

Composable simultaneous purification: when all communication scenarios reduce to spatial correlations

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In this work, we address a fundamental question in quantum information theory: under what conditions can quantum correlations arising in communication scenarios be reduced to spatial Bell correlations? Such reduction is well established for the simplest bipartite communication scenario, the prepare-and-measure scenario, as recently formalized by Wright *et al.* [1]. The key tool is the Schrödinger-Gisin-Hughston-Jozsa-Wooters (S-G-HJW) theorem [2, 3, 4], which guarantees that non-signalling assemblages can always be simultaneously purified. However, whether a similar reduction holds for multipartite scenarios has remained a long-standing open problem. Indeed, the discovery of post-quantum steering in multipartite scenarios by Sainz *et al.* [5] suggested that such a reduction might be fundamentally impossible due to the richer structure of multipartite correlations.

We resolve this tension by proving that an analogous reduction to spatial Bell correlations holds universally for any multipartite communication scenario. To achieve this, we unify and extend simultaneous purification theorems to encompass instruments, super-instruments, and general quantum processes, which are composable structures.

Non-signalling assemblages. The operational backbone of our framework are non-signalling assemblages. Just as a measurement is defined by a collection of positive operators summing to identity, a *state assemblage* is a collection of sub-normalized quantum states $\{\rho_{a|x}\}_{a,x}$, such that for any fixed $\bar{x}$ the average over the outcome a , $\sum_a \rho_{a|\bar{x}} = \rho_{\bar{x}}$, is a normalised state. Consider a scenario in which two parties, Alice and Bob, share a bipartite resource ρ_{AB} . Alice aims to remotely prepare the state assemblage $\rho_{a|x}$ for Bob by performing a measurement $M_{a|x}$ on her system A . To generate all state assemblages allowed by quantum mechanics, the shared resource would generally have to depend on the input setting x (Fig. 1, decomposition (i)). However, if only Alice has access to the variable x , the celebrated S-G-HJW theorem states that she can remotely prepare exactly the assemblages that satisfy the *no-signalling* property $\sum_a \rho_{a|x} = \rho$ for all x and some fixed state ρ . This equation encodes the causal requirement that the marginal state on Bob’s side cannot depend on Alice’s input (Fig. 1, decomposition (ii)). To be precise, the S-G-HJW theorem states that an assemblage $\{\rho_{a|x}\}_{a,x}$ on a finite-dimensional Hilbert space $\mathcal{H}$ is non-signalling if and only if there exist an auxiliary Hilbert space $\mathcal{H}_A$, a set of POVMs on this space $\{M_{a|x}\}_{a,x} \subseteq \mathcal{B}(\mathcal{H}_A)$ and a pure bipartite state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}$ such that the assemblage can be remotely prepared: $\rho_{a|x} = \text{Tr}_A[(M_{a|x} \otimes \mathbb{1})|\psi\rangle\langle\psi|]$.

An equivalent of the S-G-HJW theorem can be formulated for quantum instruments [6, 7, 8]. An instrument assemblage $\{\mathcal{I}_{a|x}\}_{a,x} \subseteq \mathcal{CP}(\mathcal{H}_I, \mathcal{H}_O)$ is non-signalling, *i.e.* $\sum_a \mathcal{I}_{a|x} = \mathcal{I}$, if and only if it admits a decomposition in the form of $\mathcal{I}_{a|x}(\sigma) = \text{Tr}_A[(M_{a|x} \otimes \mathbb{1})\Lambda(\sigma)]$, where Λ is some fixed isometric quantum channel, and $\{M_{a|x}\}$ describes a measurement in the auxiliary Hilbert space $\mathcal{H}_A$ (Fig. 1, decomposition (ii)).

