Quantifying Quantum Measurement Incompatibility via Graph Invariants

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Introduction: Many of the remarkable consequences of quantum theory—including uncertainty relations, nonlocality, contextuality, and complementarity—originate from the non-commutativity of certain quantum observables [1–3]. A basic implication of non-commutativity is the impossibility of simultaneously measuring certain physical properties of a quantum system, which precludes the existence of a joint probability distribution with marginals matching the statistics of each observable. This lack of joint measurability, known as *measurement incompatibility*, imposes fundamental limits on how information can be transmitted and retrieved [4, 5]. At the same time, it serves as a quantum resource [6], underpinning applications such as quantum steering and quantum state discrimination [7–9].

Non-commutativity, however, is not by itself sufficient for incompatibility: certain non-commuting POVMs admit a joint measurement, for instance when they are sufficiently unsharp [10]. A natural way to quantify incompatibility is to mix the measurements with noise and determine the minimal noise level required to render them jointly measurable—the *robustness of incompatibility* [6]. Beyond serving as a resource-theoretic measure of nonclassicality, robustness measures provide insights into the quantum-to-classical transition in open quantum systems [11, 12], and determine bounds on the sample complexity of estimation tasks in quantum computing [5, 13, 14]. In general, quantifying robustness requires semidefinite programming, which quickly becomes intractable as the number of measurements or dimension of the system grows [15, 16].

In this work[1], I develop a graph–theoretic framework that provides analytic and scalable methods for quantifying measurement incompatibility for collections of binary observables defined by their (anti-)commutation relations [10, 13, 14, 18–20]. We show that any pattern of anti-commutation relations can be physically realised by binary observables, and that the incompatibility robustness does not depend on the specific choice of observables realising the pattern. General upper and lower bounds on the incompatibility robustness are derived in terms of standard graph invariants, including the Lovász number [21]. To illustrate the strength and scalability of these bounds, we consider degree–k Majorana monomials—observables arising in many-body fermionic systems with applications in quantum chemistry and partial tomography [22–25]. Their anti-commutativity graphs are merged Johnson graphs [26]. Using recent asymptotic bounds on the Lovász number [27], we obtain a corresponding asymptotically tight characterisation of their incompatibility robustness in the large-system limit at fixed k.

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For classes of line graphs—which have a one-to-one correspondence with families of quadratic Majorana observables—we obtain spectral bounds and exact formulas for several families, including cycles, Johnson graphs, hypercubes and rook's graphs, as well as tight asymptotic bounds for paths. Furthermore, we determine that a line graph saturates a universal upper bound only if it is regular and its root graph admits an orientation whose skew-adjacency matrix is a skew-weighing matrix, a well-studied object in combinatorics [28]. This establishes a connection between joint measurability and classical problems in algebraic combinatorics, such as the existence of Hadamard and skew-conference matrices [28].

Framework (observables, graphs, and robustness): Let $\mathcal{A} = \{A_v \mid v \in V\}$ be a family of binary observables acting on a d-dimensional Hilbert space, with each observable A_v a Hermitian operator satisfying $A_v^2 = \mathbb{1}$. The latter constraint implies A_v is unitary with eigenvalues ± 1. The associated sharp binary POVM is

$$\mathsf{M}_v(\pm) = \tfrac{1}{2}(\mathbb{1} \pm A_v). \tag{1}$$

The algebraic relations between the observables of $\mathcal{A}$ can be described by an anti-commutativity graph.

Definition 1. The *anti-commutativity graph* $G = (V, E)$ of a set of observables $\mathcal{A}$ is defined by

$$\{v, v'\} \in E \iff A_v A_{v'} = -A_{v'} A_v, \tag{2}$$

so that adjacent vertices correspond to anti-commuting observables, and non-adjacent vertices correspond to commuting ones.

