Extended abstract for *On the experimental falsification of Real Quantum Theory*

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I. OVERVIEW

Real Quantum Theory (RQT) is a variant of standard quantum theory (QT) in which one restricts the mathematical objects used to describe states and measurements: instead of allowing arbitrary complex Hermitian matrices, one works with real symmetric matrices, while keeping the usual quantum rules for assigning probabilities and updating states after measurements. RQT was developed as a toy theory to probe the role of complex numbers in QT and has been explored extensively since the 1990s [1–3].

A key structural choice is that RQT keeps the same system-composition rule as QT (the usual Kronecker product). This makes RQT qualitatively different in principle—for instance, it can fail local tomography [1], meaning that global properties of a composite system are not always fully determined by local measurement data alone. Yet despite such theory-level differences, RQT is known to reproduce all measurement statistics achievable in QT in the usual single-party and bipartite Bell scenarios [3–5]. (Up to an innocuous enlargement of the underlying real description, which is not something one can certify experimentally in a device-independent way [6].) In this sense, RQT has long served as an experimentally indistinguishable alternative to QT in the simplest testing paradigms.

This motivated the question of whether QT and RQT can *ever* be distinguished operationally, or whether they merely offer different mathematical representations of the same observable physics [7]. Renou et al. [8] recently claimed a positive answer in a network setting (the bilocal/entanglement-swapping scenario [9]): they argue that QT and RQT make different predictions that can be witnessed in a theory-independent way via the violation of a Bell-like inequality, and they propose an experiment based on two *independent sources*. They claimed that obtaining certain measurement outcomes which are consistent with QT is sufficient to rule out a RQT explanation [8]. This result was published in Nature and was a QIP 2022 Plenary talk, in addition to receiving a lot of attention from the popular press at the time with articles in Quanta, Scientific American and APS News.

Our results show that this conclusion hinges on an additional assumption about how "independent sources" are modeled—an assumption that is not experimentally certifiable in a (semi-)device-independent manner. Consequently, once one insists on operationally testable premises, no network experiment—including the Renou et al. experiment—can distinguish QT from RQT without witnessing a violation of QT. This is to say, all network experiments whose experimental outcomes are consistent with QT with be inconclusive: both QT and RQT yield plausible explanations.

There may exist a network experiment which can certify RQT experimentally, if a statistic which violates QT is observed experimentally. However, even if such a test does exist, it is not very useful, because over 40 years of experiments have agreed with quantum theory, so it's unlikely that we will find such an outcome.

II. THE RENOU ET AL. EXPERIMENT

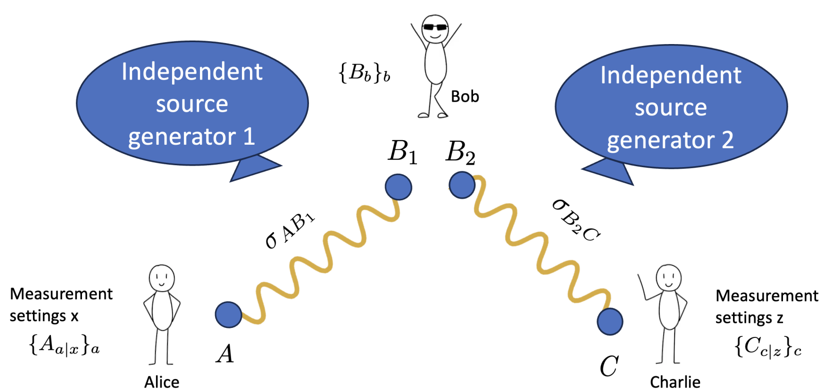

FIG. 1. **Experimental setup from [8].** A bilocal network with two independent sources distributes systems to Alice, Bob (two subsystems B_1, B_2), and Charlie. Alice (setting x) and Charlie (setting z) perform local measurements, Bob performs a fixed joint measurement, and one estimates the correlations $p(a, b, c|x, z)$. The goal is to decide whether the observed statistics are compatible with QT or with RQT.

Ref. [8] introduces a Bell-like *local game* (a Bob-conditioned variant of CHSH$_3$ [10, 11]) tailored to the bilocal/entanglement-swapping setting [9]. Let $\mathcal{I}$ denote the associated "swapped CHSH$_3$" functional. (See Ref. [8] for the explicit expression; our discussion does not depend on it.) For any local-hidden-variable (LHV) model one has [7, 8, 10, 11] $\mathcal{I}\big(\{p_{LHV}(a,b,c|x,z)\}_{a,b,c,x,z}\big) \leq 6$. Assuming standard QT, the correlations take the form

$$p_{quant.}(a,b,c|x,z) = \mathrm{tr}\left[(\sigma_{\mathrm{AB}_1} \otimes_K \sigma_{\mathrm{B}_2\mathrm{C}})(A_{a|x} \otimes_K B_b \otimes_K C_{c|z})\right] , \tag{1}$$

with bipartite source states σ_{AB_1} and $\sigma_{\mathrm{B}_2\mathrm{C}}$ and local POVMs $\{A_{a|x}\}_a$, $\{B_b\}_b$, $\{C_{c|z}\}_c$. For qubit subsystems, QT achieves [8, 10, 11]

$$\mathcal{I}\big(\{p_{quant.}(a,b,c|x,z)\}_{a,b,c,x,z}\big) \leq 6\sqrt{2} \approx 8.49 \,. \tag{2}$$

An optimal strategy uses maximally entangled sources ($\sigma_{\mathrm{AB}_1} = \sigma_{\mathrm{B}_2\mathrm{C}} = |\Phi^+\rangle\langle\Phi^+|$), a Bell measurement on B_1B_2 at Bob, and CHSH$_3$-type measurements at Alice and Charlie conditioned on x, z.

Renou et al. then ask whether the same value is attainable in RQT (i.e., with purely real operators). They show that even allowing *arbitrary local dimensions* and *classically correlated sources*, one cannot reach the quantum optimum. Concretely, they consider

$$p_{RQT}(a,b,c|x,z) = \sum_\lambda p(\lambda)\, \mathrm{tr}\left[(\sigma^{|\lambda}_{\mathrm{AB}_1} \otimes_K \sigma^{|\lambda}_{\mathrm{B}_2\mathrm{C}})(A_{a|x} \otimes_K B_b \otimes_K C_{c|z})\right] , \tag{3}$$

where λ is shared randomness, $\sigma^{|\lambda}_{\mathrm{AB}_1}$ and $\sigma^{|\lambda}_{\mathrm{B}_2\mathrm{C}}$ are (finite-dimensional) RQT states, and the POVMs are (finite-dimensional) RQT POVMs. They prove:

Proposition 1 (Paraphrase from Ref. [8], Theorem 3)**.** *The maximal value of the swapped CHSH$_3$ game obtained from a distribution $p_{RQT}(a,b,c|x,z)$ as in Eq. (3) is, up to numerical precision,*

$$\mathcal{I}\big(\{p_{RQT}(a,b,c|x,z)\}_{a,b,c,x,z}\big) \leq r \approx 7.66 \,, \tag{4}$$

regardless of the dimensions of the subsystems.

Thus, observing values near $6\sqrt{2} \approx 8.49$ would (in their interpretation) rule out RQT in favour of QT. Subsequent experiments [12, 13] reported visibilities consistent with the quantum value and inconsistent with $r \approx 7.66$.

III. OUR RESULTS

However, the result discussed at the end of the previous section does not rule out an RQT description. Some leap of faith is indeed required to pass from proposition 1 to the experimental data, which is to trust that Eq. (3) models the experiment accurately. Observe that the model (3) relies on independent sources being represented by product states.

