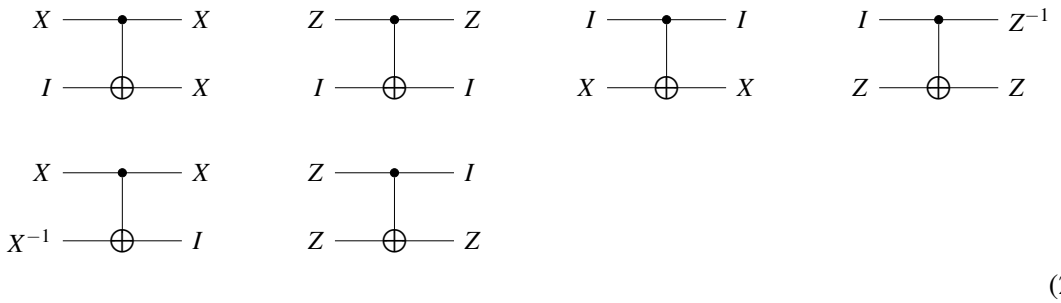
(21)

Proof. Based on Lemma 2.25, we proceed by repeatedly applying the action of H on Pauli X and Z.

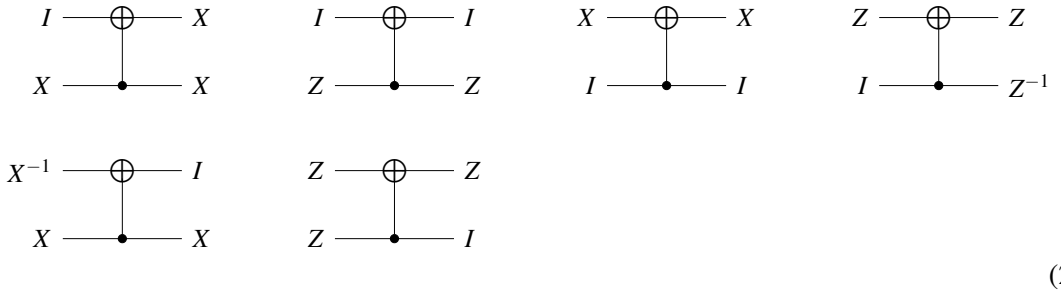


□

Lemma B.3.

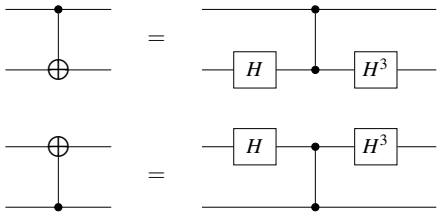


(22)



(23)

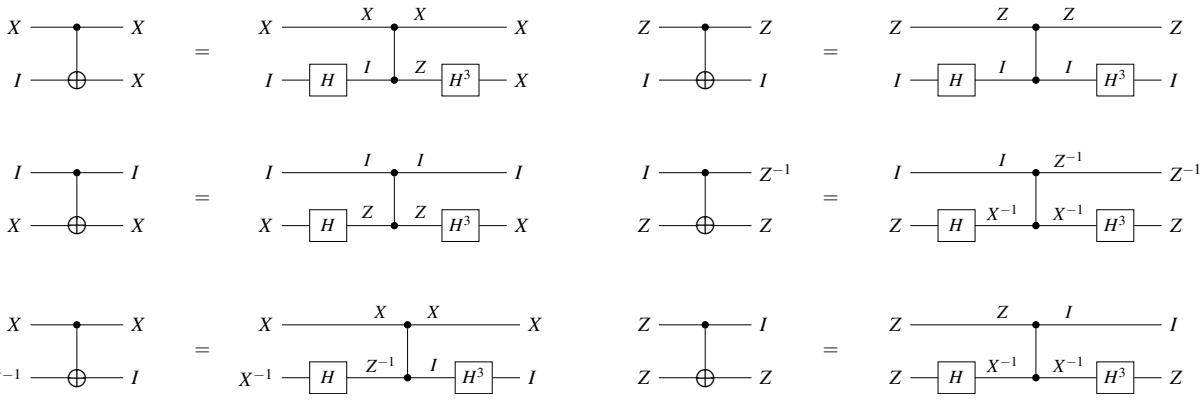
Proof. Recall that



(*D*7)

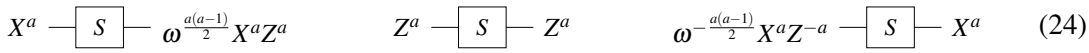
(*D*8)

Up to symmetry, it suffices to show (22). By (9), (10), and (21), we therefore have



□

Lemma B.4. *For all $a \in \mathbb{Z}_d$,*



(24)

Proof. By Lemma 2.25, $SZ^aS^\dagger = \left(SZS^\dagger\right)^a = Z^a$. For the actions of S on X^a, we proceed by induction on a.

Base Case: When $a = 1$, the statement holds by Lemma 2.25.

Induction Hypothesis (IH): Suppose the statement holds for $a \geq 1$. That is,

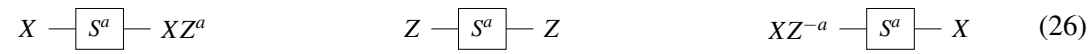
$$SX^aS^\dagger = \omega^{\frac{a(a-1)}{2}} X^a Z^a, \qquad S^\dagger X^a S = \omega^{-\frac{a(a-1)}{2}} X^a Z^{-a}. \tag{25}$$

Induction Step: We want to show that the statement holds for $a+1$.

$$SX^{a+1}S^\dagger = S\left(X^aX\right)S^\dagger = \left(SX^aS^\dagger\right)\left(SXS^\dagger\right) \overset{\text{IH}}{\underset{(9)}{=}} \left(\omega^{\frac{a(a-1)}{2}} X^aZ^a\right)(XZ)$$
$$\overset{(6)}{=} \omega^{\frac{a(a-1)}{2}} \omega^a X^{a+1}Z^{a+1} = \omega^{\frac{(a+1)a}{2}} X^{a+1}Z^{a+1}.$$

$$S^\dagger X^{a+1}S = S^\dagger\left(X^aX\right)S = \left(S^\dagger X^aS\right)\left(S^\dagger XS\right) \overset{\text{IH}}{\underset{(9)}{=}} \left(\omega^{-\frac{a(a-1)}{2}} X^aZ^{-a}\right)\left(XZ^{-1}\right)$$
$$\overset{(6)}{=} \omega^{-\frac{a(a-1)}{2}} \omega^{-a} X^{a+1}Z^{-(a+1)} = \omega^{-\frac{(a+1)a}{2}} X^{a+1}Z^{-(a+1)}.$$ □

Lemma B.5. *For all $a \in \mathbb{Z}_d$,*



(26)

Proof. By Lemma 2.25, $S^aZ\left(S^\dagger\right)^a = Z$. For the actions of S^a on X, we proceed by induction on a.

Base Case: When $a = 1$, the statement holds by Lemma 2.25.

Induction Hypothesis (IH): Suppose the statement holds for $a \geq 1$. That is,

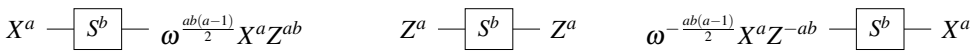
$$S^aX\left(S^\dagger\right)^a = XZ^a, \qquad \left(S^\dagger\right)^a XS^a = XZ^{-a}. \tag{27}$$

Induction Step: We want to show that the statement holds for $a+1$.

$$S^{a+1}X\left(S^\dagger\right)^{a+1} = S\left(S^aX\left(S^\dagger\right)^a\right)S^\dagger \overset{\text{IH}}{=} S\left(XZ^a\right)S^\dagger = \left(SXS^\dagger\right)\left(SZ^aS^\dagger\right) \overset{(9)}{=} XZZ^a = XZ^{a+1}$$
$$\left(S^\dagger\right)^{a+1} XS^{a+1} = S^\dagger\left(\left(S^\dagger\right)^a XS^a\right)S \overset{\text{IH}}{=} S^\dagger\left(XZ^{-a}\right)S = \left(S^\dagger XS\right)\left(S^\dagger Z^{-a}S\right) \overset{(9)}{=} XZ^{-1}Z^{-a} = XZ^{-(a+1)}.$$

□

Lemma B.6. *For all $a,b \in \mathbb{Z}_d$,*



(28)

Proof. By Lemma 2.25, $S^b Z^a \left(S^\dagger\right)^b = Z^a$. For the actions of S^b on X^a, we proceed by induction on b.

Base Case: When $b = 1$, the statement holds by Lemma B.4.

Induction Hypothesis (IH): Suppose the statement holds for $b \geq 1$. That is,

$$S^b X^a \left(S^\dagger\right)^b = \omega^{\frac{ab(a-1)}{2}} X^a Z^{ab}, \qquad \left(S^\dagger\right)^b X^a S^b = \omega^{-\frac{ab(a-1)}{2}} X^a Z^{-ab}. \tag{29}$$

Induction Step: We want to show that the statement holds for $b+1$.

$$\begin{aligned} S^{b+1} X^a \left(S^\dagger\right)^{b+1} &= S\left(S^b X^a \left(S^\dagger\right)^b\right) S^\dagger \overset{\mathsf{IH}}{=\!=} S\left(\omega^{\frac{ab(a-1)}{2}} X^a Z^{ab}\right) S^\dagger = \omega^{\frac{ab(a-1)}{2}} \left(S X^a S^\dagger\right)\left(S Z^{ab} S^\dagger\right) \\ &\overset{(24)}{=\!=} \omega^{\frac{ab(a-1)}{2}} \left(\omega^{\frac{a(a-1)}{2}} X^a Z^a\right) Z^{ab} = \omega^{\frac{a(b+1)(a-1)}{2}} X^a Z^{a(b+1)}. \\ \left(S^\dagger\right)^{b+1} X^a S^{b+1} &= S^\dagger \left(\left(S^\dagger\right)^b X^a S^b\right) S \overset{\mathsf{IH}}{=\!=} S^\dagger \left(\omega^{-\frac{ab(a-1)}{2}} X^a Z^{-ab}\right) S = \omega^{-\frac{ab(a-1)}{2}} \left(S^\dagger X^a S\right)\left(S^\dagger Z^{-ab} S\right) \\ &\overset{(24)}{=\!=} \omega^{-\frac{ab(a-1)}{2}} \left(\omega^{-\frac{a(a-1)}{2}} X^a Z^{-a}\right) Z^{-ab} = \omega^{-\frac{a(b+1)(a-1)}{2}} X^a Z^{-a(b+1)}. \end{aligned}$$

□

Lemma B.7. *For all $a,b \in \mathbb{Z}_d$,*

$$\left(XZ^{-a}\right)^b = \omega^{-\frac{ab(b-1)}{2}} X^b Z^{-ab}. \tag{30}$$

Proof. We proceed by induction on b.

Base Case: When $b = 1$, the statement holds trivially.

Induction Hypothesis (IH): Suppose the statement holds for $b \geq 1$. That is,

$$\left(XZ^{-a}\right)^b = \omega^{-\frac{ab(b-1)}{2}} X^b Z^{-ab}. \tag{31}$$

Induction Step: We want to show that the statement holds for $b+1$.

$$\begin{aligned} \left(XZ^{-a}\right)^{b+1} &= \left(XZ^{-a}\right)^b \left(XZ^{-a}\right) \overset{\mathsf{IH}}{=\!=} \left(\omega^{-\frac{ab(b-1)}{2}} X^b Z^{-ab}\right)\left(XZ^{-a}\right) \\ &\overset{(6)}{=\!=} \omega^{-\frac{ab(b-1)}{2}} \omega^{-ab} X^{b+1} Z^{-ab} Z^{-a} = \omega^{-\frac{a(b+1)b}{2}} X^{b+1} Z^{-a(b+1)}. \end{aligned}$$

□

Lemma B.8. *Let p be an odd prime and $\omega = e^{2\pi i/p}$.*

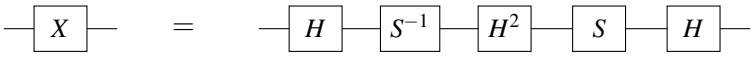
$$-1 = (-\omega)^p \tag{D1}$$
$$\omega = (-\omega)^{p+1} \tag{D2}$$

Proof.

$$(D1).\mathsf{RHS} = (-\omega)^p = -\omega^p \overset{(5)}{=\!=} -1 = (D1).\mathsf{LHS}.$$
$$(D2).\mathsf{RHS} = (-\omega)^{p+1} = (-\omega)^p(-\omega) \overset{(D1)}{=\!=} (-1)(-\omega) = \omega = (D2).\mathsf{LHS}.$$

□

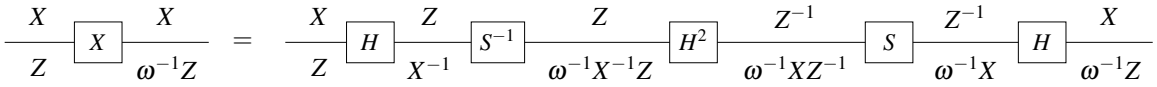
Lemma B.9.



(D4)

Proof. By (6), it suffices to check that RHS has the following action on X and Z.

Based on (9) and Lemmas B.1, B.2 and B.4 to B.6, we have



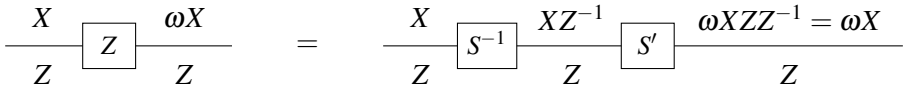
□

Lemma B.10.

(D5)

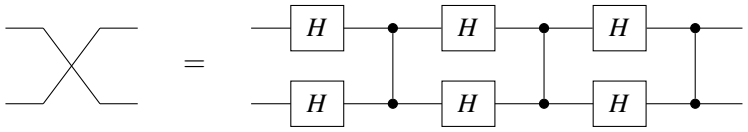
Proof. By (6), it suffices to check that RHS has the following action on X and Z.

Based on (9) and Lemma B.5, we have



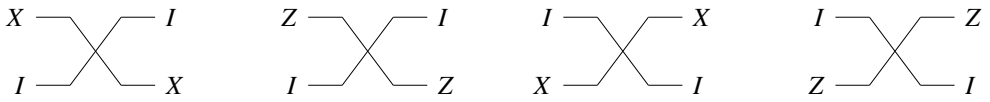
□

Lemma B.11.



