Extended abstract for *A real matrix theory consistent with both the postulates and predictions of quantum theory*

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This is an extended abstract for arXiv:2504.02808 alongside upcoming continuation work.

Quantum theory was radically different from the theories of nature that came before it in many aspects. One of its key differences was its reliance on complex numbers. In classical electrodynamics, for instance, complex numbers are simply a technical convenience that can be eliminated if desired. In quantum theory, however, they appear to be essential. This opened a longstanding debate over whether quantum theory fundamentally *requires* complex numbers, or if their use is merely a convenient choice. For nearly a century, the question remained unsettled—until Renou et al. [1] tipped the scales in favour of the latter. As they claimed in a plenary talk at QIP2022, *quantum theory needs complex numbers.*

This claim must be interpreted carefully, however. Contrary to what could have been read in the non-technical press, this is not the statement that Nature is somehow aware of the mathematics we use to describe it and prefers us to use complex numbers. Rather, this is a statement about the Hilbert space formalism: the postulates of quantum theory are to be made using complex numbers. What Renou et al. meant is *should the postulates be made using real numbers, the resulting theory would make predictions that differ in ways that could be experimentally tested.*

It is important to understand that the "real quantum theory" considered in Renou et al. is merely the name of a mathematical construct; it is obtained by redefining the Hilbert space formalism over the real numbers and is "quantum" only by its analogy with the standard formulation. RQT, by its 'construct' or 'foil' nature, must be contrasted with the alternative formulations of quantum theory of a more 'fundamental' nature. Actually, many such alternatives, such as quantum logics and algebraic quantum theory, use the reals as their base field (see Ref. [2] for the one that later evolved into the C^*-algebraic formulation and Ref. [3] for the quantum logic one). It is only when trying to recover the standard formulation out of these approaches that the complex Hilbert spaces naturally impose themselves. Still, the eventuality of using other fields than the complexes also arises in this approach [2, 4]. Yet, the real Hilbert space formulations obtained like that do not coincide with RQT. RQT then arises as a purely 'foil' theory, obtained by twisting the mathematics of QT rather than by proposing a new set of principles or formalism. Hence, what Renou et al. showed is *the foil known as "real quantum theory" can be experimentally distinguished from the complex Hilbert space formalism of quantum theory.*

Recently, we have shown that this claim relies on a model-dependent assumption. And that it no longer holds true when this assumption is replaced by a model-independent counterpart. The falsification is then confined within the mathematical model, making the experiment inconclusive [5]. (This argument is the subject of a separate contribution to this conference.) This implies that the claim must be further nuanced into '*the statistical predictions of "real quantum theory" are different from those obtained with quantum theory*'.

For this contribution, we want to shift the attention back to the need for complex numbers in QT. Before even debating the experimental falsifiability of real quantum theory, one may ask whether it is the only consistent way to define a Hilbert space quantum-like theory over the reals to begin with? Even if RQT can be experimentally falsified, if it is not the only real alternative to QT, then every other alternative must also be proven falsifiable before inferring anything about the necessity of complex numbers in QT.

Our argument is that there is a way to make the real matrix representation of the complex Hilbert space quantum formalism consistent with the postulates of the real Hilbert space quantum formalism. This real theory, that we call real-number quantum theory (RNQT) (to distinguish it from the other theory, namley RQT), is obviously indistinguishable from QT. In addition, if one wishes to disregard this real theory on the grounds that 'it is only QT in disguise' or 'it has an intrinsic complex structure', we prove that it can be embedded into a larger theory which is not in a bijection with QT.

In more technical terms, we develop RNQT as a model of real symmetric matrices that satisfies the formal postulates of quantum theory while reproducing any outcome distribution predicted by the usual (complex Hermitian) quantum theory. We then extend it as some kind of generalised probabilistic theory (GPT) based on the D_8 group.

Hence, beyond reminding the reader that the intrinsic need for a complex structure in QT is specific to its Hilbert-space formalism, our model is intended to highlight the false dichotomy between QT and RQT. Outside of its role in the QT/RQT debate, this model is by itself of genuine interest as a GPT. The remainder of this abstract outlines the key technical steps in this approach.