Definition 1 (Product-State Independence)**.** *In QT and RQT, the joint state Ψ_{XY} of a bipartite system shared by parties* X *and* Y *is said to be* ***product-state independent*** *if and only if it can be written in the form*

$$\rho_{\mathrm{X}} \otimes_K \sigma_{\mathrm{Y}}, \tag{5}$$

where ρ_{X} and σ_{Y} are (real) quantum states respectively defined over the local Hilbert spaces associated with party X *and* Y*.*

This definition is but a *prescription of how independent sources should be represented within the model.* One may ask if this prescription, pertaining to the mathematical description of the experiment, suitably represents the sources in the experiment? As argued in the main work on this matter, the experimentalist does not *declare* that she prepares a tensor product state; rather, she *concludes* (from previous experiments) that her sources produce systems with this mathematical description. In the bilocal scenario, this conclusion can be reached when assuming quantum theory: if Alice, Bob, and Charlie doubt that the systems they share have a joint state of the form $\sum_\lambda p(\lambda)(\sigma^{|\lambda}_{\mathrm{AB}_1} \otimes_K \sigma^{|\lambda}_{\mathrm{B}_2\mathrm{C}})$, they could always perform local tomography as a first way to certify their sources, before moving on to more robust techniques like heralding or self-testing. But this cannot be done when assuming RQT, because it is a non-locally tomographic theory, so certification via state tomography must be global, which is outside of what can be done in a local experiment. Therefore, the parties cannot certify the form of the joint state in Eq. (4) by performing local measurements; they must accept it as an assumption within the model. Nevertheless, the goal of the experiment is to invalidate a model. In that regard, accepting an assumption within the model under scrutiny is, at minimum, ill-advised and, at worst, fallacious. For these reasons, one may seek a description of independent sources external to the model; one that can be checked solely from the outcomes statistics.

Definition 2 (Operational Independence)**.** *In QT and RQT, the joint state Ψ_{XY} of a bipartite system shared by parties* X *and* Y *is said to be* ***operationally independent*** *if and only if any pair of local measurements performed on it results in uncorrelated measurement statistics. That is, for every pair of POVMs $\{X_\alpha\}_\alpha$ and $\{Y_\beta\}_\beta$ respectively defined over the local Hilbert spaces associated with parties* X *and* Y*, and producing the respective outcomes α and β, the resulting joint distribution of α and β,*

$$p(\alpha, \beta) = \mathrm{tr}\left[\Psi_{\mathrm{XY}}\left(X_\alpha \otimes_K Y_\beta\right)\right] , \tag{6}$$

is always uncorrelated, i.e., there exists some distributions $p(\alpha)$ and $p(\beta)$ such that $p(\alpha, \beta) = p(\alpha)p(\beta)$.

It is then instrumental to see which values of $\mathcal{I}$ are obtainable in the experiment 1 when one replaces Renou et al's assumption of product-state independence for operational independence. Let

$$p_{RQT'}(a, b, c|x, z) = \mathrm{tr}\left[\Psi_{\mathrm{AB_1B_2C}}(A_{a|x} \otimes_K B_b \otimes_K C_{c|z})\right] , \tag{7}$$

be the model of a bilocal experiment assuming RQT and operationally independent sources. There is actually an ambiguity here, operational independence can be certified by the parties performing the measurement or by the referees who locally prepare the two bipartite systems. In the first case, it means that for any quadruple of POVMs $\{A^{\mathrm{A}}_\alpha\}_\alpha$, $\{B^{\mathrm{B_1}}_\beta\}_\beta$, $\{\tilde{B}^{\mathrm{B_2}}_\gamma\}_\gamma$ and $\{C^{\mathrm{C}}_\delta\}_\delta$ representing local measurements performed of the systems A, $\mathrm{B_1}$, $\mathrm{B_2}$, and C (Bob must locally measure each of his systems to avoid inducing correlations from his measurements), the distribution

$$p(\alpha, \beta, \gamma, \delta) = \mathrm{tr}\left[\Psi_{\mathrm{AB_1B_2C}}\left(A^{\mathrm{A}}_\alpha \otimes_K B^{\mathrm{B_1}}_\beta \otimes_K \tilde{B}^{\mathrm{B_2}}_\gamma \otimes_K C^{\mathrm{C}}_\delta\right)\right] , \tag{8}$$

is uncorrelated over the partition $(A\mathrm{B_1})(\mathrm{B_2}C)$, i.e. $p(\alpha, \beta, \gamma, \delta) = p(\alpha, \beta)p(\gamma, \delta)$. Whereas in the second case, it means that for any pair of POVMs $\{X^{\mathrm{AB_1}}_\alpha\}_\alpha$ and $Y^{\mathrm{B_2C}}_\beta$ representing the local measurements performed on the systems $\mathrm{AB_1}$ and $\mathrm{B_2C}$, the distribution

$$p(\alpha, \beta) = \mathrm{tr}\left[\Psi_{\mathrm{AB_1B_2C}}\left(X^{\mathrm{AB_1}}_\alpha \otimes_K Y^{\mathrm{B_2C}}_\beta\right)\right] , \tag{9}$$

is uncorrelated. We have shown in the main paper that this ambiguity does not matter for our purposes; the following will hold for both cases. Here, we focus on the second possibility, which is the most constraining one, allowing fewer states $\Psi_{\mathrm{AB_1B_2C}}$ to be regarded as the joint state of two independent systems.

Passing the model from Eq. (3) to Eq. (7) amounts to allowing the two states in Eq. (3) to be mildly entangled, but the reader should not be fooled: this is a purely non-physical "entanglement of representation" induced by working within a non-algebraically closed field[2]. This weak form of entanglement is a speck of the maths: only the written-down formula for the state is manifestly non-separable, but this non-separability exists only in the Hilbert-space formalism; it never manifests itself physically. Indeed, despite the state being entangled, operational independence guarantees that this kind of entanglement can never result in observable correlations (let alone nonlocality) between the two systems produced by the two independent sources. Yet, the presence of this 'unusable' kind of entanglement allows one to attain the quantum bound.

Proposition 2. *In the Renou et al. experiment (Fig. 1), when the product-state independence assumption (definition 1) is replaced with its operational counterpart (definition 2), i.e., when the model Eq.* (3) *is replaced by Eq.* (7)*, there exists a strategy in RQT such that*

$$\mathcal{I}\big(\{p_{RQT'}(a, b, c|x, z)\}_{a,b,c,x,z}\big) = 6\sqrt{2}\,. \tag{10}$$

Thus, by comparing Equations (2) and (4), one observes that the difference in $\mathcal{I}$ between QT and RQT vanishes! Since the proof is by construction, it can also be verified computationally. In [14], we compute $6\sqrt{2}$ for RQT analytically to provide independent confirmation.

One may think that there is still a way to distinguish the two theories, by considering a different game $\mathcal{I}$. Unfortunately, we also rule this out. This is because our proof of the equivalency is at the levels of the statistics (i.e. for every $p_{quant.}(a, b, c|x, z)$ we find a $p_{RQT'}(a, b, c|x, z)$ such that $p_{RQT'}(a, b, c|x, z) = p_{quant.}(a, b, c|x, z)$.).

Furthermore, one may try to salvage the Renou et al. conclusion by considering a different network. Unfortunately this is not the case: in the technical manuscript, we prove a general no-go theorem which shows that for every possible network, when the independent sources are defined operationally, there always exists a RQT strategy which yields identical measurement statistics to every QT strategy.

Our proof technique consists in embedding QT into the reals via RNQT (see our other QPL2026 submission) and then showing that the R-product can be exchanged for Kronecker product for computing measurement probabilities.

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On the experimental falsification of Real Quantum Theory

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Abstract

The authors of [Nature 600, 625–629 (2021)] proposed an experiment for quantum theory whose statistics are incompatible with real quantum theory—a candidate theory for physical reality that only uses real numbers. The authors claim it constitutes an experimental falsification of real quantum theory. Here we show that their definition of independent sources is only verifiable in quantum theory. Once one replaces it with an operational and verifiable one, we prove that every conceivable quantum experiment has an explanation in real quantum theory.