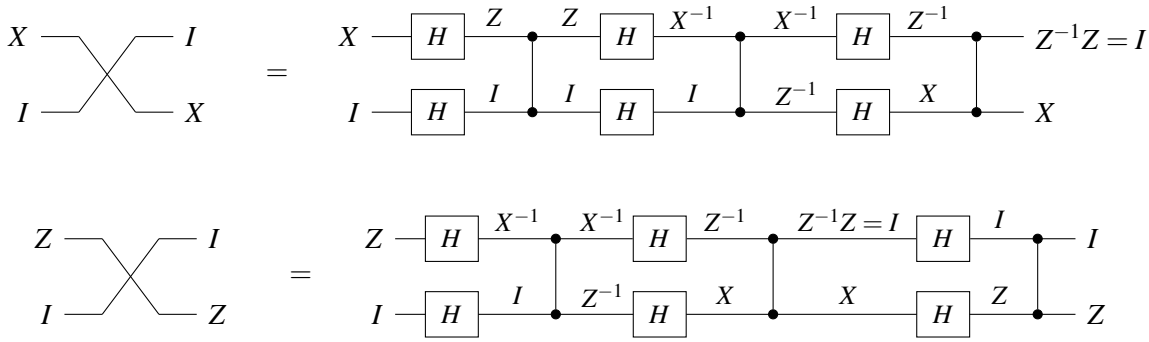
(*D*6)

Proof. We need to show that the RHS of (*D*6) acts on the Pauli generators as specified by (32).



(32)

Up to symmetry, it suffices to show the action of RHS on $X \otimes I$ and $Z \otimes I$. By (9) and (10), we can show that



□

C Actions of the Normal Boxes

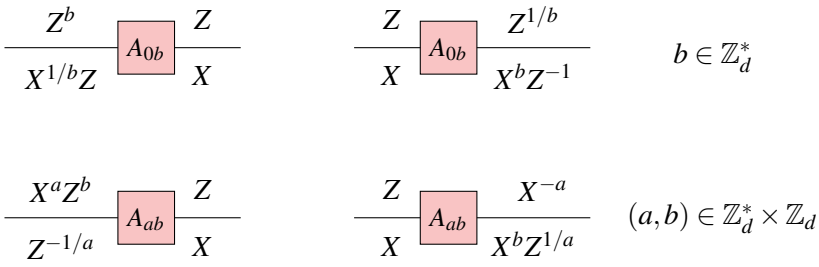


Figure 15: Up to a global phase, the action of A boxes on the Pauli generators.

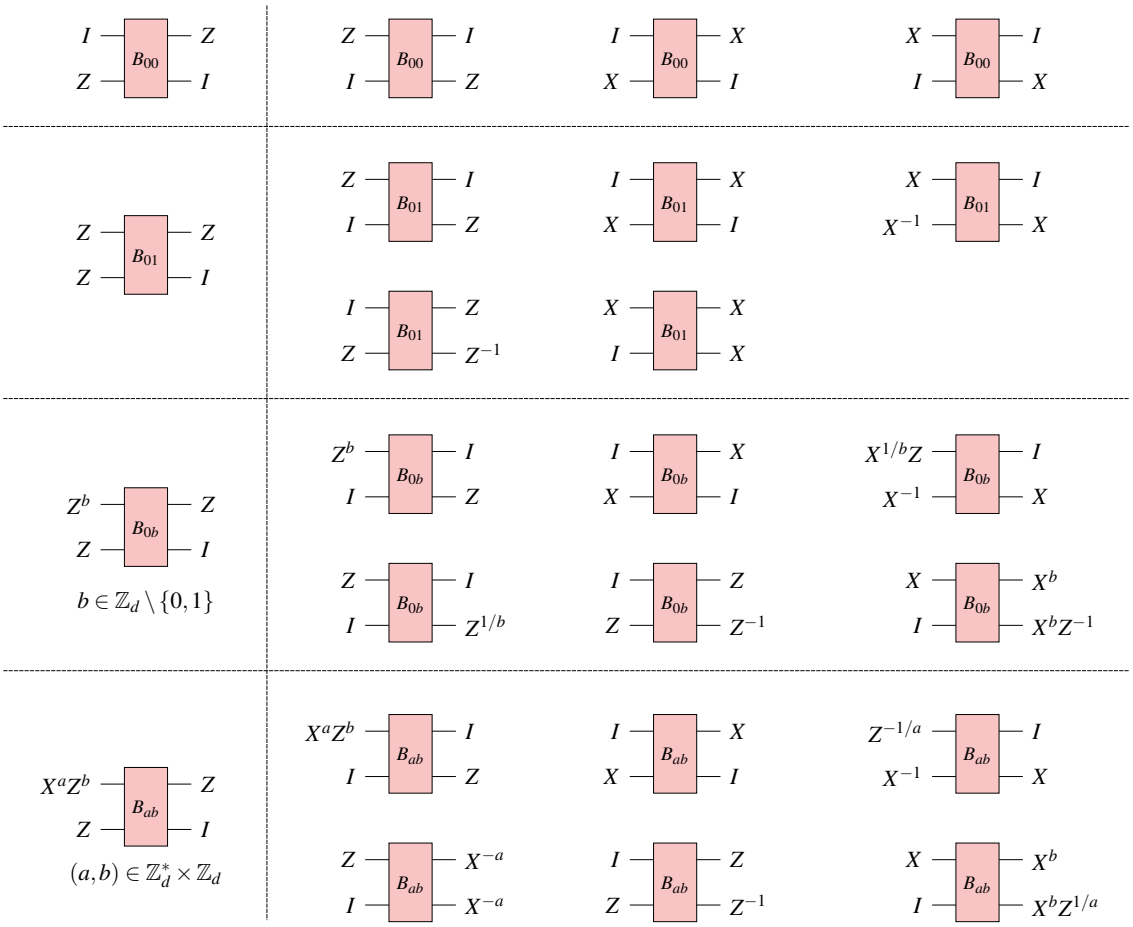


Figure 16: Up to a global phase, the action of B boxes on the Pauli generators.

D Push Through a Normal Box

Recall that we use $\mathcal{C}_n$ to denote the set of n-qudit Clifford circuits, where p is an odd prime, and we use $\mathcal{B}_n$ to denote the set of n-qudit Pauli operators which generate the n-qudit Pauli group $\mathcal{P}_n$.

Proposition D.1. *Let $k_1, k_2 \in \mathbb{N}$, and $C \in \mathcal{C}_{k_1+k_2}$. Suppose that for every $B \in \mathcal{B}_{k_1}$ there exists a $B' \in \mathcal{P}_{k_1}$ such that*

$$C(B \otimes I)C^\dagger = B' \otimes I. \tag{33}$$

Then there exist $C_1 \in \mathcal{C}_{k_1}$ and $C_2 \in \mathcal{C}_{k_2}$ such that $C = C_1 \otimes C_2$, up to scalar. Here, I is the identity map on $\mathbb{C}^{d^{k_2}}$.

Proof. By Equation (33), the mapping $\Phi : \mathcal{B}_{k_1} \to \mathcal{M}_{d^{k_1}}(\mathbb{C})$ sending $B \mapsto \mathrm{tr}_2\left(C(B \otimes I)C^\dagger\right)$ extends to an automorphism Φ of $\mathcal{P}_{k_1}$ such that $C(B \otimes I)C^\dagger = \Phi(B) \otimes I$, where tr_2 is the normalized partical

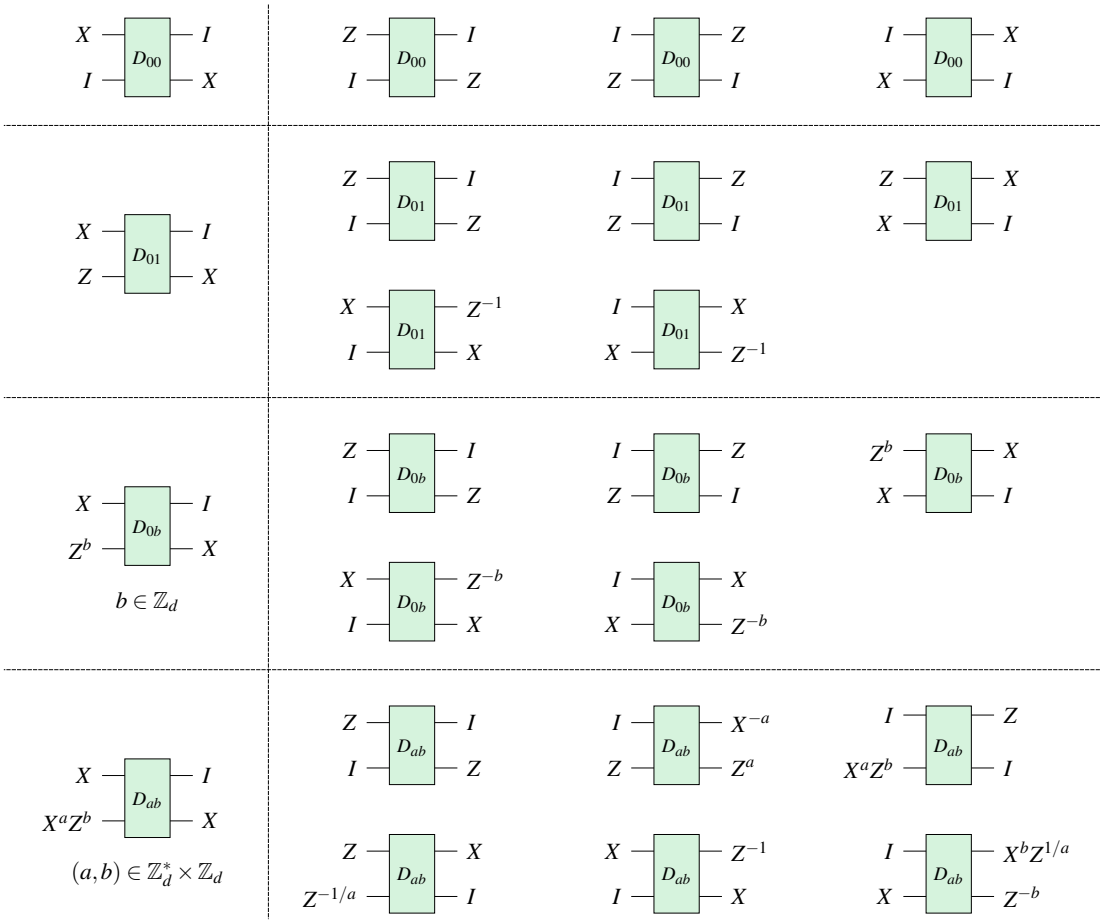


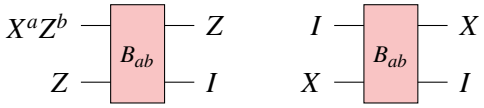
Figure 17: Up to a global phase, the action of D boxes on the Pauli generators.

Figure 18: The action of E boxes on the Pauli generators.

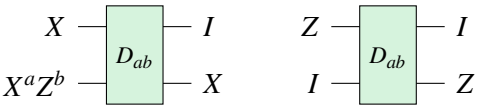
trace $\frac{1}{d^{k_2}} id \otimes \mathrm{Tr}$ that traces out the second system $\mathcal{M}_{d^{k_2}}(\mathbb{C})$. Hence, there exists a $C_1 \in \mathcal{C}_{k_1}$ such that $C_1 B C_1^\dagger = \Phi(B)$. The rest follows from [35, Lemma E.6]. □

Firstly, note that (34) and (35) follow from Figures 16 and 17. Based on this, we can derive

proof-of-principle box relations as shown in Figure 19.

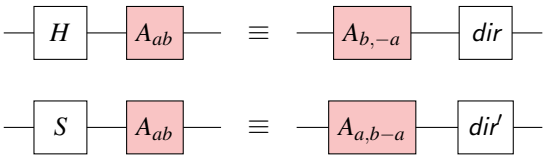


(34)



(35)

Lemma D.2. *For all* $(a,b) \in \mathbb{Z}_d \times \mathbb{Z}_d \setminus \{(0,0)\}$, *there exist* dir, dir′ $\in \mathfrak{C}_1$ *such that*

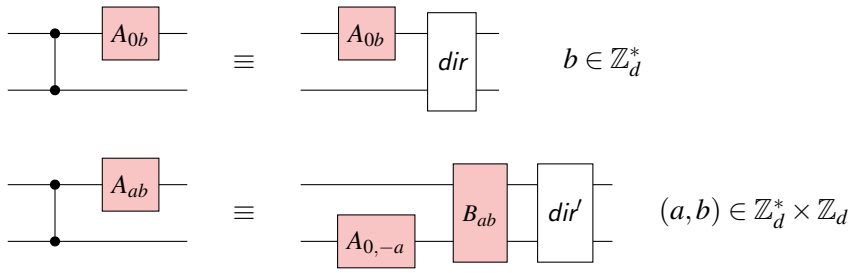


(36)

Proof. Computing the preimage of Z in the LHS of (36) yields

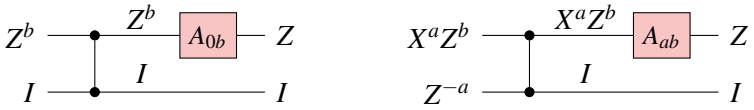
By the uniqueness of the symplectic normal form, X^bZ^{-a} and X^aZ^{b-a} imply that the updated normal boxes should be those in the RHS of (36). □

Lemma D.3. *For all* $(a,b) \in \mathbb{Z}_d \times \mathbb{Z}_d \setminus \{(0,0)\}$, *there exist* dir, dir′ $\in \mathfrak{C}_2$ *such that*