I. REAL QUANTUM THEORY CANNOT LOCALLY REPRESENT QUANTUM THEORY

Firstly, the (relevant, mixed-state) postulates of quantum theory are the following[1]: (i) For every physical system S, there corresponds a Hilbert space $\mathcal{H}_\mathsf{S}$ and a set of states $\mathcal{S}(\mathcal{H}_\mathsf{S})$. Each state is represented by an operator ρ, acting on this space. The state has positive spectrum ($\rho \geq 0$) and trace one ($\mathrm{Tr}[\rho] = 1$). (ii) A measurement in S corresponds to a Positive Operator-Valued Measure (POVM). Namely, a set of positive operators $\{e_r \geq 0\}_r$ acting on $\mathcal{H}_\mathsf{S}$, and indexed by the measurement result r, with $\sum_r e_r = I_\mathsf{S}$. The elements of a POVM are called effects and the set of all effects is denoted $\mathcal{E}(\mathcal{H}_\mathsf{S})$. (iii) Born rule: if we measure $\{e_r\}_r$ when system S is in state ρ, the probability of obtaining result r is given by the (Hilbert-Schmidt) inner product of the state with the corresponding effect, $\Pr(r) = \langle \rho\,,\, e_r\rangle$. (iv) Composition: The Hilbert space $\mathcal{H}_\mathsf{ST}$, corresponding to the composition of two systems S and T is $\mathcal{H}_\mathsf{S} \otimes \mathcal{H}_\mathsf{T}$, where $\otimes$ is the tensor product. The operators used to describe measurements or transformations in system S act trivially on ST and vice versa. Similarly, the state representing two independent preparations of the two systems is the tensor product of the two preparations.

These postulates involve abstract operators whose matrix representation is yet to be specified. By force of habit, one may expect the operators to be the matrices corresponding to linear operators on $\mathcal{H}_\mathsf{S}$, with the tensor product given by the matrix operation called the Kronecker product (that we will denote using $\otimes_K$).

In doing so, we obtain (standard) **quantum theory** (QT) if the base field is $\mathbb{C}$. **Real Quantum Theory** (RQT) is obtained analogously but with $\mathbb{R}$ as base field instead. Since a Bell inequality violation can be explained using either theory, the question which drove research on RQT is whether some local experiment could not be explained by RQT [1, 7–9]. What makes this question non-trivial is the final sentence of (iv): it is rarely stated explicitly as a postulate, but it is essential to exclude the (naive) real representation of QT contained in any RQT [7].

Definition 1 (Representation of quantum theory)**.** *Let* $\mathtt{T}$ *be a theory obeying the postulates (i)-(iv) and denote its sets of states and effects by, respectively,* $\mathcal{S}_\mathtt{T}$ *and* $\mathcal{E}_\mathtt{T}$*. Then,* $\mathtt{T}$ *provides a representation of quantum theory if, for every pair of states and effects* $(\rho \in \mathcal{S}_{Qu.}, E \in \mathcal{E}_{Qu.})$*, there exists a pair of maps* $(M, E) : \mathcal{S}_\mathtt{T} \times \mathcal{E}_\mathtt{T} \to \mathcal{S}_{Qu.} \times \mathcal{E}_{Qu.}$ *such that*

$$\langle \rho\,,\, E_r\rangle = \langle M(\rho)\,,\, E(e_r)\rangle \ . \tag{1}$$

In addition, the simulation is said to be ***representation-local*** *when applying the maps commutes with the composition rule [Postulate (iv)].*

Because of the last part of postulate (iv), representation-locality indeed becomes an important characteristic of the representation: without it, there is no hope to explain the predictions of QT with the theory in which it is represented, as the meaning of independent preparations is lost when trying to map on.