Real Quantum Theory (RQT) is obtained by removing all complex quantities from the usual (complex Hilbert-space) formulation of quantum theory (QT). In the mixed-state formulation, this corresponds to restricting the (complex) Hermitian matrices that represent density operators, observables, and positive operator-valued measures (POVMs) to (real) symmetric matrices. All other elements of the formalism remain unchanged: the rule for assigning probabilities to a state and effect pair (the Born rule), the state-update rule (the Lüders rule), and the rule for combining systems—the Kronecker product—are exactly the same as in QT. It was put forward and developed from the 90s [1–3] as a toy model to shed light on the relevance of complex numbers in QT.

A key feature of RQT is that the Kronecker product $\otimes_K$ is still used as the matrix representation of the system-combination rule. This choice leads to an important qualitative difference with QT: RQT is no longer a locally tomographic theory [3]. Unlike QT, the state of a multipartite system cannot be reconstructed by performing local measurements on each subsystem in RQT. Despite this extensively studied theoretical difference[1], RQT can nevertheless reproduces all experimental predictions of QT in single-partite [6] and bipartite (Bell) scenarios [2, 7, 8].

More precisely, any experiment that admits a (quantum) model using complex matrices also admits a model using real matrices of twice the size. Keep in mind that this apparent mismatch in dimensions is neither a qualitative nor a testable difference between RQT and QT: On the one hand, complex vector-space dimensions amount to two real vector-space dimensions. On the other hand, the dimension of a quantum system cannot be experimentally certified [9].

To illustrate this point, consider a single party X performing a measurement on a system. In QT, the system is described by a density matrix ρ_{X} and the measurement by a POVM $\{X_{\alpha|\mu}\}_\alpha$. The measurement procedure is chosen among several possibilities depending on the party's strategy, here represented by her setting μ. The probability of obtaining outcome α given the chosen setting μ is then given by the Born rule $p(\alpha|\mu) = \mathrm{tr}\left[\rho_{\mathrm{X}} X_{\alpha|\mu}\right]$. For this scenario, RQT provides an experimentally indistinguishable alternative to QT: the predicted distributions $p(\alpha|\mu)$ are identical irrespective of the model requiring ρ_{X} and $X_{\alpha|\mu}$ to be complex or real matrices; for every distribution obtained with complex matrices, the same distribution can also be one obtained with real matrices [2, 6].

Similarly, for two local parties X and Y measuring a shared system in state Ψ_{XY}, X's procedure is locally described by a POVM $\{X_{\alpha|\mu}\}_\alpha$ while Y's is described by $\{Y_{\beta|\nu}\}_\beta$. The obtained joint distribution of their respective outcomes α and β conditioned by their respective settings μ and ν is given by $p(\alpha, \beta|\mu, \nu) = \mathrm{tr}\left[\Psi_{\mathrm{XY}}(X_{\alpha|\mu} \otimes_K Y_{\beta|\nu})\right]$. Once again, any such distribution can be reproduced using either complex or real matrices [7, 8]. Consequently, Bell inequalities cannot be used to distinguish QT from RQT in a theory-independent manner. Every Bell inequality violation achievable with complex quantum operators has an equivalent realisation using purely real operators. In this sense, the QT and RQT descriptions of Bell experiments cannot be tested against each other in the same way that local hidden-variable models can be experimentally ruled out in favour of QT.

This led Gisin [10] and others to wonder whether QT and RQT are actually experimentally distinguishable via local experiments or whether these two theories differ only in their mathematical formulation. Without violating the predictions of QT, is it that there exist more complicated local scenarios that do not admit RQT description, or is it that any local experiment can always be described using real matrices?

This question was recently addressed by Renou et al. [11]: they claimed that, in the bilocal (or entanglement-swapping) scenario [12] –a tripartite network involving two independent sources– the predictions of QT and RQT differ. In particular, they showed that this difference can be revealed through the violation of a Bell-like inequality. According to their analysis, the maximal violation predicted by RQT is strictly smaller than the one predicted by QT.

Because the inequality provides a theory-independent bound, observing a value larger than the RQT maximum would imply that RQT is experimentally falsified by an experiment consistent with QT. At the very least, it would show that RQT models

[1] See e.g., Refs. [4, 5] for recent works on the matter.

are not adequate to describe this bilocal scenario, in the same sense that models with local hidden variables are not adequate to describe the Bell scenario. Subsequent experiments performed by several groups have reported violations consistent with the QT prediction, thereby supporting the claim that the RQT description is inadequate in this context [13, 14].

However, this result crucially relies on how the system-composition rule is represented in RQT. Some of the original authors have since discussed the importance of this assumption for interpreting the experiment in a recent work [15]. From a theoretical perspective, the choice of composition rule itself has recently been questioned even within QT [16]. In parallel, several teams including ours have investigated alternative real-matrix formulations based on different composition rules that remain consistent with the predictions of QT [17–19].

In this article, we revisit the claimed falsification of RQT and prove that it relies on an experimentally untestable assumption. Unlike the Bell scenario, the considered experimental protocol uses *two independent sources*. But how does one describe independent sources within a theory? And can this description be certified in a (semi-)device-independent manner?

For both QT and RQT, there is a first way to describe 'independent sources', which is relative to the operator representing the state of the bipartite system produced by the two sources:

Definition 1 (Product-State Independence) *A state on the joint Hilbert space of two parties X and Y, is said to be* ***(Kronecker-)product-state independent*** *if and only if it can be written in the form*

$$\rho_{\mathrm{X}} \otimes_K \rho_{\mathrm{Y}}, \tag{1}$$

where ρ_{X}*,* ρ_{Y} *are two density matrices on the Hilbert spaces associated with parties* X *and* Y*, respectively.*

This definition should be contrasted with the operational description of independence, which is relative only to the statistics of local measurements:

Definition 2 (Operational Independence) *A state on the joint Hilbert space of two parties* X *and* Y*, is said to be* ***operationally independent*** *when any pair of local measurements performed on it results in independent measurement statistics.*

Specifically, let Ψ *be the joint state of two sources, it is operationally independent if and only if for every POVM* $\{X_{\alpha|\mu}\}_\alpha$ *and* $\{Y_{\beta|\nu}\}_\beta$ *applied by, respectively, parties* X *and* Y*, depending on settings* μ *and* ν *and producing outcomes* α *and* β*, the resulting joint distribution*

$$p(\alpha, \beta|\mu, \nu) = \mathrm{tr}\left[\Psi(X_{\alpha|\mu} \otimes_K Y_{\beta|\nu})\right] \tag{2}$$

always factorises as a product of marginals,

$$p(\alpha, \beta|\mu, \nu) = p(\alpha|\mu)p(\beta|\nu)\,. \tag{3}$$

These two definitions of independence aim to translate an experimental fact (independent sources) into a mathematical constraint on the model. Inevitably, such constraints cannot be phrased without assuming the mathematical description of the states and, for definition 2, also the one of measurements and the probability rule. Yet, while product-state independence (definition 1) is an assumption made on the mathematical description of the model (i.e., independent sources have a state in a pure Kronecker-product form; this assumption applies to the right-hand side of Eq. (2), *within the Born rule*), operational independence (definition 2) is an assumption made on its predictions (i.e., independent sources lead to independent outcomes for all commuting measurements; this assumption applies to the left-hand side of Eq. (2), *outside of the Born rule*)[2]. Hence, while product-state independence is defined from an a priori on the (theory-dependent) mathematical structure of the model, operational independence is rather inferred from an a priori on the (theory-independent) correlations between the observed outcomes.

In QT, definitions 1 and 2 coincide. This is to say, a state Ψ is operationally independent if and only if it is product-state independent. Contrastingly, a state of RQT satisfies definition 2 if, but not only if, it satisfies definition 1. As a consequence, defining independent sources in RQT by direct analogy with the QT formalism (i.e., by assuming that definition 1 can also be applied to RQT) is misleading: although independent sources can be defined via definition 1 in QT 'for free', doing the same in RQT requires 'paying the price' of excluding certain states that nevertheless satisfy operational independence. But is the price justified? Operationally independent states are never (observably) correlated by definition, so for what reason would these states not be describing the state of two independent sources, if not an attachment to how things are in the QT model? This extra restriction constitutes the implicit assumption made in the analysis of Renou et al. [11].