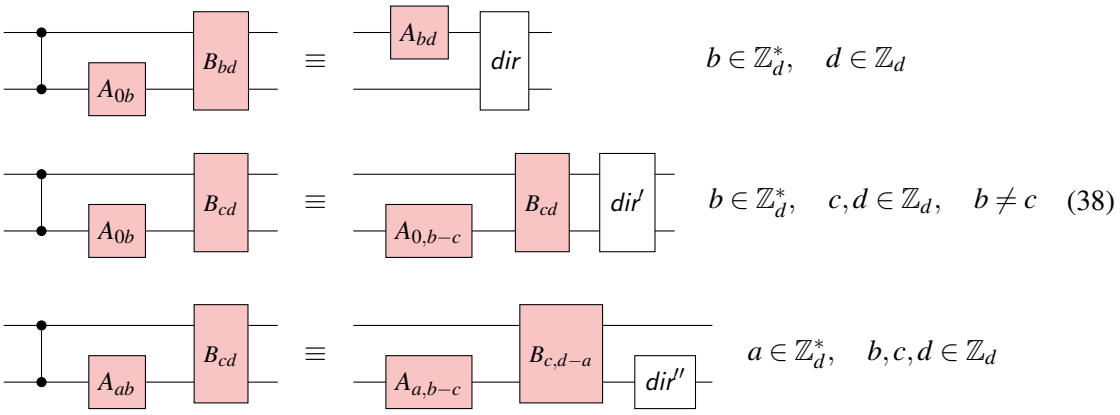
(37)

Proof. Computing the preimage of $Z \otimes I$ in the LHS of (37) yields

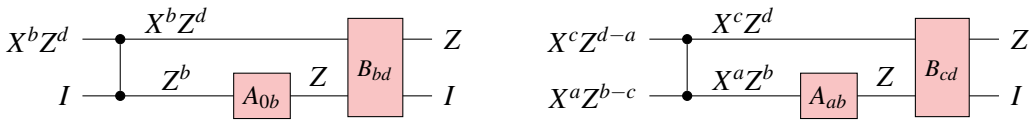


By the uniqueness of the symplectic normal form, $Z^b \otimes I$ and $X^aZ^b \otimes Z^{-a}$ imply that the updated normal boxes should be those in the RHS of (37). □

Lemma D.4. *For all $(a,b) \in \mathbb{Z}_d \times \mathbb{Z}_d \setminus \{(0,0)\}$ and $(c,d) \in \mathbb{Z}_d \times \mathbb{Z}_d$, there exist dir, dir$'$ $\in \mathfrak{C}_2$ and dir$''$ $\in \mathfrak{C}_1$ such that*

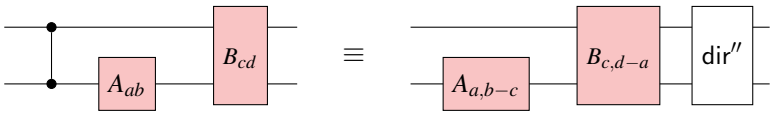


Proof. Computing the preimage of $Z \otimes I$ in the LHS of (38) yields

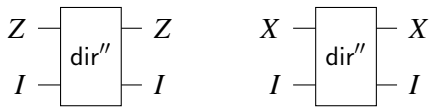


By the uniqueness of the symplectic normal form, $X^bZ^d \otimes I$ and $X^cZ^{d-a} \otimes X^aZ^{b-c}$ imply that the updated normal boxes should be those in the RHS of (38).

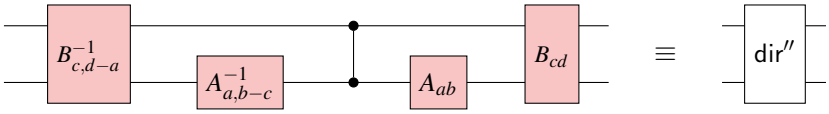
Next, we consider the case when $a \neq 0$. Suppose that for some dir$'' \in \mathfrak{C}_2$,



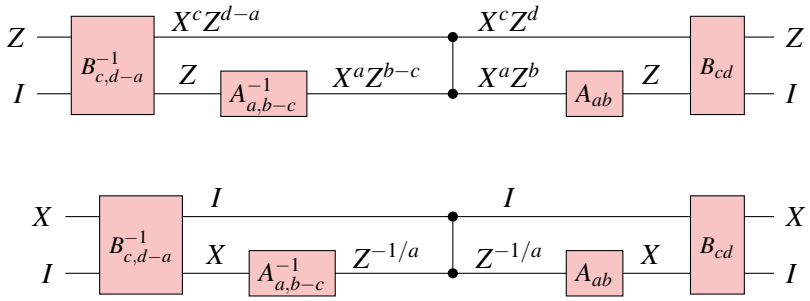
By Proposition D.1, it remains to show that



Equivalently, we can compute the preimage of $Z \otimes I$ and $X \otimes I$ in the LHS of (39).

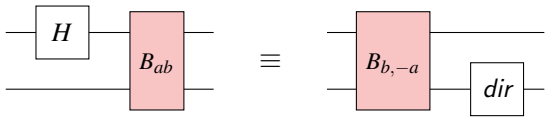


By (34), we have



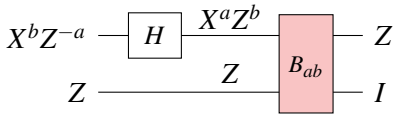
This completes the proof. □

Lemma D.5. *For all $a,b \in \mathbb{Z}_d$, there exists $dir \in \mathfrak{C}_1$ such that*

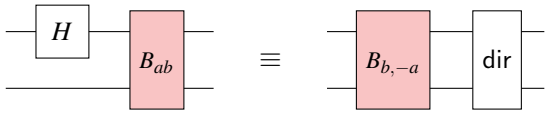


(40)

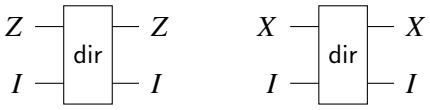
Proof. Computing the preimage of $Z \otimes I$ in the LHS of (40) yields



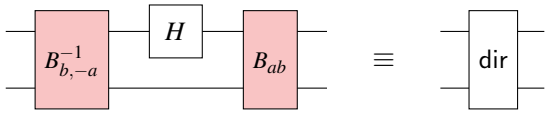
By the uniqueness of the symplectic normal form, $X^bZ^{-a} \otimes Z$ implies that the updated normal box should be the one in the RHS of (40). Suppose that for some $\mathsf{dir} \in \mathfrak{C}_2$,



By Proposition D.1, it remains to show that

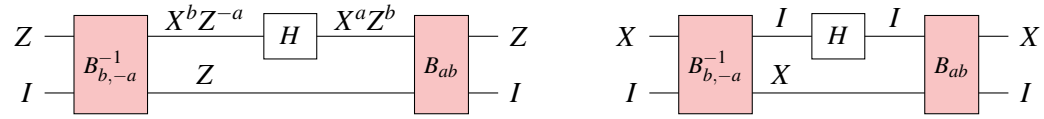


Equivalently, we can compute the preimage of $Z \otimes I$ and $X \otimes I$ in the LHS of (41).



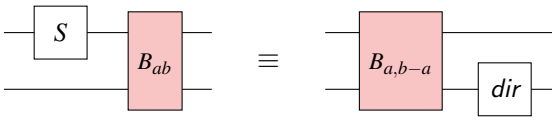
(41)

By (34), we have



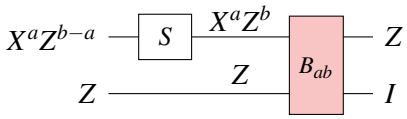
This completes the proof. □

Lemma D.6. *For all $a,b \in \mathbb{Z}_d$, there exists $\mathsf{dir} \in \mathfrak{C}_1$ such that*

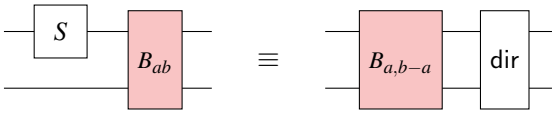


(42)

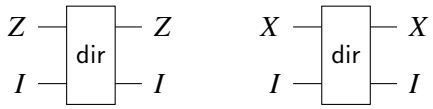
Proof. Computing the preimage of $Z \otimes I$ in the LHS of (42) yields



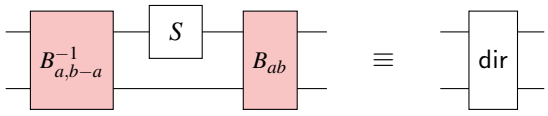
By the uniqueness of the symplectic normal form, $X^aZ^{b-a} \otimes Z$ implies that the updated normal box should be the one in the RHS of (42). Suppose that for some $\mathsf{dir} \in \mathfrak{C}_2$,



By Proposition D.1, it remains to show that

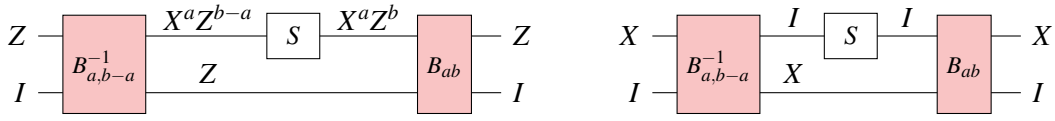


Equivalently, we can compute the preimage of $Z \otimes I$ and $X \otimes I$ in the LHS of (43).



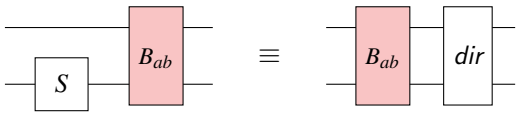
(43)

By (34), we have



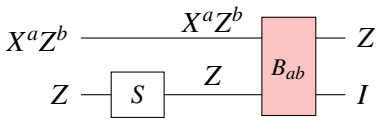
This completes the proof. □

Lemma D.7. *For all $a, b \in \mathbb{Z}_d$, there exists $dir \in \mathfrak{C}_2$ such that*



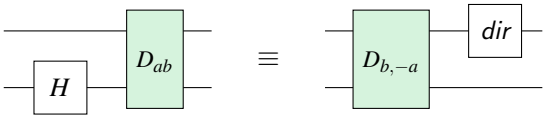
(44)

Proof. Computing the preimage of $Z \otimes I$ in the LHS of (44) yields



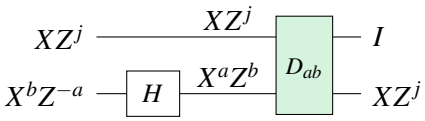
By the uniqueness of the symplectic normal form, $X^aZ^b \otimes Z$ implies that the updated normal box should be the one in the RHS of (44). □

Lemma D.8. *For all $a, b \in \mathbb{Z}_d$, there exists $dir \in \mathfrak{C}_1$ such that*

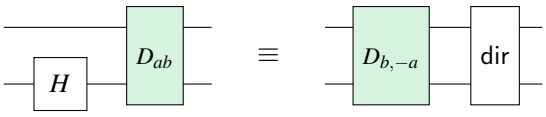


(45)

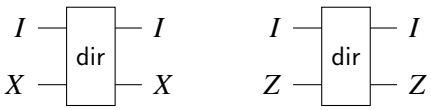
Proof. For any $j \in \mathbb{Z}_d$, computing the preimage of $I \otimes XZ^j$ in the LHS of (45) yields



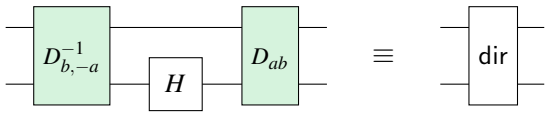
By the uniqueness of the symplectic normal form, $XZ^j \otimes X^bZ^{-a}$ implies that the updated normal box should be the one in the RHS of (45). Suppose that for some $\mathsf{dir} \in \mathfrak{C}_2$,



By Proposition D.1, it remains to show that

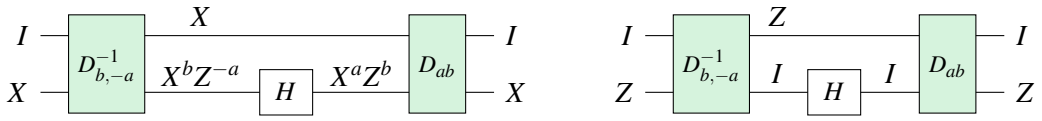


Equivalently, we can compute the preimage of $I \otimes X$ and $I \otimes Z$ in the LHS of (46).



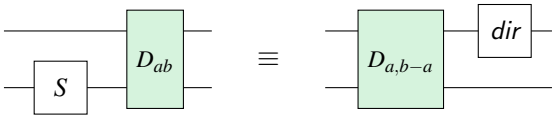
(46)

By (35), we have



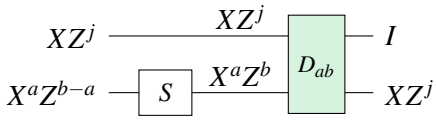
This completes the proof. □

Lemma D.9. *For all $a,b \in \mathbb{Z}_d$, there exists $\mathsf{dir} \in \mathfrak{C}_1$ such that*

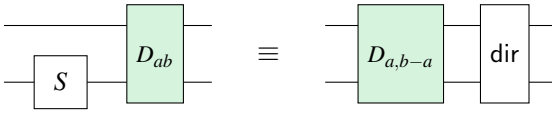


(47)

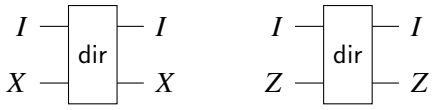
Proof. For any $j \in \mathbb{Z}_d$, computing the preimage of $I \otimes XZ^j$ in the LHS of (47) yields



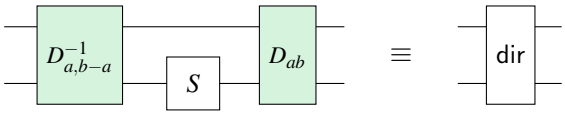
By the uniqueness of the symplectic normal form, $XZ^j \otimes X^aZ^{b-a}$ implies that the updated normal box should be the one in the RHS of (47). Suppose that for some $\mathsf{dir} \in \mathfrak{C}_2$,



By Proposition D.1, it remains to show that

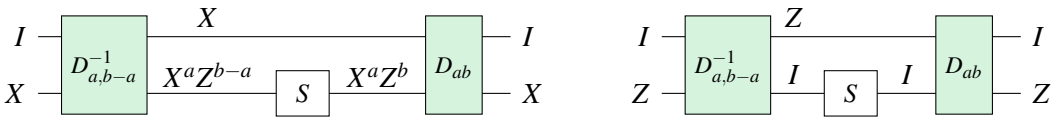


Equivalently, we can compute the preimage of $I \otimes X$ and $I \otimes Z$ in the LHS of (48).