The structural similarity between these known results is suggestive, and the first key contribution of this work is the rigorous extension of this framework to more general structures like quantum super-instruments [9] and, most significantly, to arbitrary quantum processes [10]. Indeed, consider an assemblage of general quantum processes $\{O_{a|x}\}_{a,x}$. Taking advantage of the linear structure of quantum mechanics, the assemblage is said to be non-signalling if $\sum_a O_{a|x} = O$ for all x and some fixed process O . We prove that assemblages satisfying this property can always be decomposed as a dilated quantum process of the same type, followed by a delayed measurement on some auxiliary space; Fig. 1 graphically illustrates this pattern. The formal proof is based on the Choi–Jamiołkowski isomorphism, and can be found in the technical manuscript.

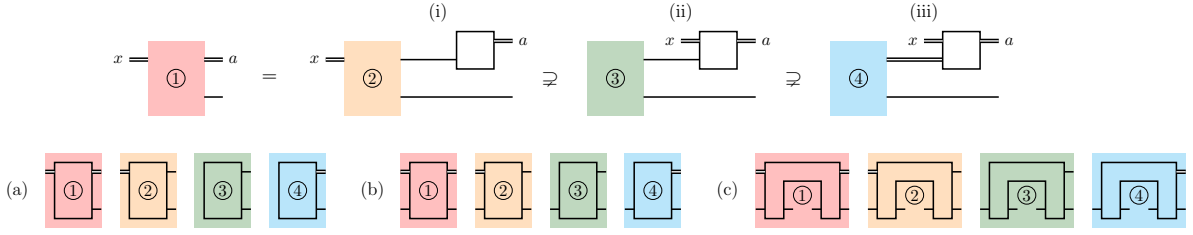
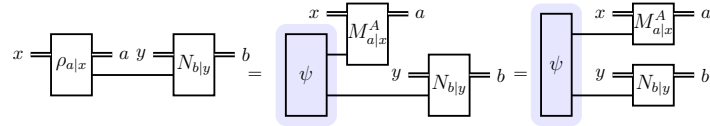


Figure 1: Decompositions for (a) state, (b) instrument, and (c) super-instrument assemblages. Single wires represent quantum and double wires classical information. Time is flowing from left to right. Equality (i) is true for any quantum assemblage. Decomposition (ii) is valid if and only if the assemblage is non-signalling. Decomposition (iii) refers to unsteerable assemblages. These separations reflect the separations between non-signalling, quantum, and classical sets of correlations. The decompositions can be generalized beyond super-instruments to arbitrary quantum objects.

Communication scenarios and composability. Bell non-locality studies the correlations that non-communicating players can achieve when constrained by different physical models. The most well-studied cases are classical, quantum, and non-signalling correlations, where the latter encodes solely constraints from causality; interestingly they form a strict hierarchy of sets. However, the introduction of communication often results in the collapse of this hierarchy. For instance, in a two-players scenario with a one-way communication channel, a trivial classical protocol suffices to saturate the maximum correlations allowed by causality (the one-way non-signalling polytope). This raises the central question that motivates this work: *can one impose restrictions on communication scenarios such that the fundamental separations between classical, quantum, and non-signalling correlation sets are preserved?*

The S-G-HJW theorem for states finds a natural application in answering this question, by characterising the necessary and sufficient condition to relate non-local and communication scenarios. This connection has been known for the bipartite case, and formalised in [1]. Using the diagrammatic approach introduced in Fig. 1, the proof is immediate; if and only if the state assemblage prepared by Alice is non-signalling, the following decomposition is valid:

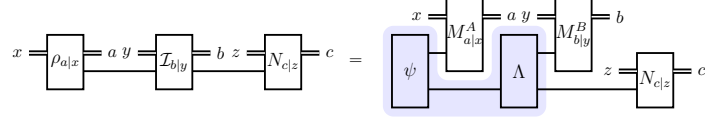


where the right-hand side is nothing but a bipartite Bell scenario – meaning that every such prepare-and-measure scenario has a Bell bipartite realization.