Abandoning the assumption of product-state independence in favour of operational independence amounts to allowing the state of a system produced by several independent sources to be mildly entangled. But the reader should not be fooled: this is a purely non-physical "entanglement of representation" induced by working within a non-algebraically closed field [1]. This weak form of entanglement is an artefact of the maths: only the written-down formula for the state is manifestly non-separable, but this non-separability exists only in the Hilbert-space formalism; it never manifests itself physically. Indeed, despite the state being entangled, operational independence guarantees that this kind of entanglement can never result in observable correlations (let alone nonlocality) between the two systems produced by the two independent sources. One interpretation of these properties within RQT, is that the measurements are less capable at detecting correlations than in QT.

A recent study has shown that a substantial subset of these mildly entangled states does not affect the conclusion of Renou et al. [15]. In the technical section below, however, we demonstrate how our previous work on a real representation of quantum theory [18] allows us to identify precisely the class of operationally-independent-yet-non-product states required to reproduce every prediction of QT within RQT. Unlike

[2] One may remark that the Kronecker-product form is assumed for the POVM effects in Eq. (2). But this is a different situation than with the states: the Kronecker-product form of the effects actually comes from a different heuristic than independence, namely locality. The operations performed by two local parties must always commute, and, assuming finite dimensions, the commuting measurements of X and Y can always be represented by measurements in a Kronecker-product form without loss of generality. Requiring the effects to be in product form thus come from a different constraint which precedes the independence constraint.

previous works, which focused on specific scenarios to determine whether an equivalent RQT description exists, our construction is fully general and applies to arbitrary network scenarios. In other words, the presence of this 'unusable' form of entanglement in RQT is exactly what enables the theory to reproduce the predictions of QT for any conceivable experiment.

Theorem 1 *Given the operational notion of independent systems, any experiment that has an explanation in QT also has one in RQT.*

This theorem implies that the only way to distinguish RQT from QT in an experiment would be to observe a violation of the predictions of QT, and not the other way around. This means that RQT cannot be experimentally falsified.

Our result indicates that, for RQT, the (model-dependent) product-state independence assumption puts a stronger constraint on the model than the (model-independent) operational independence, and that the former assumption must necessarily be made in order to 'falsify' RQT models without observing a violation in the predictions of the equivalent QT models. Indeed, the states required to reproduce the same violations of local inequalities on QT product states always exist within RQT. But while they obey the same independence heuristic as the QT product state, they are not RQT product states. This occurs because RQT product states do not exactly coincide with the notion of independent sources in RQT.

On a broader scope, our results highlight the complexity of testing a toy theory against quantum theory. The very accommodating formalism of quantum theory cannot be taken for granted in alternative (foil) theories. Before proposing experiments to challenge such alternatives, one must first verify that each element of the alternative framework carries the same operational meaning as its counterpart in QT. Without doing so, the foil is condemned to remain a toy theory, and its scope cannot pretend to extend from the blackboard to the lab bench.

One can still ask whether the model-dependent assumption could itself be certified. I.e., could one experimentally demonstrate that a nature governed by RQT somehow restricts the states accessible by independent systems to only the product states? Within a QT description, the parties could attempt to do so by performing various local measurements on their respective subsystems and then comparing their measurement statistics gathered from multiple experimental runs. From the data, they could conclude they were indeed in the presence of a product state. In the RQT description, however, this procedure cannot be attempted since the theory is not locally tomographic. As we further show in the technical section, the RQT states required to reproduce the predictions of QT are locally indistinguishable from RQT product states. This issue is discussed further in Appendix A.

As a direction for future work, it would be interesting to explore whether there exists a stronger notion of operational independence that would coincide with the product-state assumption for RQT. And, if it exists, whether this stronger notion still coincides with the product-state assumption for QT.

Another interesting direction would be to identify local inequalities which hold for QT but are violated for RQT models. Our result does not rule out the existence of experiment that has an explanation in RQT but not in QT. These would allow for an experimental verification of RQT at the expense of experimentally observing a violation of QT.

Framework and Setup

This technical section include a high-level proof of the main statement as well as a more rigorous formulation of the claims of the main text. The technical core of the proof relies on the theory of real matrices isomorphic to QT we introduced in a previous work [18]. We refer to this framework as Real-Number Quantum Theory (RNQT). The overall proof strategy is to show that RNQT models (which are isomorphic to QT models) are always contained within RQT models, which explains why RQT predictions can always explain QT predictions.

As a warm-up, we first revisit the cases where an RQT explanation is already known to exist, namely single-partite [2, 6] and local models [7, 8]. Recovering these proofs using RNQT is almost immediate: one simply needs to identify the subset of states and effects of RQT that corresponds to the states and effects of RNQT.

The main difficulty to overcome when extending the argument to arbitrary network scenarios (which we shall call 'multilocal' in opposition to local scenarios) is the treatment of independent sources. In particular, assuming product-state independence is insufficient to guarantee that RNQT appears as a submodel of RQT; one needs to extend it to operational independence.

Once this point is recognised, the general proof consists of the following steps: 1) Product-state-independent RQT states cannot be locally distinguished from product-state-independent RNQT states, and both classes satisfy operational independence. Therefore, replacing the former by the latter entails no loss of generality. 2) The existence of these product-state-independent RNQT states within the RQT state space is sufficient to ensure that the RQT model contains an RNQT submodel. 3) Since the RNQT submodel is isomorphic to the corresponding QT model, any bound on local inequalities achievable within complex quantum theory can also be achieved within a real quantum theory model that assumes operational independence.

The detailed constructions required for the proof (how to build the RNQT model and how it is isometrically isomorphic to a QT model) are largely technical. For this reason, these have been delayed into two supplementary appendices.

Preliminaries and the definitions of independent sources

Let us first formalise our language: by a '**theory**', we mean here the general set of rules believed to provide an accurate mathematical description of physical phenomena. Following the recent RQT literature, we limit the scope of the theory to a definition of the set of accessible states of systems, the set of possible measurements performed on a system, the prescription of how to associate settings and outcomes to the measurements, and the rule to obtain probabilities out of a state and effect pair. Other considerations –like the evolution rule (the 'dynamics' of the theory)– are disregarded[3].

[3] With the exception of Appendix A, where RQT evolutions are considered in the discussion.

This is because the theories under consideration in this paper –quantum theory (QT), real quantum theory (RQT), and real number quantum theory (RNQT)– are all *foils* of quantum theory: alternative theories that are sufficiently close to be essentialised as variations of the density-matrix formalism of quantum theory. Passing from one theory to the other simply amounts to changing the constraints defining the set of valid states and the set of valid effects, and, for RNQT, also to changing the composition rule.

Definition 3 *In the following,* ***Quantum Theory (QT)*** *refers to the usual quantum-mechanical density-operator description of systems with a finite number of degrees of freedom, as described for example in Ref. [20].*

Real Quantum Theory (RQT) *refers to the restriction of QT under the following rule: For a chosen basis, the operators of RQT are only the operators of QT that are represented as real matrices in that basis.*

Real-Number Quantum Theory (RNQT) *refers to the further modification of RQT under the following rules: 1) Every density operator and effect must commute with a given matrix J such that $J^2 = -\mathbb{1}$; 2) The state and effects composition rules are respectively changed by the bilinear forms $\otimes_{\mathrm{R}}$ and $\otimes_{\mathrm{R}^*}$. (The exact forms of J, $\otimes_{\mathrm{R}}$, and $\otimes_{\mathrm{R}^*}$ are representation-dependent specificities of RNQT explicited in Appendix B.)*

In other words, RQT can always be seen as a sub-theory of QT, and RNQT as a sub-theory of RQT with modified composition rules. Nevertheless, while RNQT might be 'smaller' than RQT, it has been proven to yield the same predictions as QT [17, 18]; the difference is that a system with d degrees of freedom in QT is described as a system with $2d$ degrees of freedom in RNQT. This is not a problem as the number of degrees of freedom of a system is not something that can be experimentally bounded [9], and so it is treated as a parameter of the model of an experiment.