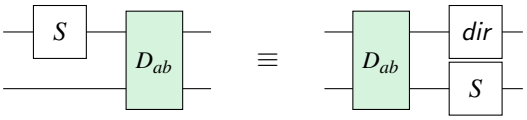
(48)

By (35), we have



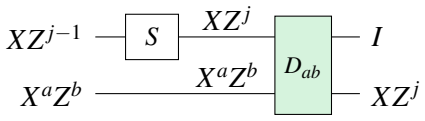
This completes the proof. □

Lemma D.10. *For all $a,b \in \mathbb{Z}_d$, there exists $\mathsf{dir} \in \mathfrak{C}_1$ such that*

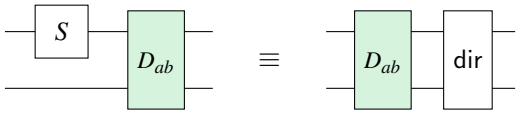


(49)

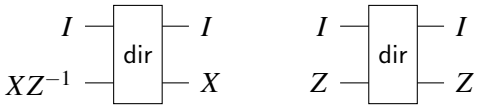
Proof. For any $j \in \mathbb{Z}_d$, computing the preimage of $I \otimes XZ^j$ in the LHS of (49) yields



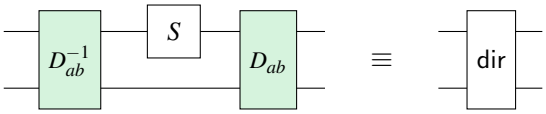
By the uniqueness of the symplectic normal form, $XZ^{j-1} \otimes X^aZ^b$ implies that the updated normal box should be the one in the RHS of (49). Suppose that for some $\mathsf{dir} \in \mathfrak{C}_2$,



By Proposition D.1, it remains to show that

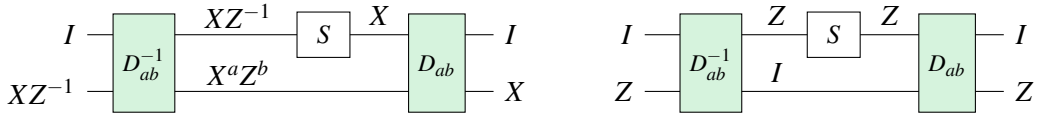


Equivalently, we can compute the preimage of $I \otimes X$ and $I \otimes Z$ in the LHS of (50).



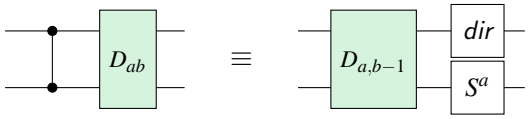
(50)

By (35), we have



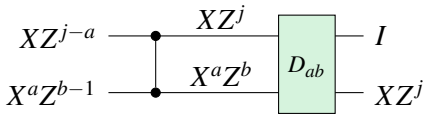
Since S sends XZ^{-1} to X and Z to Z, this completes the proof. □

Lemma D.11. *For all $a,b \in \mathbb{Z}_d$, there exists $dir \in \mathfrak{C}_1$ such that*

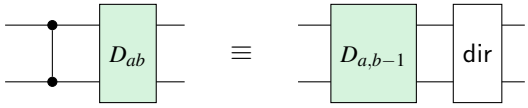


(51)

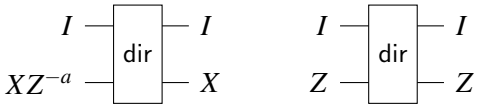
Proof. For any $j \in \mathbb{Z}_d$, computing the preimage of $I \otimes XZ^j$ in the LHS of (51) yields



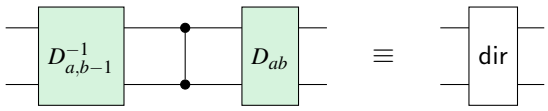
By the uniqueness of the symplectic normal form, $XZ^{j-a} \otimes X^aZ^{b-1}$ implies that the updated normal box should be the one in the RHS of (51). Suppose that for some $\mathsf{dir} \in \mathfrak{C}_2$,



By Proposition D.1, it remains to show that

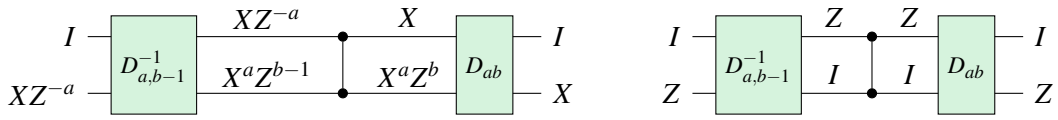


Equivalently, we can compute the preimage of $I \otimes X$ and $I \otimes Z$ in the LHS of (52).



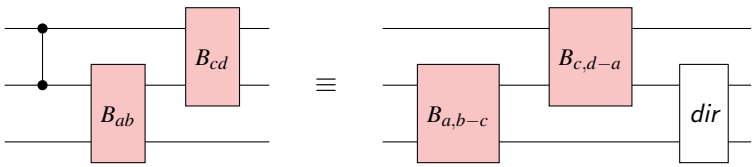
(52)

By (35), we have



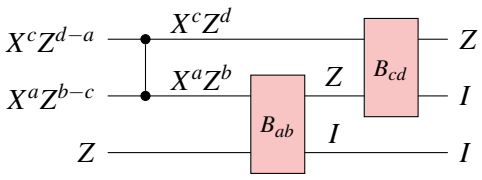
Since S^a sends XZ^{-a} to X and Z to Z, this completes the proof. □

Lemma D.12. *For all $a,b \in \mathbb{Z}_d$, there exists $dir \in \mathfrak{C}_2$ such that*

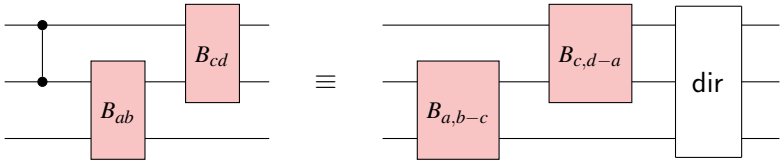


(53)

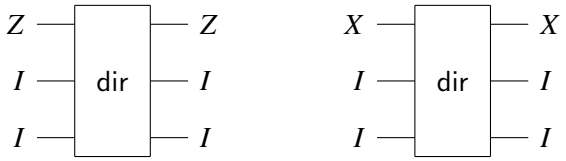
Proof. Firstly, find preimage of $Z \otimes I \otimes I$ in the LHS of (53).



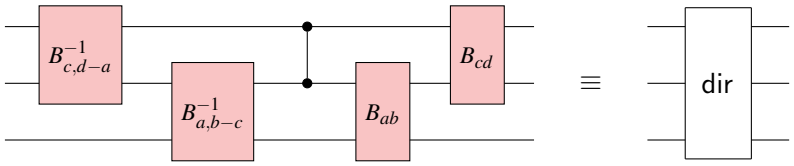
By the uniqueness of the symplectic normal form, $X^cZ^{d-a} \otimes X^aZ^{b-c} \otimes Z$ implies that the updated normal boxes should be the ones in the RHS of (53). Suppose that for some $\mathsf{dir} \in \mathfrak{C}_3$,



By Proposition D.1, it remains to show that



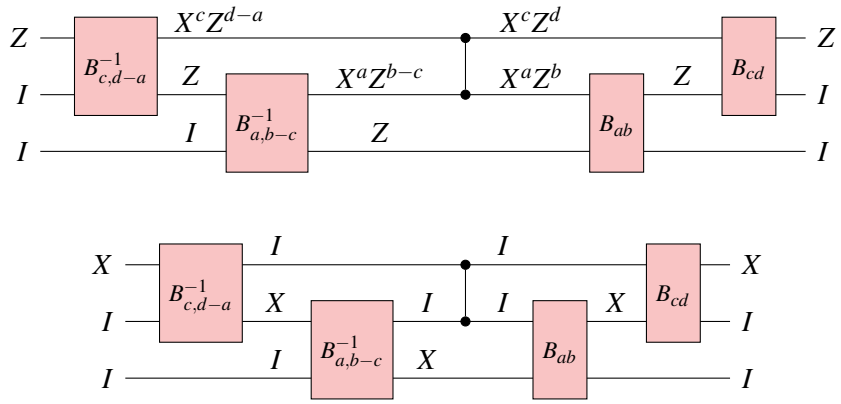
Equivalently, we can compute the preimage of $Z \otimes I \otimes I$ and $X \otimes I \otimes I$ in the LHS of (54).



(54)

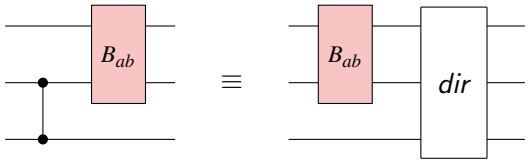
According to Figure 16,

we have



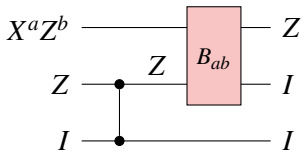
This completes the proof. □

Lemma D.13. *For all $a,b \in \mathbb{Z}_d$, there exists $dir \in \mathfrak{C}_3$ such that*



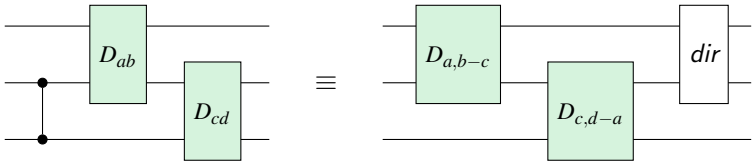
(55)

Proof. Computing the preimage of $Z \otimes I \otimes I$ in the LHS of (55) yields



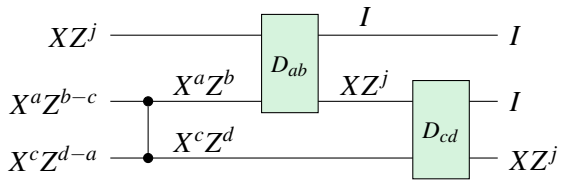
By the uniqueness of the symplectic normal form, $X^aZ^b \otimes Z \otimes I$ implies that the updated normal box should be the one in the RHS of (55). □

Lemma D.14. *For all $a,b \in \mathbb{Z}_d$, there exists $dir \in \mathfrak{C}_2$ such that*

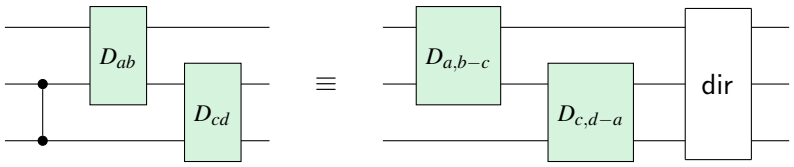


(56)

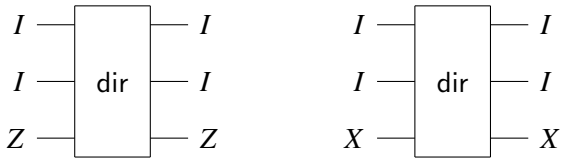
Proof. For any $j \in \mathbb{Z}_d$, computing the preimage of $I \otimes I \otimes XZ^j$ in the LHS of (56) yields



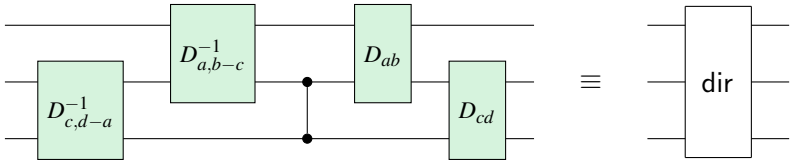
By the uniqueness of the symplectic normal form, $XZ^j \otimes X^aZ^{b-c} \otimes X^cZ^{d-a}$ implies that the updated normal boxes should be the ones in the RHS of (56). Suppose that for some $\mathsf{dir} \in \mathfrak{C}_3$,



By Proposition D.1, it remains to show that



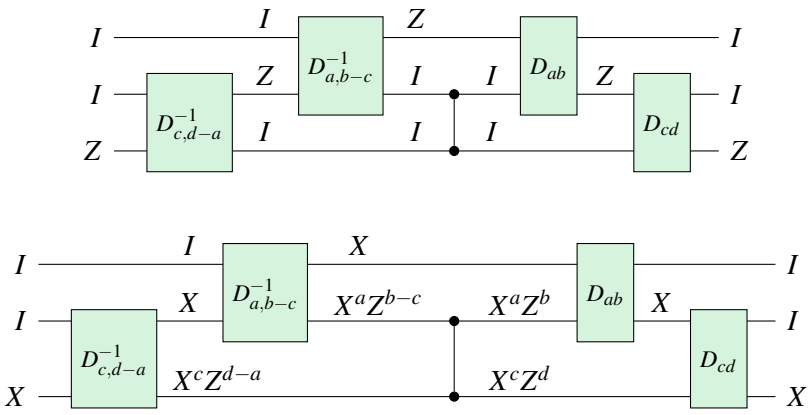
Equivalently, we can compute the preimage of $I \otimes I \otimes Z$ and $I \otimes I \otimes X$ in the LHS of (57).