A similar result is not known for any other communication scenario. Consider, for example, three sequential players connected by a one-way quantum channel, a prepare-transform-measure scenario. Following the bipartite proof, one might expect that if all the states exchanged by the parties are non-signalling state assemblages, this would suffice to reconstruct a tripartite Bell realisation. The S-G-HJW theorem allows to construct two separate bipartite models, one for each set of non-signalling state assemblages; in the first, the middle player’s action is grouped with the state preparation, in the second, it is grouped with the measurement. However it is unclear how to combine these to obtain a Bell tripartite model. Even more, the existence of post-quantum steering [5] seems to suggest that this is not possible. The bottleneck is that non-signalling assemblages of states, and their decomposition, are not-composable.

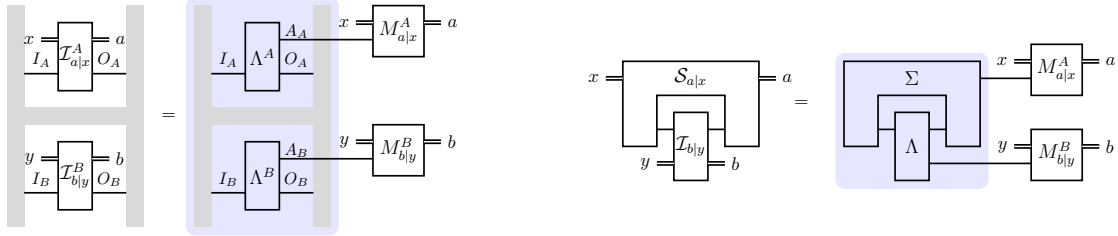
Our second key contribution is to overcome this barrier, by exploring the potential of the simultaneous purification framework for composable quantum structures. More precisely, we propose to model the intermediate player as a quantum instrument, and consider non-signalling assemblages of instruments instead. This allows us to sequentially apply above decomposition

of no-signalling states for the first player, and the decomposition of no-signalling instruments for the second:



where the state ψ followed by the isometry Λ can be interpreted as the shared resource of a tripartite Bell scenario. The same reasoning extends recursively to a sequence of k players, all implementing non-signalling composable processes.

In this work we go beyond the sequential case, showing that simultaneous purification of general quantum processes is the decisive tool to determine when correlations in arbitrary communication scenarios reduce to spatial Bell correlations. Using the diagrammatic approach presented above, we establish a direct link between this reduction and the non-signalling nature of the players' operations. Remarkably, this argument holds universally, regardless of the causal structure of the communication network: it encompasses all directed acyclic graphs, and even more exotic communication topologies like indefinite causal order and cyclic communication loops, represented in the following two diagrams.



This leads to the following theorem, our main result.

Theorem 1. *Let $k \in \mathbb{N}$. The arbitrary composition of k non-signalling quantum assemblages always produces correlations which admit a k -partite Bell quantum model.*

As the contraposition, it follows that any interactive quantum realisation of k -partite correlations that are outside the Bell quantum set must involve signalling communication. This holds even if the no-signalling principle is satisfied at the level of correlations, and can be seen as a generalisation of the well-established fact that signalling is necessary to realise PR box correlations in one-way communication scenarios [11].

An analogous classical result follows immediately by replacing the non-signalling decompositions with the unsteerable decompositions [12, 8] depicted in Fig. 1 (iii).

Theorem 2. *Let $k \in \mathbb{N}$. The arbitrary composition of k unsteerable assemblages always produces correlations which admit a k -partite Bell local realisation.*

Discussion. These findings represent a significant advance by clarifying the structure of correlations in general quantum multipartite communication scenarios, revealing a universal reduction to spatial correlations that had previously been observed only in restricted settings, and conjectured to fail in the multipartite regime. Beyond this fundamental result, our work provides a versatile operational toolkit. Just as the original S-G-HJW theorem for states is extensively used across a plethora of subfields, we expect our generalization to quantum processes and composable applications to become a standard technique for analysing complex quantum networks, having potential impact in multiple areas of quantum information, including steering, generalized contextuality, operational probabilistic theories, and operator algebra. We refer to the technical manuscript for a more detailed discussion of related work and further potential applications of our results.

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