By a '**model**', we mean here the specific form of the mathematical expression used to generate the probability distribution associated with a given experiment. It prescribes which and how systems are prepared at their source(s), how they are split into subsystems, and how these subsystems are subsequently distributed among the 'labs' of the 'parties' that will measure them. At the formal level, the structure of the model is the same whether the underlying theory is QT, RQT, or RNQT.

For instance, consider an experiment where Alice and Bob share a bipartite system. Irrespective of whether the underlying is QT, RQT, or RQNT, the outcomes-given-settings probability distribution is modelled by an expression of the form

$$p(a, b|x, y) = \mathrm{tr}\left[\rho_{\mathrm{AB}}\, F_{a,b|x,y}\right] \,, \tag{4}$$

where ρ_{AB} is the state of the shared system, whereas $F_{a,b|x,y}$ is the POVM element (the effect) representing Alice observing outcome a and Bob observing outcome b from a POVM $\{F_{a,b|x,y}\}_{a,b}$ (the measurement) they chose according to their respective settings x and y. In this case, the information relative to the model of the experiment is the fact that there are two parties, hence two Hilbert spaces labelled by A and B, two pairs of random variables (a, x) and (b, y) respectively representing the outcome and setting pair of each party.

More generally, the model assumes a specific underlying theory, but specialises it to the experimental context. This specialisation takes the form of extra constraints: it can be a constraint on the dimensions of the systems involved in the experiment, or, as we will see next, on the locality of parties, or on the independence of sources.

By '**locality**', we mean here the constraint imposed on the model to capture the experimental fact that the parties operate in spacelike-separated labs[4]. Under this condition, the parties cannot influence each other's actions. In the model, this requirement is expressed through the commutativity assumption: any measurement performed in one laboratory must commute with measurements performed in another laboratory.

Following the previous literature on RQT, we also assume that all systems have finite dimensions for simplicity. This will allow us to represent commuting sets of measurements as pairs of POVMs in tensor product [5]. This avoids many of the subtleties that arise when equating locality with the commutativity assumption[6].

For example, in an experiment where Alice and Bob are in local labs but share a bipartite system, the model of the experiment is again formally the same for QT, RQT and RNQT:

$$p(a,b|x,y) = \mathrm{tr}\left[\rho_{\mathrm{AB}}\left(A_{a|x}\otimes B_{b|y}\right)\right]\ . \tag{5}$$

In this case, the joint POVM is now $\{A_{a|x}\otimes B_{b|y}\}_{a,b}$ where $\{A_{a|x}\}_a$ (respectively, $\{B_{b|y}\}_b$) is the local POVM picked by Alice (resp., Bob) in her (resp., his) lab, and where $\otimes$ is (a matrix representation of) the tensor product of the theory at hand (for QT and RQT it is represented by $\otimes_K$, the Kronecker product; for RNQT it is represented by another bilinear form noted $\otimes_{\mathrm{R}^*}$).

Finally, by '**independent sources**', we mean here the constraint imposed on the model to reflect the experimental fact that certain systems are assumed uncorrelated. For example, when the systems are locally prepared in separate laboratories. As discussed in the main text, this idea has two distinct translations into a mathematical constraint on the states modelling the independent sources. One possibility is to impose independence directly at the level of the model, requiring the states to factor into a product state; we refer to this as '*product-state independence*' (definition 1). The alternative is to define independence operationally, by requiring the measurement statistics to be uncorrelated; we refer to this as '*operational independence*' (definition 2).

Definition 4 (Independent sources) *Let there be a model that assumes an n-partition between parties A(lice), B(ob), ..., and underlaid by a theory in which the tensor product is noted $\otimes$ such that the distributions predicted by the model have the form*

$$p(a,b,...|x,y,...) = \mathrm{tr}\left[\rho_{\mathrm{AB}...}\left(A_{a|x}\otimes B_{b|y}\otimes\ldots\right)\right] \tag{6}$$

*for any joint state of the shared system $\rho_{\mathrm{AB...K}}$ and for any POVMs $\{A_{a|x}\}_a$, $\{B_{b|y}\}_b$, ... Then, the model assumes **product-state independent states** with respect to the n-partition if and only if the joint state has the form*

$$\rho_{\mathrm{AB}...} = \rho_{\mathrm{A}}\otimes\rho_{\mathrm{B}}\otimes\cdots, \tag{7}$$

[4]Or at least a situation where the parties refrain from performing measurements that affect other parties' systems in any observable way.

[5]Representing commuting measurements as tensor-product measurements requires the Tsirelson conjecture to hold, which in practice amounts to restricting the model to systems of finite dimension or to measurements with finitely many outcomes [21].

[6]See some relevant reviews [22–24] of these issues and the sources within.

where ρ_{A} is a valid state of Alice's subsystem, ρ_{B} is one of Bob's, etc.

Alternatively, the model assumes ***operationally independent states*** *with respect to the n-partition if and only if the distribution is uncorrelated for any local measurement with respect to the n-partition. That is, if and only if there exists n conditional distributions $p(a|x)$, $p(b|y)$, ... such that*

$$p(a, b, ...|x, y, ...) = p(a|x)p(b|y)... \,. \tag{8}$$

Like locality, remark that independence is defined with respect to a partitioning of the subsystems. Be aware that some models tend to have different partitioning between systems preparation and the subsequent local measurements. For example, in the Bell scenario, the bipartite state is global during preparation, i.e. it comes from a single independent source prepared w.r.t. the partition (AB), whereas there are two measurements local with respect to the partitions (A)(B). A more complicated scenario is the bilocal scenario [12] (the one considered by Renou et al. [11]): it models a four-partite system which is prepared by two independent sources w.r.t. the partition $(\mathrm{AB}_1)(\mathrm{B}_2\mathrm{C})$ and then locally measured by three parties w.r.t. the partition $(\mathrm{A})(\mathrm{B}_1\mathrm{B}_2)(\mathrm{C})$.

As mentioned in the main text, the two meanings of 'independent sources' can be confused in models assuming QT (or the equivalent RNQT) without loss of generality. But the same cannot be said about RQT [1].

Proposition 5 *In QT, the independent sources assumption is equivalently represented by a model with either product-state or operationally independent states.*

In RQT, product-state independence is only sufficient for operational independence, making the independent sources assumption stronger when represented by a model with product states rather than operationally independent states.

Proof We prove it in the bipartite case; the multipartite case is a straightforward generalisation of the proof.

It is direct to see that product-state independence is sufficient for operational independence since

$$\mathrm{tr}\left[(\rho_{\mathrm{A}} \otimes_K \sigma_{\mathrm{B}})(A_{a|x} \otimes_K B_{b|y})\right] = \mathrm{tr}\left[\rho_{\mathrm{A}}\, A_{a|x}\right] \mathrm{tr}\left[\sigma_{\mathrm{B}}\, B_{b|y}\right] \,, \tag{9}$$

for every pair of local states and POVMs, leading to the operational independence condition $p(a, b|x, y) = p(a|x)p(b|y)$.