(57)

According to Figure 17,

we have



This completes the proof. □

E Proofs from Section 3.1

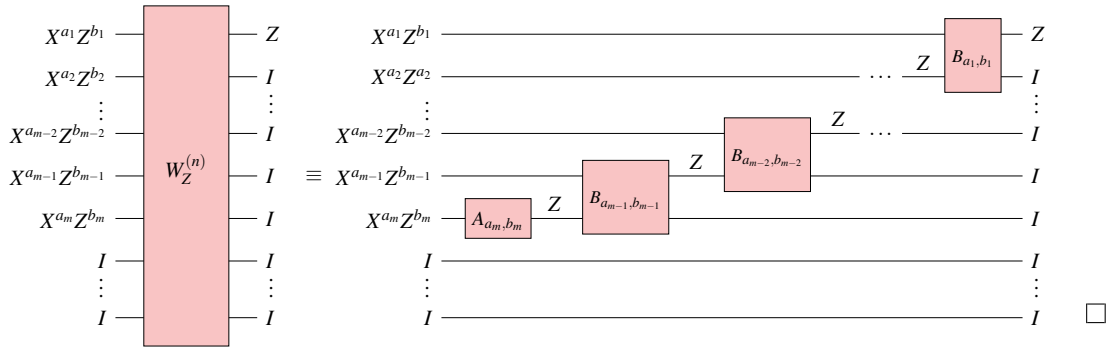
Lemma 3.5. *For every $P \in \mathcal{P}_n \setminus \{\omega^t I;\ t \in \mathbb{Z}_d\}$ be a non-scalar n-qudit Pauli operator, there exists a unique Z-normal circuit W_Z such that $[\![W_Z]\!]_u \bullet P \equiv_p Z \otimes I \otimes \cdots \otimes I$.*

Proof. There are unique $a_j, b_j \in \mathbb{Z}_d, 1 \leq j \leq n$ such that $P \equiv_p \bigotimes_{j=1}^n X^{a_j} Z^{b_j}$. Since P is not a scalar, there exists j such that $X^{a_j} Z^{b_j}$ is not an identity. Let m be the largest such index, then we can write

$$P \equiv_p \left(X^{a_1} Z^{b_1}\right) \otimes \cdots \otimes \left(X^{a_m} Z^{b_m}\right) \otimes I \otimes \cdots \otimes I. \tag{58}$$

Next, we show how to construct a unique Z-normal circuit W_Z based on (58) and the unique normal box actions specified in the second column of Figure 5. Starting from the qudit wire m, A_{a_m,b_m} is uniquely determined by $X^{a_m} Z^{b_m}$. As a result, the unitary implementation of this A box maps $X^{a_m} Z^{b_m}$ to Z up to a global phase[1]. Consequently, $B_{a_{m-1},b_{m-1}}$ is uniquely determined by $X^{a_{m-1}} Z^{b_{m-1}} \otimes Z$. Then, this B box maps $X^{a_{m-1}} Z^{b_{m-1}} \otimes Z$ to $Z \otimes I$. Continue this process by concatenating B normal boxes upward, until reaching the top two qudit wires, where B_{a_1,b_1} is uniquely determined by $X^{a_1} Z^{b_1} \otimes Z$. As a result, $X^{a_1} Z^{b_1} \otimes Z$ is mapped to $Z \otimes I$. This gives us a Z-normal circuit W_Z that is displayed below, which is uniquely determined by P. Since all the Pauli propagations are tracked up to global phases, we conclude that $[\![W_Z]\!]_u \bullet P \equiv_p Z \otimes I \otimes \cdots \otimes I$.

[1] Recall that the Pauli propagation is tracked up to a phase



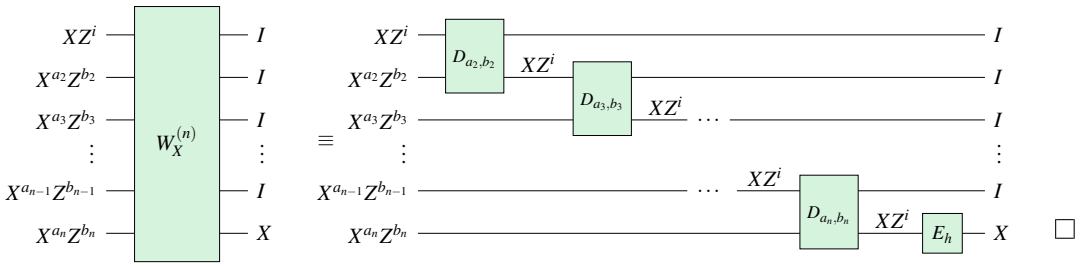
Given a Z-normal circuit W_Z, we can read off its preimage of $Z \otimes I \otimes \cdots \otimes I$ by reading the indices of each normal box in W_Z from right to left.

Lemma 3.6. *For every $Q \in \mathcal{P}_n$ such that $(Z \otimes I \otimes \cdots \otimes I)Q = \omega Q(Z \otimes I \otimes \cdots \otimes I)$, there exists a unique X-normal circuit W_X such that $[\![W_X]\!]_u \bullet Q \equiv_p I \otimes \cdots \otimes I \otimes X$.*

Proof. There are unique $a_j, b_j \in \mathbb{Z}_d, 1 \leq j \leq n$ such that $Q \equiv_p \bigotimes_{j=1}^n X^{a_j} Z^{b_j}$. Since $(Z \otimes I \otimes \cdots \otimes I)Q = \omega Q(Z \otimes I \otimes \cdots \otimes I)$, we must have

$$Q \equiv_p \left(XZ^i\right) \otimes \left(X^{a_2} Z^{b_2}\right) \otimes \cdots \otimes \left(X^{a_n} Z^{b_n}\right). \tag{59}$$

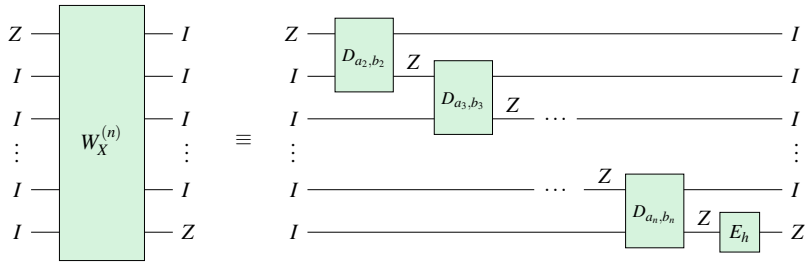
Next, we show how to construct a unique X-normal circuit W_X based on (59) and the unique normal box actions specified in the second column of Figure 5. Starting from qudit wires 1 and 2, D_{a_2,b_2} is uniquely determined by $(XZ^i) \otimes (X^{a_2} Z^{b_2})$. As a result, this D box maps $(XZ^i) \otimes (X^{a_2} Z^{b_2})$ to $I \otimes (XZ^i)$. Consequently, D_{a_3,b_3} is uniquely determined by $(XZ^i) \otimes (X^{a_3} Z^{b_3})$. Again, this D box maps $(XZ^i) \otimes (X^{a_3} Z^{b_3})$ to $I \otimes (XZ^i)$. Continue this process by concatenating D normal boxes downward, until reaching the bottom two qudit wires, where D_{a_n,b_n} is uniquely determined by $(XZ^i) \otimes (X^{a_n} Z^{b_n})$. Thus, $(XZ^i) \otimes (X^{a_n} Z^{b_n})$ is mapped to $I \otimes (XZ^i)$ by this last D box. Then, E_i is uniquely determined by XZ^i, and it maps XZ^i to X. This gives us an X-normal circuit W_X that is displayed below. It is uniquely determined by Q and $[\![W_X]\!]_u \bullet Q \equiv_p I \otimes \cdots \otimes I \otimes X$.



Given an X-normal circuit W_X, we can read off its preimage of $I \otimes \cdots \otimes I \otimes X$ by reading the indices of each normal box in W_X from right to left.

Lemma 3.7. *Every X-normal circuit W_X satisfies $[\![W_X]\!]_u \bullet (Z \otimes I \otimes \cdots \otimes I \otimes I) \equiv_p I \otimes I \otimes \cdots \otimes I \otimes Z$.*

Proof. This follows from the additional actions of D and E boxes on certain Pauli operators (see the third column of Figure 5). In particular, every D box maps $Z \otimes I$ to $I \otimes Z$, every E box maps Z to Z. It is visualized below.



□

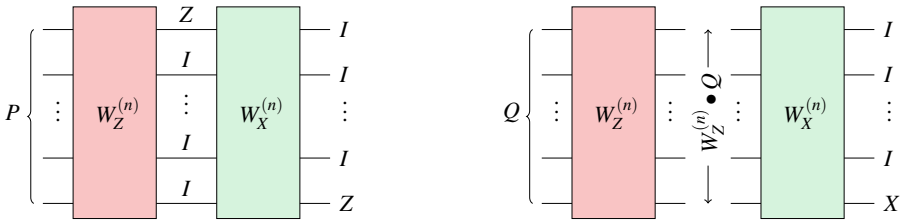
Lemma 3.8. *Let $P, Q \in \mathcal{P}_n$ such that $PQ = \omega QP$. Then there exist unique normal circuits W_Z and W_X such that $[\![W_X W_Z]\!]_u \bullet P \equiv_p I \otimes \cdots \otimes I \otimes Z$ and $[\![W_X W_Z]\!]_u \bullet Q \equiv_p I \otimes \cdots \otimes I \otimes X$.*

Proof. Since $PQ = \omega QP$, we must have $P \notin \langle \omega \rangle$. By Lemma 3.5, there exists a unique Z-normal circuit W_Z such that $[\![W_Z]\!]_u \bullet P \equiv_p Z \otimes I \otimes \cdots \otimes I$. Moreover, $([\![W_Z]\!]_u \bullet P)([\![W_Z]\!]_u \bullet Q) = [\![W_Z]\!]_u \bullet (PQ) = [\![W_Z]\!]_u \bullet (\omega QP) = \omega([\![W_Z]\!]_u \bullet Q)([\![W_Z]\!]_u \bullet P)$. Hence $(Z \otimes I \otimes \cdots \otimes I)([\![W_Z]\!]_u \bullet Q) = \omega([\![W_Z]\!]_u \bullet Q)(Z \otimes I \otimes \cdots \otimes I)$. By Lemma 3.6, there exists a unique X-normal circuit W_X such that $[\![W_X W_Z]\!]_u \bullet Q = [\![W_X]\!]_u \bullet ([\![W_Z]\!]_u \bullet Q) \equiv_p I \otimes \cdots \otimes I \otimes X$.

By Lemma 3.7, $[\![W_X W_Z]\!]_u \bullet P = [\![W_X]\!]_u \bullet ([\![W_Z]\!]_u \bullet P) \equiv_p [\![W_X]\!]_u \bullet (Z \otimes I \otimes \cdots \otimes I) \equiv_p I \otimes \cdots \otimes I \otimes Z$. It follows that

$$[\![W_X W_Z]\!]_u \bullet P \equiv_p I \otimes \cdots \otimes I \otimes Z, \qquad [\![W_X W_Z]\!]_u \bullet Q \equiv_p I \otimes \cdots \otimes I \otimes X. \tag{60}$$

Diagrammatically, we can express this process:



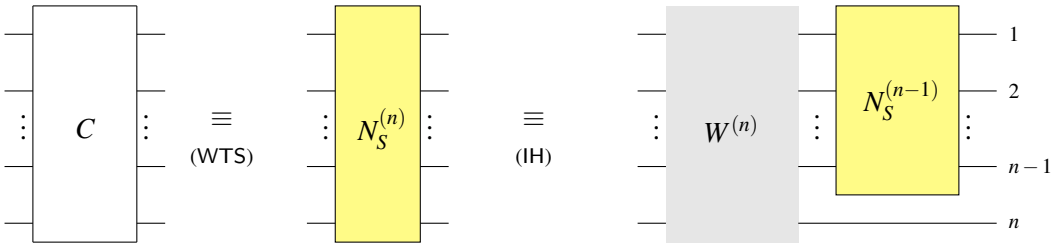
This proves the existence of the desired Z-normal circuit W_Z and X-normal circuit W_X. Now suppose towards contradiction that there exist another Z-normal circuit W_Z' and X-normal circuit W_X' satisfying (60). Since $[\![W_X']\!]_u \bullet ([\![W_Z']\!]_u \bullet P) \equiv_p I \otimes \cdots \otimes I \otimes Z$, Lemma 3.7 implies that $[\![W_X']\!]_u \bullet P \equiv_p Z \otimes I \cdots \otimes I$. By the uniqueness of the Z-normal circuit in Lemma 3.5, $W_Z \equiv W_Z'$. Then

$$[\![W_X']\!]_u \bullet ([\![W_Z']\!]_u \bullet Q) \equiv_p [\![W_X']\!]_u \bullet ([\![W_Z]\!]_u \bullet Q) \equiv_p I \otimes \cdots \otimes I \otimes X.$$

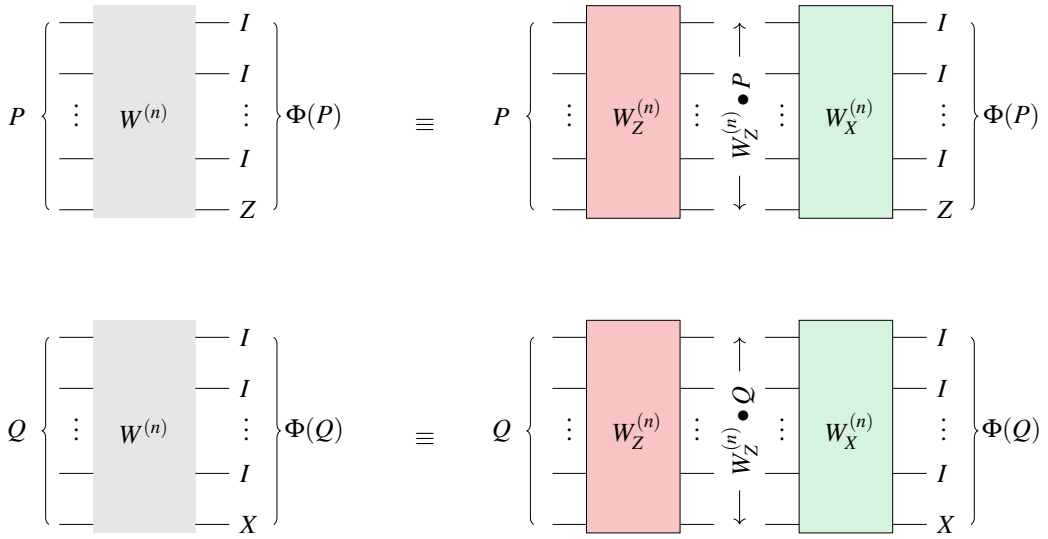
By the uniqueness of the X-normal circuit in Lemma 3.6, $W_X' \equiv W_X$. □

Proposition 3.9. *Let $\phi : \mathcal{P}_n \to \mathcal{P}_n$ be an automorphism of the Pauli group such that ϕ fixes scalars. There exists a unique Clifford circuit $C \in \mathfrak{C}_n$ in symplectic normal form such that for all $P \in \mathcal{P}_n$, $[\![C]\!]_u \bullet P \equiv_p \phi(P)$.*

Proof. We proceed by induction on n. When $n = 0$, the Pauli operators are scalars of the form ω^t, $t \in \mathbb{Z}_d$. In this case, ϕ is the identity. So C must be an empty wire without any gates. Now suppose that our claim is true for $n-1$ and consider the case of n. First we prove that the symplectic normal circuit $W^{(n)}$ on the n-th layer exists.