Two notable examples of observables that admit anti-commutativity graphs are Pauli strings and many-body fermionic Majorana observables. On m qubits, Pauli strings of the form $P = \otimes_{i=1}^m P_i$, with $P_i \in \{\mathbb{1}, \sigma_x, \sigma_y, \sigma_z\}$, anti-commute if they differ on an odd number of sites by single-qubit Paulis that anti-commute. Fermionic Majorana observables are built from Majorana operators $\Gamma_1, \ldots, \Gamma_m$ satisfying

$$\Gamma_j^\dagger = \Gamma_j, \qquad \Gamma_j^2 = \mathbb{1}, \qquad \{\Gamma_j, \Gamma_k\} = 2\delta_{jk}\mathbb{1}, \tag{3}$$

The anti-commutativity graph of these m operators is the complete graph $G = K_m$. Products of an even number of distinct Majorana operators, such as the quadratic observables $A_{jk} := i\,\Gamma_j\Gamma_k$, with $j < k$, anti-commute if and only if they share an odd number of Majorana operators. Graph representations of Pauli and Majorana

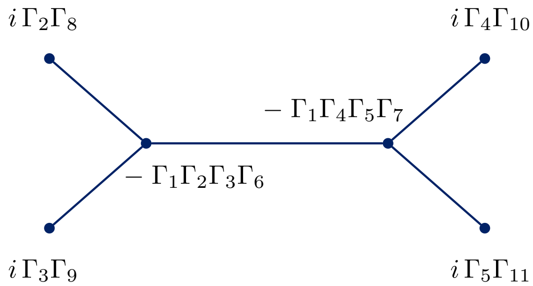


FIG. 1. Example of a six-vertex anti-commutativity graph realised by quadratic and quartic Majorana monomials. Observables anti-commute if and only if they share one or three common Majorana operators from the set $\{\Gamma_j \,|\, j = 1, \ldots, 11\}$.

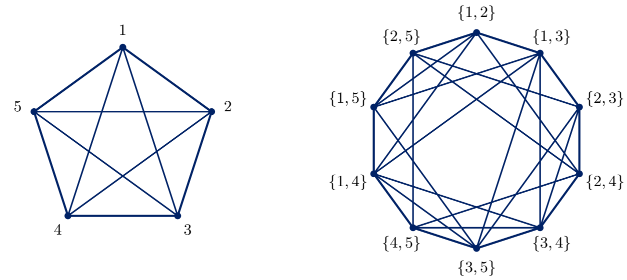


FIG. 2. The complete graph K_5 (left) and its line graph $L(K_5)$ (right) which is the Johnson graph $J(5, 2)$. Each edge $\{u, v\}$ of K_5 corresponds to a vertex $\{u, v\}$ of $L(K_5)$; two such vertices are adjacent in $L(K_5)$ if and only if the corresponding edges in K_5 are incident.

observables have been used in various contexts, such as uncertainty relations [4, 29] and free-fermion spin models [30, 31].

For every simple connected graph $G = (V, E)$, there exists a set of binary observables whose anti-commutativity graph is G (see, e.g. Fig. 1). To quantify the degree of incompatibility of $\mathcal{A}$—the extent to which it fails to admit a joint measurement—we define the *incompatibility robustness* as

$$\eta(G) \equiv \max\{\eta \in [0, 1] \mid \mathcal{A}^\eta \text{ is jointly measurable}\}, \quad (4)$$

where

$$\mathcal{A}^\eta = \{A_v^\eta \,|\, v \in V\}, \qquad A_v^\eta = \eta A_v + \frac{\mathrm{tr}(A_v)}{d}(1-\eta)\mathbb{1}, \quad (5)$$

are noisy (unsharp) versions of the original observables [1, 10]. The associated binary POVM elements are

$$\mathsf{M}_v^\eta(\pm) = \tfrac{1}{2}(\mathbb{1} \pm \eta A_v), \qquad \eta \in [0, 1]. \quad (6)$$

The incompatibility robustness of $\mathcal{A}$ can be viewed as a graph invariant, in the sense that it only depends on G.