To show that the condition is necessary for QT, we can use the affine decomposition of bipartite states, $\rho_{\mathrm{AB}} = \sum_i q_i(\rho_i \otimes_K \sigma_i)$ where $\{\rho_i\}_i$ (respectively, $\{\sigma_i\}_i$) is a collection of local states of Alice's system (resp., Bob) and where $\{q_i\} \subset \mathbb{R}$ are affine coefficients. Injecting this into a local model, one gets

$$\mathrm{tr}\left[\rho_{\mathrm{AB}}(A_{a|x} \otimes_K B_{b|y})\right] = \sum_i q_i \mathrm{tr}\left[\rho_i A_{a|x}\right] \mathrm{tr}\left[\sigma_i B_{b|y}\right] = \sum_i q_i p(a|x, i)p(b|y, i) \,. \tag{10}$$

Because the choice of POVMs are arbitrary, one can always pick one such that there is a subset $\{A_{a'|x} \otimes_K B_{b'|y}\}_{a',b'} \subseteq \{A_{a|x} \otimes_K B_{b|y}\}_{a,b}$ of the POVM elements such that $A_{a'|x} \otimes_K B_{b'|y} = c_i\rho_i \otimes_K \sigma_i$ with c_i a positive constant. Because of that, the only way for $\sum_i q_i p(a|x, i)p(b|y, i) = p(a|x)p(b|y)$ to be true for any POVM is to require that $\{q_i\}_i$

only possesses one member, meaning that $\mathrm{tr}\left[\rho_{\mathrm{AB}}(A_{a|x}\otimes_K B_{b|y})\right] = p(a|x)p(b|y)$ only if $\rho_{\mathrm{AB}} = (\rho_{\mathrm{A}}\otimes_K \sigma_{\mathrm{B}})$ for some states ρ_{A} and σ_{B} of Alice and Bob.

For RQT, the above necessity proof no longer holds because the affine decomposition requires a complete number field; in other words, the affine decomposition cannot be performed over $\mathbb{R}$, as it may involve local matrices that are positive semi-definite and trace-one but that cannot be expressed as purely real matrices, making them invalid RQT states. An example of such a bipartite state of two two-dimensional systems is given by any state of the form [1]:

$$\begin{aligned}&\frac{1}{4}\left(I\otimes_K I-\alpha J\otimes_K J\right)\\&\qquad=\frac{1}{2}\left[\frac{1}{2}(I+i\sqrt{\alpha}J)\otimes_K\frac{1}{2}(I+i\sqrt{\alpha}J)\right]+\frac{1}{2}\left[\frac{1}{2}(I-i\sqrt{\alpha}J)\otimes_K\frac{1}{2}(I-i\sqrt{\alpha}J)\right]\ ,\end{aligned}\tag{11}$$

with $0\le\alpha\le 1$, $I=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ and $J=\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$. As emphasised by the presence of an i in the right-hand side of the equation, the decomposition can only be done using complex matrices, hence within QT. This state is separable when seen as a QT state, but entangled when seen as an RQT state. However, if the only accessible local POVMs are RQT ones (i.e., positive semi-definite real matrices), remark that $\mathrm{tr}\left[(J\otimes_K J)(A_{a|x}\otimes_K B_{b|y})\right]=0$ because any pair from $\{A_{a|x}\}_a$ and $\{B_{b|y}\}_b$ are real symmetric matrices, which are orthogonal to J since J is an antisymmetric matrix. Thus,

$$\mathrm{tr}\left[\frac{1}{4}\left(I\otimes_K I-\alpha J\otimes_K J\right)\left(A_{a|x}\otimes_K B_{b|y}\right)\right]=\frac{1}{2}\mathrm{tr}\left[A_{a|x}\right]\ \frac{1}{2}\mathrm{tr}\left[B_{b|y}\right]\ .\tag{12}$$

While this state is an entangled state –therefore not a product state–, it is yet an operationally independent one. This provides a counterexample to product-state independence being necessary for operational independence in the RQT case, which concludes the proof. □

Single source experiments have an RQT description

We start by reminding the reader that in experiments involving a single source and a single measurement, QT can always be alternatively explained by RQT. This has been known since the beginning of quantum theory and formally established by the seminal work of Stückleberg [6] (who defined a theory that can be seen as single-partite RNQT; see also the relevant but more abstract approach taken by Dyson [25]). This construction was later studied by McKague et al. as a way to prove that a single-partite RQT model always contains a sub-model isomorphic to a QT model [2]. We recover this result through the following (see Section B.1 for a proof).

Proposition 6 (Single-partite QT models are sub-models of single-partite RQT models.) *A single-partite QT model involving a single source producing a state ρ measured by a POVM $\{A_{a|x}\}_a$ can always be embedded into an RQT model of twice the matrix size.*

That is, there exists a pair of real-linear injective maps $(M,E):\mathbb{C}^{d_A\times d_A}\to\mathbb{R}^{2d_A\times 2d_A}$ such that for any QT state ρ, $M(\rho)$ is a valid RQT state, such that for any QT POVM $\{A_{a|x}\}_a$, $E(A_{a|x})$ is a valid RQT POVM effect and $\{E(A_{a|x})\}_a$ a valid RQT POVM, and such that every distribution of the QT model is mapped to one of the RQT model as:

$$\forall a:\quad \mathrm{tr}\left[\rho A_{a|x}\right]=\mathrm{tr}\left[M(\rho)E(A_{a|x})\right]\ .\tag{13}$$

The image of the QT model through the pair of maps (M, E) is what we refer to as the 'QT-image' of the RQT. The QT-image normally differs from the RNQT model by its system-composition rule. But since single-partite models do not involve the system-composition rule, these are the same. Hence, any single-partite, single-source RQT model always contains a sub-model isomorphic to a QT model of half the matrix size. It means that any distribution of outcomes obtained with QT states and effects could have been obtained with RQT states and effects of twice the matrix size. Therefore, any single-source, single-partite QT model can be alternatively explained by an RQT model, making the two indistinguishable from experimental statistics alone.

But what about multipartite models[7]? If these are viewed as global scenarios without any locality constraints, they fall under the umbrella of the above proposition. In the presence of locality and independence constraints, we face the problem that there does not exist a pair of maps that can preserve all pure Kronecker products of states and effects while keeping the distributions invariant. This is what motivated us to change the system-combination rule of RQT to define RNQT.

Proposition 7 (QT models are RNQT models.) *A n-partite QT model can always be embedded into an RNQT model of 2^n times the matrix size.*

That is, there exists a pair of real-linear injective maps $(M_{\mathrm{AB}\dots}, E_{\mathrm{AB}\dots}) : \mathbb{C}^{d_A d_B \dots \times d_A d_B \dots} \to \mathbb{R}^{2d_A 2d_B \dots \times 2d_A 2d_B \dots}$ that respectively map any valid QT state and effect to a valid RQT state and effect that commute with a given J matrix, such that every distribution predicted by the QT model has a real-matrix counterpart and such that the system-composition rules of QT are mapped to those of RNQT. I.e.,

$$M_{\mathrm{XY}}(\rho_{\mathrm{X}} \otimes_K \sigma_{\mathrm{Y}}) = M_{\mathrm{X}}(\rho_{\mathrm{X}}) \otimes_{\mathrm{R}} M_{\mathrm{Y}}(\sigma_{\mathrm{Y}})\,; \tag{14a}$$

$$E_{\mathrm{XY}}(X_{\alpha|\mu} \otimes_K Y_{\beta|\nu}) = E_{\mathrm{X}}(X_{\alpha|\mu}) \otimes_{\mathrm{R}^*} E_{\mathrm{Y}}(Y_{\beta|\nu})\,; \tag{14b}$$

where X and Y are any two subsets of the parties $\mathrm{AB}\dots$, ρ_{X} and σ_{Y} (respectively, $X_{\alpha|\mu}$ and $Y_{\beta|\nu}$) are valid QT states (resp. effetcs) of X and Y, and $(M_{\mathrm{X}}, E_{\mathrm{X}})$ denotes a local application of the maps on the X subset.