Since ϕ is bijective, there exist $P, Q \in \mathcal{P}_n$ such that $\phi(P) = I \otimes \cdots \otimes I \otimes Z$ and $\phi(Q) = I \otimes \cdots \otimes I \otimes X$. That is, $P = \phi^{-1}(I \otimes \cdots \otimes I \otimes Z)$ and $Q = \phi^{-1}(I \otimes \cdots \otimes I \otimes X)$. Then $PQ = \phi^{-1}(I \otimes \ldots \otimes I \otimes ZX)$. Since $I \otimes \ldots \otimes I \otimes Z$ ω-anticommutes with $I \otimes \ldots \otimes I \otimes X$, $PQ = \omega QP$. By Lemma 3.8, there exist a unique X-circuit W_X and a unique Z-circuit W_Z such that $[\![W_X W_Z]\!]_u \bullet P \equiv_p I \otimes \ldots \otimes I \otimes Z = \phi(P)$ and $[\![W_X W_Z]\!]_u \bullet Q \equiv_p I \otimes \ldots \otimes I \otimes X = \phi(Q)$.



Let $\phi' : \mathcal{P}_n \to \mathcal{P}_n$ be a new automorphism and ϕ' fixes scalars such that

$$\phi'(U) = \phi\left([\![W_X W_Z]\!]_u^{-1} \bullet U\right). \tag{61}$$

Then $I\otimes\cdots\otimes I\otimes Z$ and $I\otimes\cdots\otimes I\otimes X$ are fixed points of ϕ', since for $P\in\{X,Z\}$,

$$\begin{aligned}\phi'(I\otimes\cdots\otimes I\otimes P) &= \phi\left(\llbracket W_X W_Z\rrbracket_u^{-1}\bullet(I\otimes\cdots\otimes I\otimes P)\right)\\ &= \phi\left(\llbracket W_X W_Z\rrbracket_u^{-1}\bullet\llbracket W_X W_Z\rrbracket_u\bullet P\right)\equiv_p \phi(P)=I\otimes\cdots\otimes I\otimes P.\end{aligned}$$

For any $R\in\mathcal{P}_{n-1}$, since $R\otimes I$ commutes with $I\otimes\cdots\otimes I\otimes Z$ and $I\otimes\cdots\otimes I\otimes X$, $\phi'(R\otimes I)$ commutes with $\phi'(I\otimes\cdots\otimes I\otimes Z)=I\otimes\cdots\otimes I\otimes Z$ and $\phi'(I\otimes\cdots\otimes I\otimes X)=I\otimes\cdots\otimes I\otimes X$. This implies that $\phi'(R\otimes I)=S\otimes I$ for some $S\in\mathcal{P}_{n-1}$. Then there exists an automorphism $\phi'':\mathcal{P}_{n-1}\to\mathcal{P}_{n-1}$ such that $\phi''(R)=S$. Since ϕ' fixes $I\otimes\cdots\otimes I\otimes Z$ and $I\otimes\cdots\otimes I\otimes X$,

$$\phi'=\phi''\otimes I. \tag{62}$$

By the induction hypothesis, for all $R\in\mathcal{P}_{n-1}$, there exists $C'\in\mathfrak{C}_{n-1}$ in normal form such that

$$\llbracket C'\rrbracket_u\bullet R\equiv_p \phi''(R), \tag{63}$$

Then $C=(C'\otimes I)W_X W_Z$ is in normal form. Next, we show that $\llbracket C\rrbracket_u\bullet U\equiv_p\phi(U)$ for all $U\in\mathcal{P}_n$. By (61),

$$\llbracket W_X W_Z\rrbracket_u^{-1}\bullet U=\phi^{-1}\phi'(U). \tag{64}$$

It follows that

$$\begin{aligned}\llbracket C\rrbracket_u\bullet U=\llbracket(C'\otimes I)W_X W_Z\rrbracket_u\bullet U &\overset{(64)}{=\!=} \llbracket C'\otimes I\rrbracket_u\bullet\left(\phi'^{-1}(\phi(U))\right)\\ &\overset{(62)}{=\!=} \llbracket C'\otimes I\rrbracket_u\bullet(\phi''^{-1}\otimes I)(\phi(U))\overset{(63)}{=}_p\phi(U).\end{aligned}$$

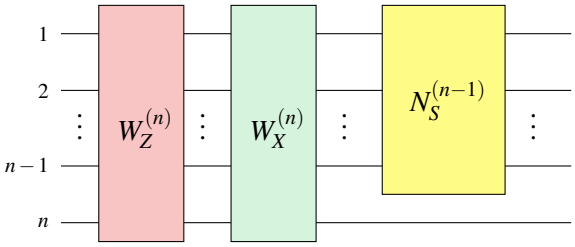
To prove uniqueness, suppose towards contradiction that $D\in\mathfrak{C}_n$ is another Clifford circuit in symplectic normal form such that $\llbracket D\rrbracket_u\bullet U\equiv_p\phi(U)$ for all $U\in\mathcal{P}_n$. By induction, $D=(D'\otimes I)(W_X'W_Z')$, where W_X' is an X-normal circuit, W_Z' is a Z-normal circuit, and D' is a symplectic normal Clifford circuit on $n-1$ qudits. Since $\llbracket D\rrbracket_u\bullet P\equiv_p\phi(P)=I\otimes\cdots\otimes I\otimes Z$, $(\llbracket D'\otimes I\rrbracket_u\llbracket W_X'W_Z'\rrbracket_u)\bullet P\equiv_p I\otimes\cdots\otimes I\otimes Z$. Then

$$\llbracket W_X'W_Z'\rrbracket_u\bullet P\equiv_p\llbracket D'\otimes I\rrbracket_u^{-1}\bullet(I\otimes\cdots\otimes I\otimes Z)=(\llbracket D'\rrbracket_u^{-1}\otimes I)\bullet(I\otimes\cdots\otimes I\otimes Z)=I\otimes\cdots\otimes I\otimes Z.$$

Similarly, $\llbracket D\rrbracket_u\bullet Q\equiv_p I\otimes\cdots\otimes I\otimes X$ implies that $\llbracket W_X'W_Z'\rrbracket_u\bullet Q\equiv_p I\otimes\cdots\otimes I\otimes X$. By the uniqueness of the X- and Z-normal circuits in Lemma 3.8, $W_X'\equiv W_X$ and $W_Z'\equiv W_Z$. By induction hypothesis, $C'\equiv D'$. Thus, the same is true for C and D. This completes the proof. □

Lemma 3.10. $|\mathrm{Sp}(2n,\mathbb{Z}_d)|=\prod_{k=1}^n\left(d^{2k}-1\right)\cdot\left(d^{2k-1}\right)$.

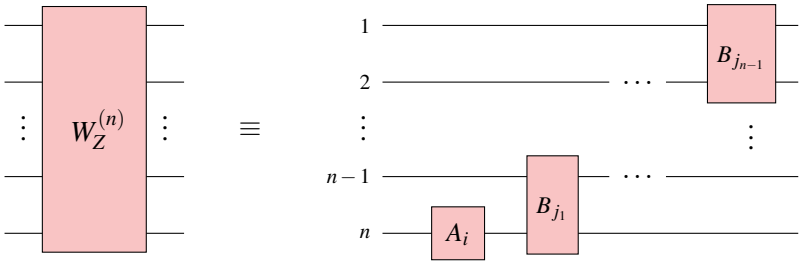
Proof. Since $\mathrm{Sp}(2n,\mathbb{Z}_d) \cong \mathcal{C}_n/\mathcal{P}_n$, by Lemma 4.6 and proposition 3.9, it is sufficient to count the total number of distinct symplectic normal forms. Recall the normal form of an arbitrary n-qudit Clifford operator up to a global base ω^t, $t \in \mathbb{Z}_d$.



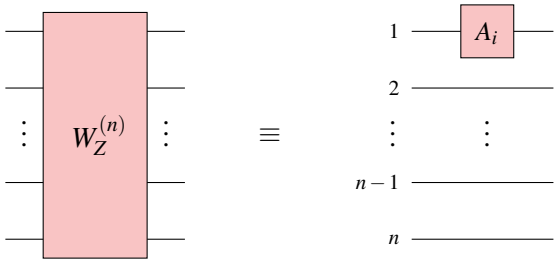
By induction, our problem is reduced to counting different ways of constructing $W_Z^{(n)}$ and $W_X^{(n)}$.

- For $W_Z^{(n)}$, according to Figure 6, there are d^2-1 choices for an A box. For B boxes, we can start concatenating them upward from any qudit wire. According to Figure 7a, we proceed by cases.

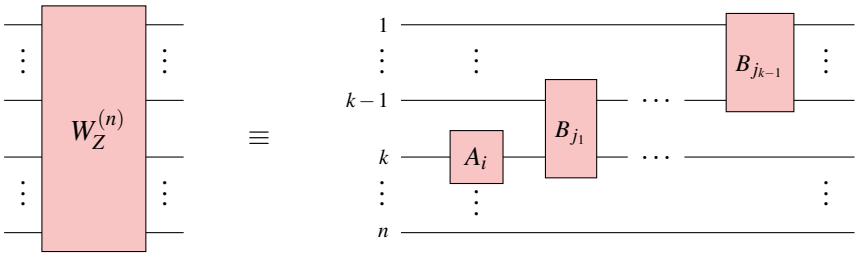
  In one extreme case: The ladder of B boxes starts from the bottom qudit wire, as shown below. There are $n-1$ B boxes in $W_Z^{(n)}$. According to Figure 6, there are d^2 choices for a B box. Since each choice of a normal box is independent of each other, there are $(d^2-1)\cdot d^{2(n-1)} = d^{2n} - d^{2(n-1)}$ distinct $W_Z^{(n)}$ when it expands over all n qudits.



  In the other extreme case: The ladder of B boxes starts from the top qudit wire, as shown below. There is no B box in $W_Z^{(n)}$. Since each choice of a normal box is independent of each other, there are d^2-1 distinct $W_Z^{(n)}$ when it expands over 1 qudit.



In all other remaining cases: The ladder of B boxes starts from qudit wire k, $1 < k < n$. In the illustration below, there are $(k-1)$ B boxes in $W_Z^{(n)}$. Reasoning analogously as before, there are $(d^2-1)\cdot d^{2(k-1)} = d^{2k} - d^{2(k-1)}$ distinct $W_Z^{(n)}$ when it expands over k qudits.



Considering all cases of concatenating B boxes in $W_Z^{(n)}$, the number of distinct $W_Z^{(n)}$ is

$$\sum_{k=1}^{n}\left(d^{2k}-d^{2(k-1)}\right) = \left(\sum_{k=1}^{n} d^{2k}\right) - \left(\sum_{k=1}^{n} d^{2(k-1)}\right) = \frac{\left(d^2-1\right)\left(1-d^{2n}\right)}{1-d^2} = d^{2n}-1.$$

- For $W_X^{(n)}$, we consider all possible ways of concatenating D and E boxes. According to Figure 6, there are p choices for E boxes respectively. For each D box, there are d^2 choices. According to Figure 7b, there are $n-1$ D boxes in $W_X^{(n)}$. Since each choice of a normal box is independent of each other, there are $p\cdot\left(d^2\right)^{(n-1)} = d^{2n-1}$ distinct $W_X^{(n)}$.

Since the choice of $W_Z^{(n)}$ and $W_X^{(n)}$ is independent of each other, there are $\left(d^{2n}-1\right)\cdot\left(d^{2n-1}\right)$ distinct $W_X^{(n)}W_Z^{(n)}$. According to the symplectic normal form outlined in Figure 8, we can calculate the total number of distinct $\widetilde{N^{(n)}}$:

$$\prod_{k=1}^{n}\left(d^{2k}-1\right)\cdot\left(d^{2k-1}\right).$$

Since they are in one-to-one correspondence with the elements in $\mathcal{C}_n/\mathcal{P}_n$, this completes the proof. □

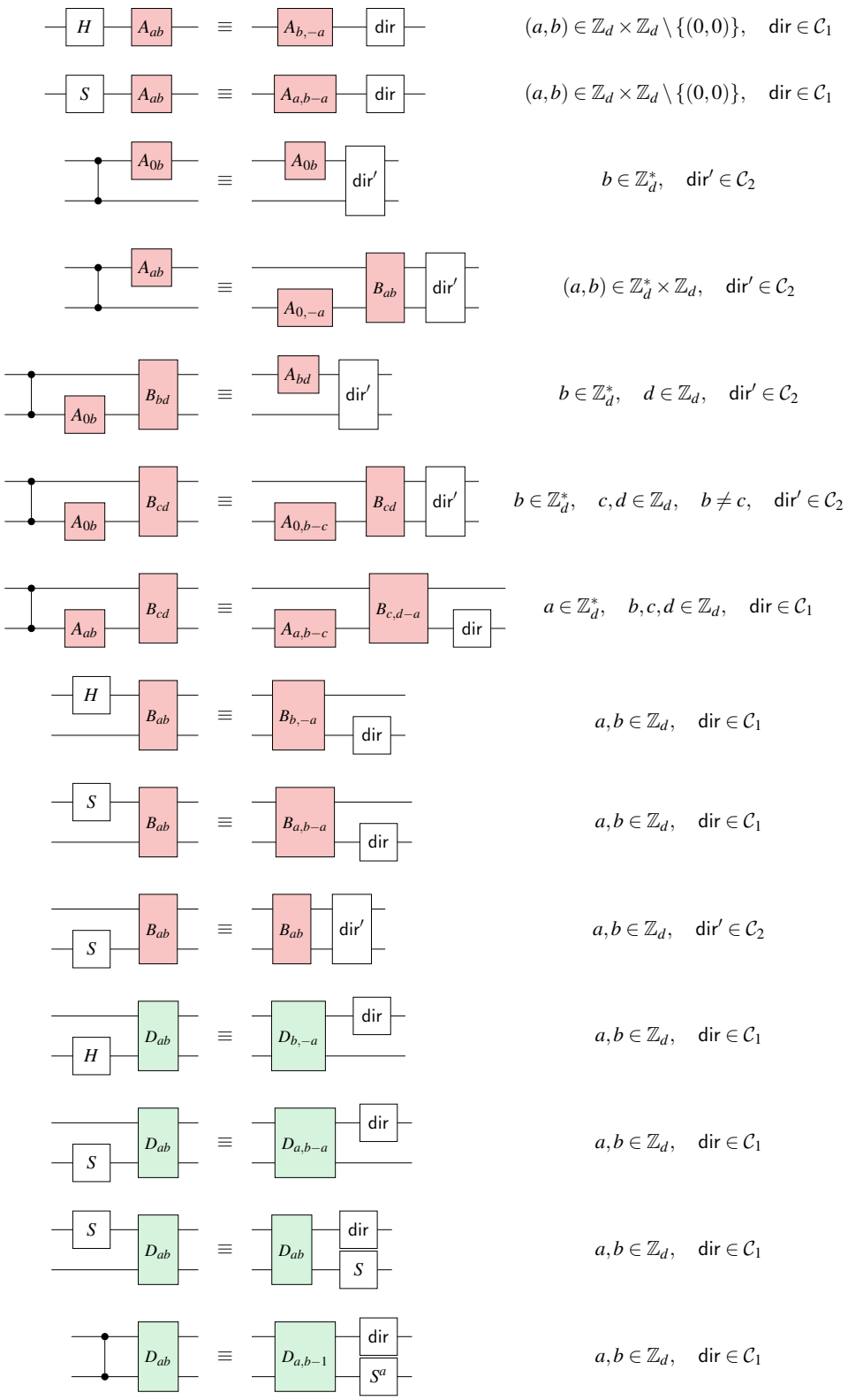


Figure 19: Proof-of-principle single- and two-qudit box relations. They demonstrate the structure of residual dirty gates in all scenarios when we normalise a two-qudit Clifford operator through "pushing-through-a-normal box".