For example, let $\mathbb{1}$ be the 2×2 identity matrix and σ_y the Pauli y matrix. Then, the well-known mapping

$$\begin{aligned} &\Gamma : \mathbb{C}^{d\times d} \to \mathbb{R}^{2d\times 2d}\,, \\ &\qquad A \mapsto \mathbb{1} \otimes_K \mathrm{Re}(A) - i\sigma_y \otimes_K \mathrm{Im}(A) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes_K \mathrm{Re}(A) + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes_K \mathrm{Im}(A)\,, \end{aligned} \tag{2}$$

maps every state and effect of QT to a counterpart in RQT while preserving all statistical predictions (i.e., $E = M = \Gamma$ up to some normalisation factors). However, this map is notoriously badly-behaved w.r.t. the Kronecker product: $\Gamma(M_1 \otimes_K M_2) \neq \Gamma(M_1) \otimes_K \Gamma(M_2)$, so it cannot be used to define a representation-local real QT. Indeed, under Γ, some bipartite states in pure tensor product will be turned into separable or entangled states, and some separable states into entangled ones. Because of that, Γ cannot be used to obtain a trivial RQT explanation for all the bounds on local games predicted by QT.

II. EXISTENCE OF A LOCAL REPRESENTATION OF QUANTUM THEORY

But no physical principle asserts that the tensor product *must be* represented by the Kronecker product. (Actually, it has been proven recently that the composition rule does not even have to be the tensor product [10].) This assumption can be relaxed by defining the real map $\otimes_r$ satisfying

$$\Gamma(A \otimes_K B) = \Gamma(A) \otimes_r \Gamma(B)\,. \tag{3}$$

[1] This presentation, adapted from Ref. [1], is close to what can be found for example in Ref. [6], and is fairly standard in the RQT literature.

The image of QT through Γ is a theory obeying postulates (i) to (iii) but using $\otimes_r$ as its composition rule. We call it real-number quantum theory (RNQT). Importantly, this $\otimes_r$ map is also a (representation of a) tensor product. One should now be wondering how it is possible that two representations of the tensor product yield different theories. The catch is that the postulates are unclear on what $\mathcal{S}(\mathcal{H}_S)$ is. Consider the following.

Proposition 1 (Characterisation of RNQT [11, 12][2]). *Let* $\mathtt{T}$ *be a theory of real matrices satisfying postulates* $(i)-(iv)$*. Then, the following are equivalent:*

1. *The pure-state formalism is defined up to an* $U(1) \cong SO(2)$ *phase;*
2. *The representation of the tensor product is* $\otimes_r$*;*
3. *Every element of* $\mathtt{T}$ *commute with a given matrix* J *s.th.* $J^2 = -I$ *and called its complex structure;*
4. $\mathtt{T}$ *is a bijective local representation of QT.*

Proposition 2 (Characterisation of RQT [12]). *Let* $\mathtt{T}$ *be a theory of real matrices satisfying postulates* $(i) - (iv)$*. Then, the following are equivalent:*

1. *The pure-state formalism is defined up to an* $\pm$ *sign;*
2. *The representation of the tensor product is* $\otimes_K$*.*

We see that the state spaces (as convex sets) of RQT and RNQT are different, and the composition rule (the tensor product) is defined on the state space, not the (unphysical) space $\mathcal{H}_\mathsf{S}$ underlying the representation. By item 1. of Propositions 1 and 2, it becomes clear that RNQT is, like QT, the theory defined over complex projective spaces ($U(1)$ phase freedom), whereas RQT is the theory defined over real projective spaces ($\pm$ sign freedom).

This raises the question whether it is still meaningful to compare RQT and QT, since we find that one is based on orientable manifolds while the other is not. Stückleberg, who was the first to formalise a quantum theory based on real numbers without a notion of representation locality [Postulate (iv)] [14–16], seemed aware of that, since he explicitly required his theory to verify item 3. of Proposition 1. Other foil theories using the Kronecker product representation have recently been explored in [17].

III. EXTENDED REAL-NUMBER QUANTUM THEORY

Nevertheless, one can argue that RNQT is but QT in the guise of real matrices, and/or that the falsification of RQT still shows that QT is fundamentally complex, since its phase freedom is $U(1)$. But even besides the loophole making the falsification impossible to test experimentally (see our other contribution), this argument still does not apply. The falsification only rules out RQT as an alternative explanation for QT; it does not imply that the predictions of QT can be made only by theories with a complex structure.

As an example for this point, we consider the extension of RNQT that uses the state and effect spaces of RQT (we relax item 3 of Proposition 1), and whose phase freedom is extended to a new symmetry that contains both the sign and phase freedom. As it turns out, this extension can be defined in a manner consistent with the wording of postulates (i)-(iv).