Using the above(see Section B.2.3 for a proof), finding the RQT sub-model equivalent to a QT model is reduced to finding a way to replace the RNQT composition rules on the left-hand side of Eq. (14) by the RQT composition rules, i.e., by the Kronecker product $\otimes_K$. This is easier than it appears because the replacement should not hold in general, but only within the Born rule and under the locality constraints. For instance, in the local (or Bell-like) scenario, since there is only a single source, one only needs to replace the composition rule on the effect side of every Born rule.

Proposition 8 (Local QT models are sub-models of local RQT models.) *A multipartite QT model with a single source producing a state $\rho_{\mathrm{AB}\dots}$ and n local measurements represented by the POVMs $\{A_{a|x}\}_a$, $\{B_{b|y}\}_b$, ..., can always be embedded into an RQT model of 2^n the matrix size.*

[7]Remark that this question arises because the tensor product structure is assumed. If the action of local parties were represented by subsets of commuting maps all defined over the same Hilbert space instead, we could simply use proposition 6 to prove that every QT model has an explanation as an RQT model. This would be the case because the pair of maps (M, E) is actually a *-algebra homomorphism and thus preserves sets of commuting maps.

That is, there exists a pair of real-linear injective maps $(M_{\mathrm{AB}...}, E_{\mathrm{A}} \otimes_K E_{\mathrm{B}} \otimes_K ...)$ *from* $\mathbb{C}^{d_A d_B ... \times d_A d_B ...}$ *to* $\mathbb{R}^{2d_A 2d_B ... \times 2d_A 2d_B ...}$ *such that* $M_{\mathrm{AB}...}(\rho_{\mathrm{AB}...})$ *is a valid RQT state of the multipartite system; such that* $E_{\mathrm{X}}(X_{\alpha|\mu}) \in \mathbb{R}^{2d_{\mathrm{X}} \times 2d_{\mathrm{X}}}$ *is a valid RQT effect on the local system associated with party* X *and* $\{E_{\mathrm{X}}(X_{\alpha|\mu})\}_\mu$ *is a valid RQT POVM for every party* X*; and such that every distribution of the QT model is mapped to one of the RQT model as:*

$$\begin{aligned} \forall a, \forall b, ... : \mathrm{tr}\left[\rho_{\mathrm{AB}...}(A_{a|x} \otimes_K B_{b|y} \otimes_K ...)\right] \\ = \mathrm{tr}\left[M_{\mathrm{AB}...}(\rho_{\mathrm{AB}...})(E_{\mathrm{A}}(A_{a|x}) \otimes_K E_{\mathrm{B}}(B_{b|y}) \otimes_K ...)\right] . \end{aligned} \tag{15}$$

The proof is presented in Section B.2.4. As mentionned, this does not imply that the mapping verifies $E_{\mathrm{AB}...}(A_{a|x} \otimes_K B_{b|y} \otimes_K ...) = E_{\mathrm{A}}(A_{a|x}) \otimes_K E_{\mathrm{B}}(B_{b|y}) \otimes_K ...$, only that they are equivalent when evaluated over every QT-image state. But this is sufficient to conclude that the RQT local models with a single source can reproduce the predictions of their corresponding QT models. Thus, this proposition generalises the known result [7, 8] that every single-source Bell-like scenario has an RQT explanation.

As a remark, notice that the POVMs can be more generally m-partite for $m \leq n$. However, one can always regroup the subsystems corresponding to the same local party in order to ensure that $m = n$. Also, notice that we use a matrix-size rescaling factor of 2 growing with the power of the number of parties, instead of the factor of 2 used in our previous work [18]. A factor that grows with the number of parties might appear unreasonable; worse, it appears to make the proposition partition-dependent. While this is correct in the absolute, it is merely a formulation chosen to simplify the proof of our main statement. Using the results of our previous work (specifically by replacing the $\otimes_R$-product on which this proposition is based by the isomorphic $\otimes_r$-product), one could prove a partition-independent version of it that only requires a factor of 2 between the dimensions of the RQT and QT systems. However, doing so would induce more technicalities to overcome, unnecessarily lengthening the proof.

Description of multiple sources in RQT

We saw that in experiments involving a single source, every model assuming QT can be explained by a model assuming RQT. But can the same be said about experiments with multiple sources? Renou et al. showed that in the bilocal scenario [12], the models assuming RQT cannot explain all distributions predicted by the corresponding models [11] *if independent sources are represented by product states*. However, as we argued above and in the main text, their product-state independence assumption is model-dependent.

In QT, one would use a model-independent approach to circumvent the issue, representing the joint state of two independent systems via an operational independence (definition 2) rather than product-state independence (definition 1; we also recalled these two notions in definition 4). As we showed in proposition 5, the same cannot be done in RQT: the product-state independence is stronger than operational independence. This implies that not all independent state-preparation procedures correspond to product states in RQT. Requiring the state of independent sources to be in a product form, therefore, amounts to restricting the set of states that the parties can prepare solely on the basis of model-dependent considerations.

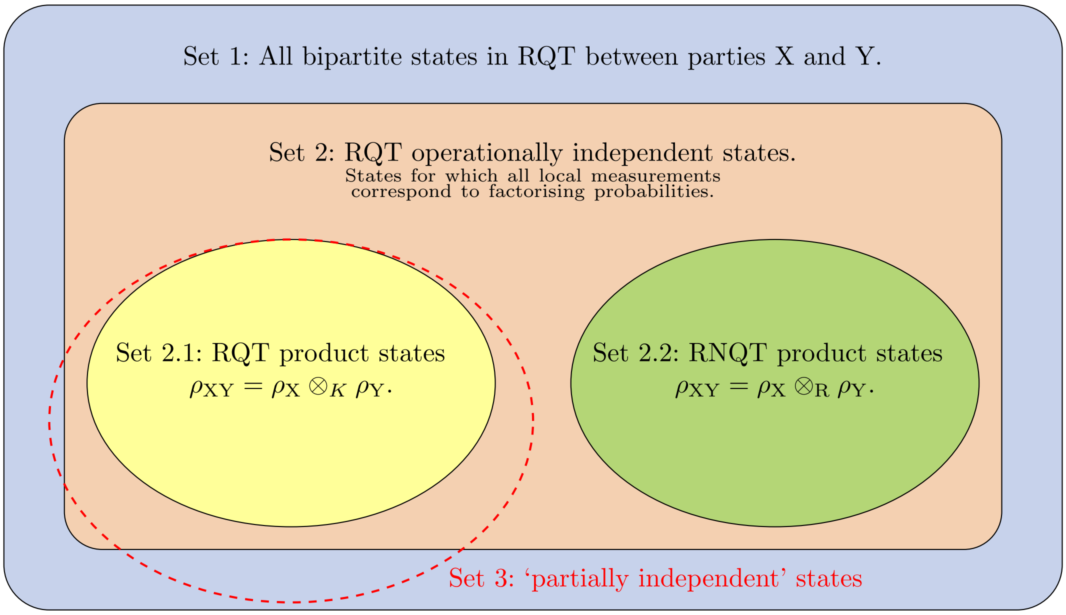

Fig. 1 Relations between sets of bipartite states in real quantum theory (RQT). Proposition 5 shows that in RQT, the sets 2 and 2.1 are disjoint. Proposition 9 further shows that many states are locally indistinguishable from the product states. An example of such locally equivalent non-product-yet-non-correlated states is provided by set 2.2: the image of QT product states in RNQT (set 2.2) is valid set of RQT states (within set 1) that are operationally independent (within set 2), but distinct from the set of product states (outside of set 2.1), yet locally indistinguishable from it. The red circle, set 3, sketches the relaxation of the product-state assumption considered in Refs. [11, 15]. Our argument is that the operationally motivated relaxation of set 2.1 should rather be set 2.

To what extent is it a problem? To begin with, it implies that the sources cannot be locally certified: in RQT, there are states that cannot be distinguished from the product states by local measurements alone. An important class of these are the operationally independent RQT states that are in the QT-image as in proposition 8. Even when they are entangled, these states must correspond to a QT product state because of proposition 5. However, the RQT product states must already correspond to the QT product states, so it implies that more than one state locally describes the same situation.