Proposition 1. *Let $G = (V, E)$ be a simple graph. The incompatibility robustness $\eta(G)$ depends only on the (anti-)commutation relations encoded by G, and is invariant under the particular choice of binary observables $\mathcal{A} = \{A_v \,|\, v \in V\}$ realising them.*

General bounds from graph invariants: Viewing $\eta(G)$ as a graph invariant, we can bound it in terms of well-studied graph parameters. In particular, we obtain an efficiently computable upper bound in terms of the Lovász number $\vartheta(G)$ [21], and a lower bound from colouring-based joint measurements in terms of the fractional chromatic number $\chi_f(G)$.

Theorem 1. *Let $G = (V, E)$ be the anti-commutativity graph of $\mathcal{A} = \{A_v : v \in V\}$, and let $\vartheta(G)$ be its Lovász number and $\chi_f(G)$ be the fractional chromatic number. Then*

$$1/\chi_f(G) \le \eta(G) \le \sqrt{\vartheta(G)/|V|}\,. \quad (7)$$

For several graph families of interest, we prove that the upper bound is tight, i.e., $\eta(G) = \sqrt{\vartheta(G)/|V|}$. The simplest example is the complete graph $G = K_n$ which has $\vartheta(K_n) = 1$, and hence $\eta(K_n) = \frac{1}{\sqrt{n}}$.

Let $\mathcal{A}_k$ denote the family of degree–k Majorana monomials on an n-mode fermionic system. Their anti-commutativity graphs are merged Johnson graphs [26], and the Lovász-number bound captures the asymptotic behaviour of the robustness for fixed k.

Proposition 2. *Assume n and k are even, with k fixed and independent of n. Then the incompatibility robustness of the observables $\mathcal{A}_k$ with anti-commutativity graph G_k satisfies $\eta(G_k) = \Theta(n^{-k/4})$.*

Line graphs and spectral characterisations: The *line graph* of a (root) graph G is the graph $L(G)$ whose vertex set is the edge set of G, and where two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are incident. For example, the line graph of a complete graph K_n is the Johnson graph $J(n, 2)$, as illustrated in Fig. 2. Other examples of line graphs include paths are cycles.

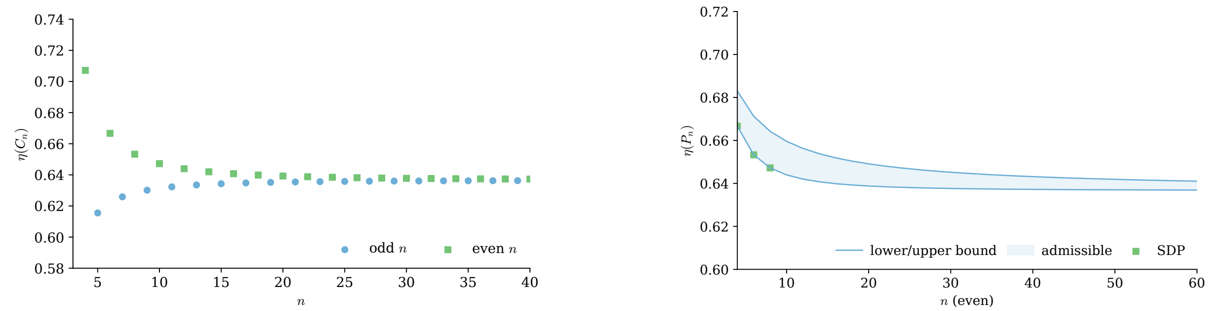
Proposition 3. *Every anti-commutativity graph that is a line graph can be realised by a family of quadratic Majorana observables. Conversely, every family of quadratic Majorana observables has a line graph as its anti-commutativity graph, provided no repeated observables.*

For a graph $G = (V, E)$, let $L(G)$ denote its line graph. The incompatibility robustness on $L(G)$ admits spectral bounds via the skew-adjacency matrix of G (for a choice of orientation) and its skew-energy, defined as the sum of the absolute values of its eigenvalues, maximised over orientations (the maximum skew-energy) [32].

Theorem 2. *Let $L(G)$ be the line graph of a graph $G = (V, E)$ with maximum skew-energy $\mathcal{E}_s^{\max}(G)$. Then*

$$\eta(L(G)) \;\le\; \frac{1}{2|E|}\mathcal{E}_s^{\max}(G)\,. \quad (8)$$

If G is edge-transitive (i.e., its symmetries act transitively on edges) then equality holds.