F A Complete Set of Box Relations

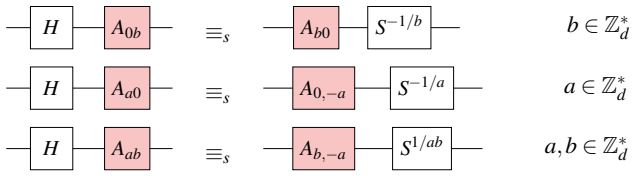


Figure 20: Pushing H through an A box, up to Pauli correction.

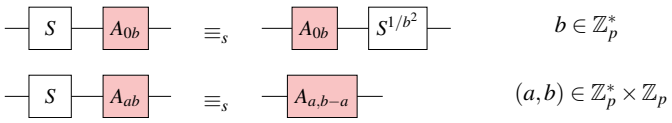


Figure 21: Pushing S through an A box, up to Pauli correction.

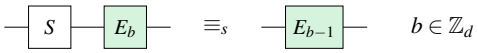


Figure 22: Pushing S through an E box.

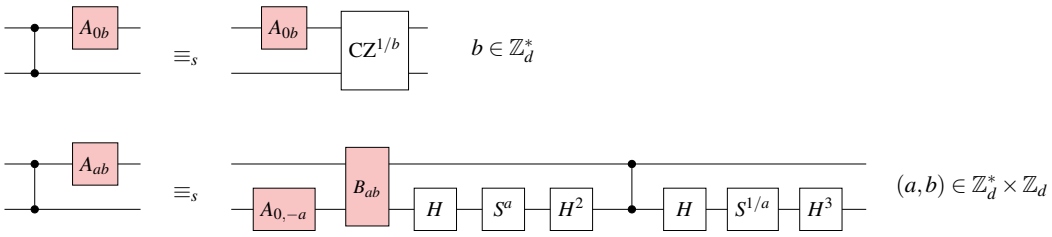


Figure 23: Pushing CZ through an A box, up to Pauli correction.

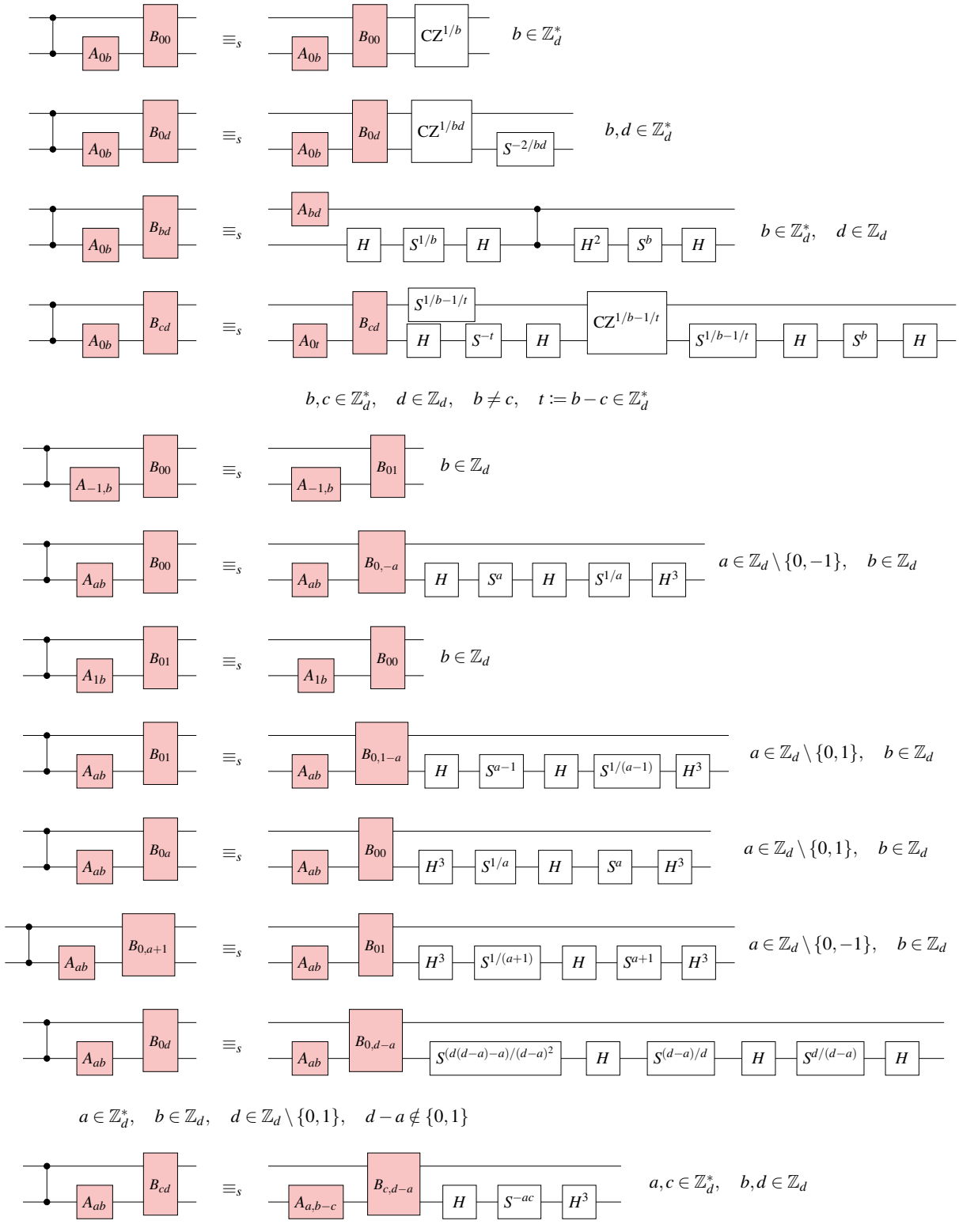


Figure 24: Pushing CZ through A and B boxes, up to Pauli correction.

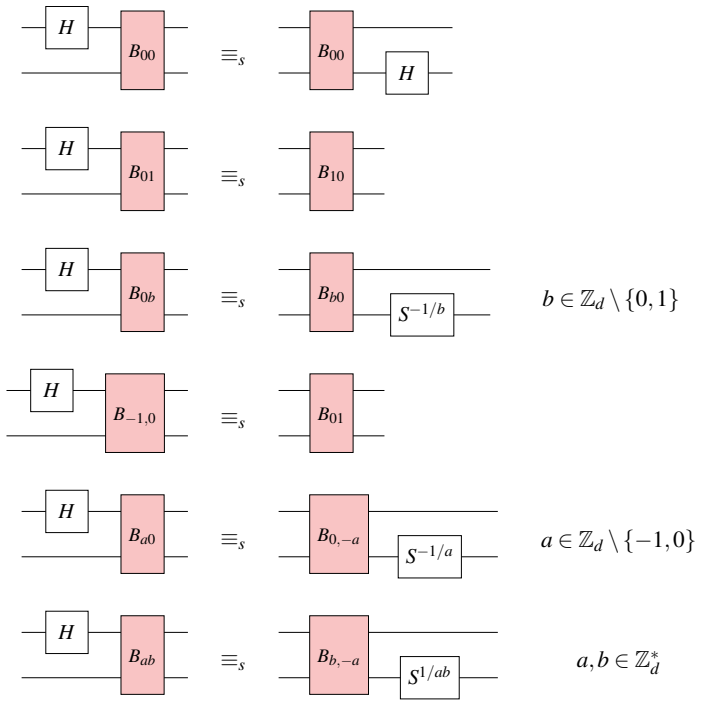


Figure 25: Pushing $H \otimes I$ through a B box, up to Pauli correction.

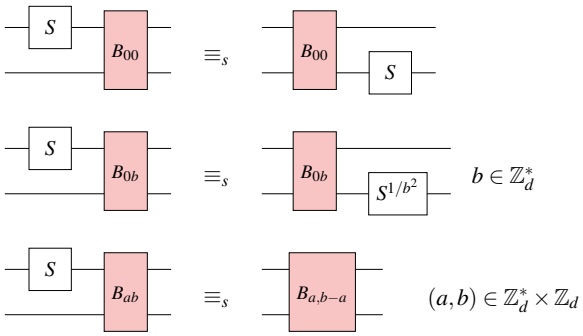


Figure 26: Pushing $S \otimes I$ through a B box, up to Pauli correction.

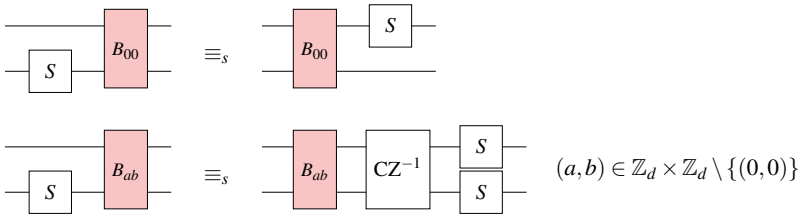


Figure 27: Pushing $I \otimes S$ through a B box, up to Pauli correction.

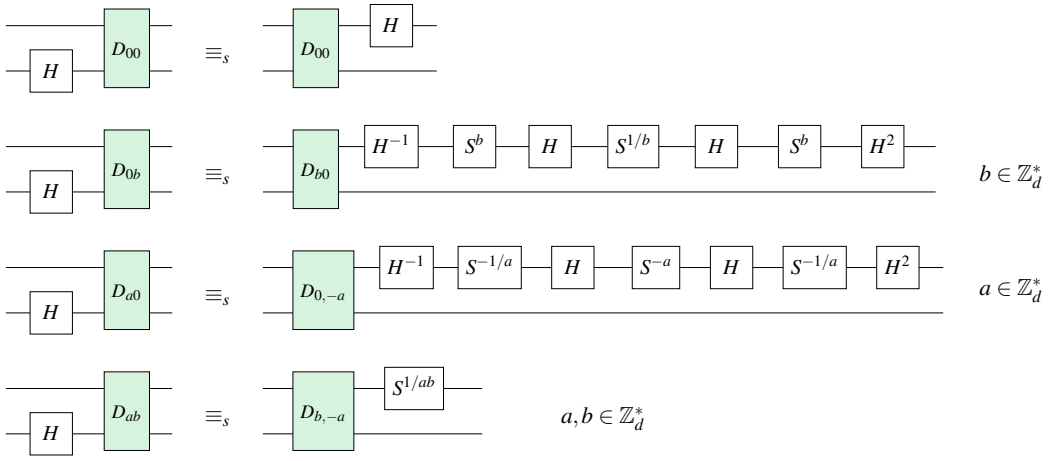


Figure 28: Pushing $I \otimes H$ through a D box, up to Pauli correction.

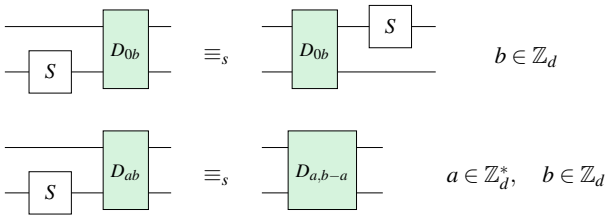


Figure 29: Pushing $I \otimes S$ through a D box, up to Pauli correction.

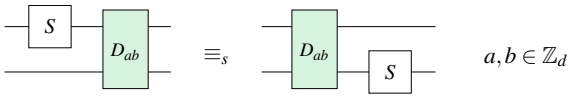


Figure 30: Pushing $S \otimes I$ through a D box, up to Pauli correction.

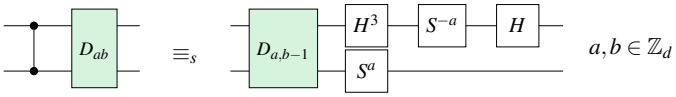


Figure 31: Pushing CZ through a D box, up to Pauli correction.