Proposition 3 (Characterisation of extended RNQT [11, 12]). *There exists a sub-theory*[3] $\mathtt{T}$ *of real matrices with the following properties*

1. *The pure-state formalism is defined up to a* $e^{\sum_i u_i G_i}$ *phase, where* $\{u_i\}_i \subset \mathbb{R}$ *and* $\{G_i\}_i$ *is a matrix representation of the finite group* D_8*;*
2. *The representation of the tensor product is* $\otimes_r$*;*
3. $\mathtt{T}$ *satisfies postulates* $(i) - (iv)$*.*
4. $\mathtt{T}$ *contains a local representation of QT.*

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[2] See also Ref. [13].

[3] To be consistent with postulate (iv), the set of states has to be restricted to a set strictly included between the sets of states of RNQT (trace-one positive semi-definite matrices commuting with some matrix J) and those of RQT (trace-one positive semi-definite matrices).

[1] M.-O. Renou, D. Trillo, M. Weilenmann, T. P. Le, A. Tavakoli, N. Gisin, A. Acín, and M. Navascués, Quantum theory based on real numbers can be experimentally falsified, Nature **600**, 625 (2021), arXiv:2101.10873.

[2] P. Jordan, J. von Neumann, and E. Wigner, On an algebraic generalization of the quantum mechanical formalism, Annals of Mathematics **35**, 29 (1934).

[3] G. Birkhoff and J. von Neumann, The logic of quantum mechanics, Annals of Mathematics **37**, 823 (1936).

[4] M. P. Solèr, Characterization of hilbert spaces by orthomodular spaces, Communications in Algebra **23**, 219 (1995).

[5] T. Hoffreumon and M. P. Woods, Real quantum theory is experimentally indistinguishable from quantum theory (2026), in preparation.

[6] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information: 10th Anniversary Edition* (Cambridge University Press, 2012).

[7] M. McKague, M. Mosca, and N. Gisin, Simulating quantum systems using real hilbert spaces, Phys. Rev. Lett. **102**, 020505 (2009).

[8] N. Gisin, Bell inequalities: many questions, a few answers, in *Quantum Reality, Relativistic Causality, and Closing the Epistemic Circle: Essays in Honour of Abner Shimony* (Springer, 2007) p. 125–138, arXiv:quant-ph/0702021 [quant-ph].

[9] M. Weilenmann, N. Gisin, and P. Sekatski, Partial independence suffices to rule out real quantum theory experimentally, Physical Review Letters **135**, 10.1103/3fv7-p8cs (2025).

[10] M. Erba and P. Perinotti, The composition rule for quantum systems is not the only possible one (2025), arXiv:2411.15964 [quant-ph].

[11] T. Hoffreumon and M. P. Woods, Quantum theory does not need complex numbers (2025), arXiv:2504.02808 [quant-ph].

[12] T. Hoffreumon and M. P. Woods, On a real version of the postulates of quantum theory (pending title, 2026), in preparation; see also the version 2 of quant-ph/2504.02808 which contains preliminary results.

[13] P. B. Hita, A. Trushechkin, H. Kampermann, M. Epping, and D. Bruß, Quantum mechanics based on real numbers: A consistent description (2025), arXiv:2503.17307 [quant-ph].

[14] E. C. G. Stückelberg, Quantum theory in real hilbert-space, Helvetica Physica Acta **33**, 727 (1960).

[15] E. C. G. Stückelberg and M. Guénin, Quantum theory in real hilbert space. ii, addenda and errats, Helvetica Physica Acta **34**, 621 (1961).

[16] E. C. G. Stückelberg, M. Guénin, and C. Piron, Quantum theory in real hilbert space. iii, fields of the first kind (linear field operators), Helvetica Physica Acta **34**, 675 (1961).

[17] Y. Yīng, M. C. Alañón, D. Centeno, J. Surace, M. M. Ansanelli, R. Liu, D. Schmid, and R. W. Spekkens, On whether quantum theory needs complex numbers: the foil theories perspective (2025), arXiv:2506.08091 [quant-ph].