(a) Cycles

(b) Paths

FIG. 3. Incompatibility robustness for (a) cycles and (b) paths. Left: exact values of $\eta(C_n)$ for odd (blue) and even (green) n. As n increases, the incompatibility of odd (even) cycles decreases (increases), with both approaching $2/\pi$ as $n \to \infty$. Right: bounds on $\eta(P_n)$ for even n, together with exact values for $n \leq 9$ obtained via semidefinite programming. Similar bounds also hold for odd n, and both cases exhibit the same asymptotic behaviour as cycles.

The theory yields closed expressions (or exact values) for several graph families. For the Johnson graph $J(n,2)$ (see, e.g., Fig. 2), which corresponds to the set of all quadratic Majorana observables $\{\, i\Gamma_j\Gamma_k \;:\; 1 \leq j < k \leq n\}$, we obtain:

Corollary 1. *For Johnson graphs $J(n,2)$, the incompatibility robustness of the corresponding set of $\binom{n}{2}$ observables satisfies*

$$\eta\big(J(n,2)\big) \;\leq\; \frac{1}{\sqrt{n-1}}\,, \tag{9}$$

with equality if and only if a skew–conference matrix of order n exists.

A cycle C_n—which is a line graph of itself—has n vertices arranged in a single closed loop. A simple formula for their incompatibility is given by the following result, and is graphically illustrated in Fig. 3.

Corollary 2. *For cycle graphs C_n, incompatibility robustness is given by*

$$\eta(C_n) = \begin{cases} \frac{1}{n}\,\cot(\pi/2n), & n \text{ odd},\\ \frac{2}{n}\,\csc(\pi/n), & n \text{ even}. \end{cases} \tag{10}$$

As $n \to \infty$, $\eta(C_n) \to \frac{2}{\pi}$.

The proof relies on Theorem 2 and calculations of the skew-energies for the two distinct switching classes of oriented cycles, given in [32].

We also show that there is a simple universal upper bound for line graphs, expressed in terms of its maximum degree Δ (i.e. the largest number of edges incident to any vertex):

Proposition 4. *Let $G = (V,E)$ be a simple graph with with $|V| = n$, $|E| = m$, and maximum degree Δ. Then its line graph satisfies*

$$\eta(L(G)) \leq \frac{n}{2m}\sqrt{\Delta}\,. \tag{11}$$

We call $L(G)$ *optimally incompatible* if it saturates this bound. Saturation forces strong algebraic-combinatorial structure:

Proposition 5. *Let G be a simple graph with n vertices, m edges and maximum degree Δ. If $L(G)$ is optimally incompatible, then there exists an orientation of G such that its skew-adjacency matrix S satisfies*

$$S^{\top} S \;=\; \Delta\, \mathbb{1}_n\,. \tag{12}$$

Consequently, G must be Δ–regular.

We use this result to show that the line graph of the hypercube Q_d is optimally incompatible for every d; the Johnson graph $J(n,2)$ is optimally incompatible if and only if a skew–conference matrix of order $n-1$ exists; and the rook's graph $K_r \square K_s$ is optimally incompatible if and only if $r = s$ and a Hadamard matrix of order s exists.

Outlook: By introducing graph theory into the problem of quantifying measurement incompatibility, we obtain exact formulas for several graph families. So far, we restrict to unitary binary observables, but it is natural to ask what changes when these assumptions are relaxed. Further directions include understanding how $\eta(G)$ varies under modifications of the anti-commutation pattern and under standard graph operations (edge/vertex deletions, complementation, products, joins), where already paths and cycles can behave differently (cf. Fig. 3).

Our analysis of line graphs reveals connections between joint measurability and extremal problems in combinatorics and matrix theory, involving weighing matrices, Hadamard matrices, and skew-conference matrices. It remains open whether, and under what conditions, these objects arise in explicit constructions of joint measurements saturating the incompatibility bounds. Such constructions would be useful in estimation tasks in quantum computing (see, e.g., [5, 13, 14]).

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