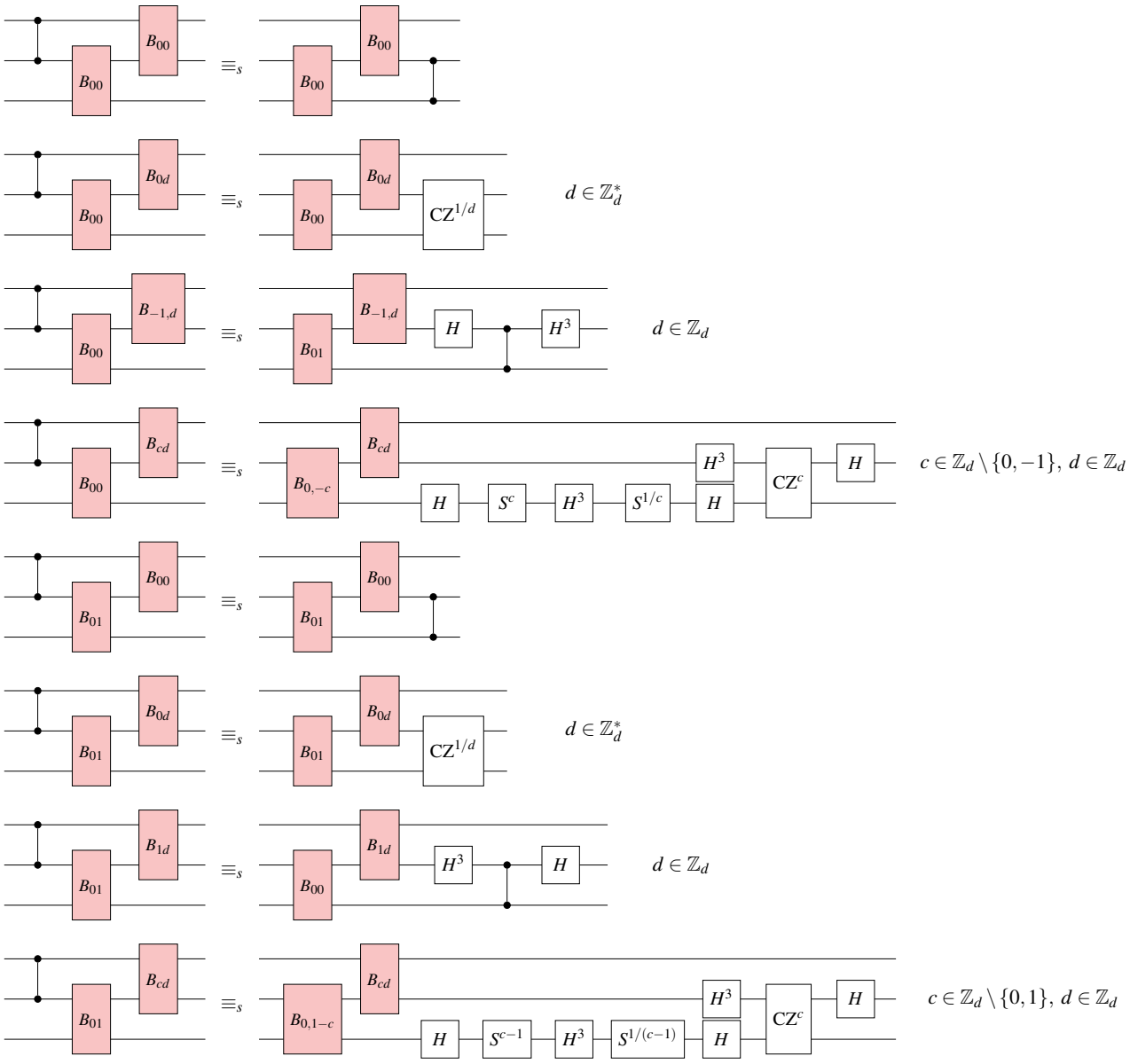


Figure 32: Pushing $\mathrm{CZ} \otimes I$ through two B boxes—B_{00} or B_{01} followed by another B box, up to Pauli correction.

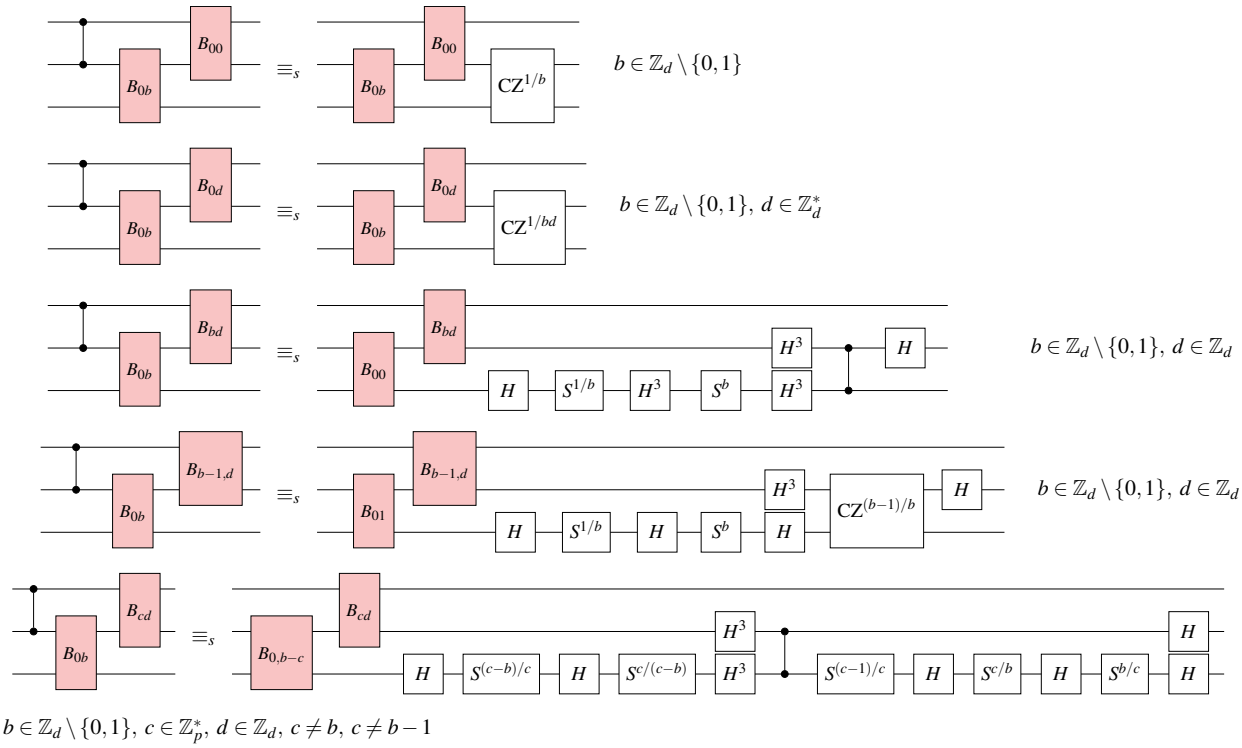


Figure 33: Pushing $\mathrm{CZ} \otimes I$ through two B boxes—B_{0b} followed by another B box, up to Pauli correction. Here, $b \in \mathbb{Z}_d \setminus \{0,1\}$.

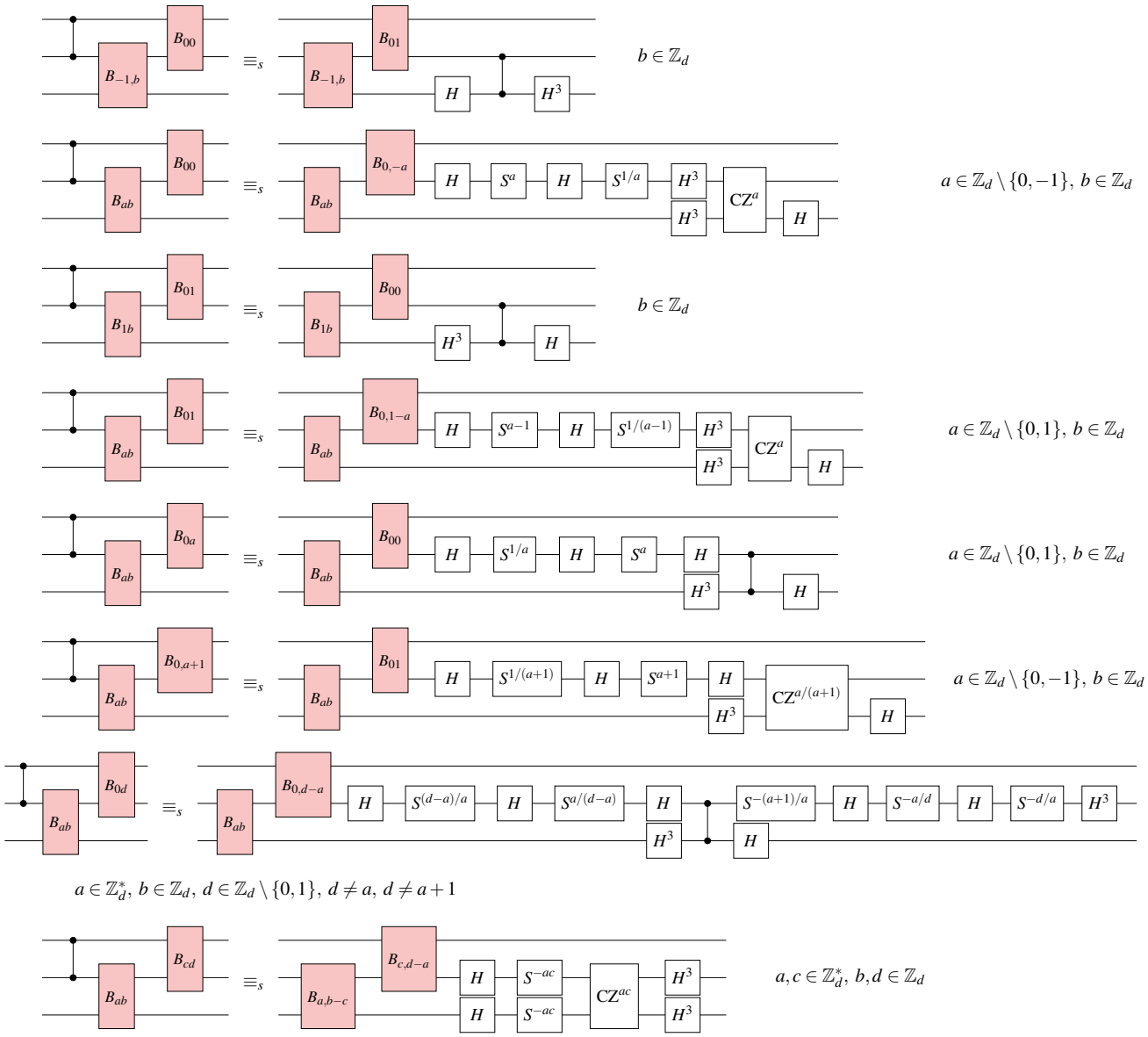


Figure 34: Pushing $\mathrm{CZ} \otimes I$ through two B boxes—B_{ab} followed by another B box, up to Pauli correction. Here, $a \in \mathbb{Z}_d^*$ and $b \in \mathbb{Z}_d$.

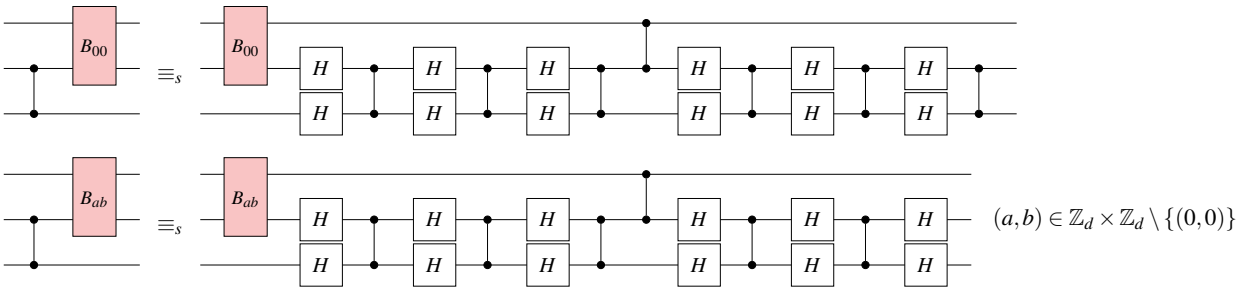


Figure 35: Pushing CZ through a B box, up to Pauli correction.

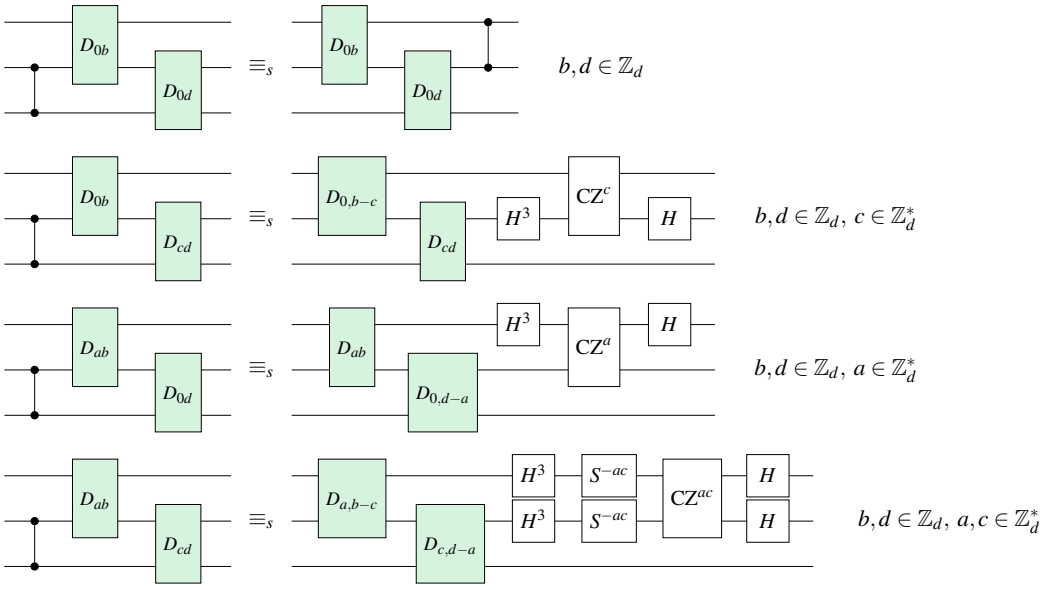


Figure 36: Pushing $I \otimes CZ$ through two D boxes, up to Pauli correction.