Proposition 9 *Let $\rho_{\mathrm{AB}\ldots}$ be an n-partite RQT state that is operationally independent as in definition 4. Then, there exists n single-partite RQT states ρ_{A}, ρ_{B}, $\ldots$ such that $\rho_{\mathrm{AB}\ldots}$ is locally indistinguishable from $\rho_{\mathrm{A}} \otimes_K \rho_{\mathrm{B}} \otimes_K \ldots$*

That is, for all RQT POVMs $\{A_{a|x}\}_a, \{B_{b|y}\}_b, \ldots$, the following equality holds:

$$\operatorname{tr}\left[\rho_{\mathrm{AB}\ldots}(A_{a|x} \otimes_K B_{b|y} \otimes_K \ldots)\right] = \operatorname{tr}\left[(\rho_{\mathrm{A}} \otimes_K \rho_{\mathrm{B}} \otimes_K \ldots)(A_{a|x} \otimes_K B_{b|y} \otimes_K \ldots)\right] . \quad (16)$$

Proof If $\rho_{\mathrm{AB}\ldots}$ is operationally independent, it verifies $\operatorname{tr}\left[\rho_{\mathrm{AB}\ldots}(A_{a|x} \otimes_K B_{b|y} \otimes_K \ldots)\right] = p(a|x)p(b|y)\ldots$. One can then set $\rho_{\mathrm{A}} = \operatorname{tr}_{\mathrm{BC}\ldots}[\rho_{\mathrm{AB}\ldots}], \rho_{\mathrm{B}} = \operatorname{tr}_{\mathrm{AC}\ldots}[\rho_{\mathrm{AB}\ldots}]$, etc. such that $p(a|x) = \text{tr}\left[\rho_A A_{a|x}\right]$, $p(b|y) = \text{tr}\left[\rho_B B_{b|y}\right]$, etc., which implies that $p(a|x)p(b|y)\cdots = \text{tr}\left[(\rho_A \otimes_K \rho_B \otimes_K \dots)(A_{a|x} \otimes_K B_{b|y} \otimes_K \dots)\right] = \text{tr}\left[\rho_{AB\dots}(A_{a|x} \otimes_K B_{b|y} \otimes_K \dots)\right]$. $\square$

This does not happen in locally tomographic theories like QT. In that case, Eq. (16) necessarily implies $\rho_{AB\dots} = \rho_A \otimes_K \rho_B \otimes_K \dots$ because operational independence implies product-state independence. But for RQT, the product states might be locally indistinguishable from entangled states.

A class of such locally indistinguishable entangled states is actually given by proposition 7: the image of QT product states, i.e., the RNQT product states which are of the form $\rho_A \otimes_R \rho_B \otimes_R \dots$, are indistinguishable from the corresponding RQT product states. Indeed, for every RQT POVMs, $\{A_{a|x}\}_a$, $\{B_{b|y}\}$, etc., one has (see the corollaries of lemma B.14):

$$\begin{aligned}\text{tr}\left[(\rho_A \otimes_R \rho_B \otimes_R \dots)(A_{a|x} \otimes_K B_{b|y} \otimes_K \dots)\right] \\ = \text{tr}\left[(\rho_A \otimes_K \rho_B \otimes_K \dots)(A_{a|x} \otimes_K B_{b|y} \otimes_K \dots)\right] \ . \end{aligned} \tag{17}$$

Yet, the state $\rho_A \otimes_R \rho_B \otimes_R \dots$ is actually entangled with respect to the Kronecker product (the bilinear map $\otimes_R$ is entangling; see the definition B.9 for the exact form of this map).

This proposition entails that assuming the systems have been prepared in a product state cannot be certified locally. In a local experiment, it only makes sense to require independent preparation procedures to yield operationally independent states, since being uncorrelated under every local measurement is the only experimentally accessible way to define independence to begin with[8]. Yet, Renou et al.'s proof *requires* the state of independent sources to be a product state. This uncertifiable assumption amounts to removing from the model every non-product state in the equivalence classes implied by proposition 9. But if a state is locally indistinguishable from a product state, why should it not correspond to a valid preparation procedure of independent sources as well, since it is as uncorrelated as a product state can be?

Aware of this limitation, the authors of Ref. [11] allowed separable but non-product states in their proof. Going further, some of these authors sought to strengthen the result by studying how allowing entangled states to correspond to independent preparation procedures narrows the gap between QT and RQT for the Renou et al. game [15]. They did not realise that some RQT entangled states –the operationally independent ones– correspond to uncorrelated systems and thus verify the heuristics of independent preparation procedures. These are precisely the states of independent systems needed to show (using our proof technique) that the RQT models can always explain the prediction of the QT models, as we will prove in the next section. What the Renou et al. result [11] actually showed, therefore, is that in the RQT model of entanglement swapping scenarios, the optimal state to saturate their bilocal game is not of the Kronecker-product form. The only conclusion that can be drawn from the

[8] One may still remark that, using global tomography, product states are distinguishable from non-product but operationally independent ones. While this is true, it does not imply that the latter states are correlated; it only means they differ from the former in a way that can be revealed only by using correlated measurements. This is nonetheless an intriguing property of RQT, and one might use it to study a stronger notion of independence using entropic or resource considerations.

experimental data [13, 14], therefore, is that the RQT description of the state in which the system started the experiment was not a product state.

Multiple sources experiments have an RQT description

As announced, RQT models with operationally independent sources can explain QT models. Our result holds for every possible network scenario, which we will call 'multilocal' as a shorthand for 'local scenarios with multiple sources'. Since single-source networks do not require defining independent sources, this result recovers the local and single-partite cases previously mentioned as special cases.

Proposition 10 (Multilocal QT models are sub-models of multilocal RQT models.) *A multipartite QT model involving n parties, $m \leq n$ measurements, and $p \leq n$ operationally independent sources, can always be embedded into an RQT model of 2^n-th the matrix size.*

Specifically, let $\mathcal{N} := \{A, B, ...\}$ be the set of parties, $|\mathcal{N}| = n$; let $\mathcal{M} = \{M_1, M_2, ..., M_m\}$ be the set of subsets of $\mathcal{N}$ representing how the parties are regrouped to perform the measurements (i.e., $M_i \subseteq \mathcal{N}$ and $\bigcup_{i=1}^m M_i = \mathcal{N}$); and let the sources be represented by a joint state $\rho_{\mathrm{AB}...}$ operationally independent w.r.t. the given partitionning into p. Then, there exists a pair of map $(M_{\mathrm{AB}...}, E_{\mathrm{AB}...})$ from $\mathbb{C}^{d_A d_B ... \times d_A d_B ...}$ to $\mathbb{R}^{2d_A 2d_B ... \times 2d_A 2d_B ...}$ such that $M_{\mathrm{AB}...}$ injectively sends the QT state to an RQT state while preserving operational independence; such that $E_{\mathrm{AB}...}$ surjectively sends the QT POVMs to RQT POVMs while preserving locality, (i.e., $E_{\mathrm{AB}...} = E_{M_1} \otimes_K E_{M_2} \otimes_K \cdots \otimes_K E_{M_m}$), and such that the pair of map preserves all inner products between any states and effects.

The proof combines all the points discussed so far: it relies on showing that for every QT model, the equivalent RNQT model, given by proposition 7, can be turned into a sub-model of an RQT model. To do so requires combining three observations: first, that the RNQT model is a sub-model of the RQT model but with different system-composition rules. Second, that the RNQT composition rule of effects $\otimes_{\mathrm{R}^*}$ can always be turned into a RQT composition rule $\otimes_K$, like in proposition 8. And third, that the RNQT product states correspond to operationally independent RQT states, as discussed in the previous section. The technical phrasing of this intuitive proof is provided in Section B.3.

This quite technical proposition immediately implies our main result.

Theorem 1 *Given the operational notion of independent systems, any experiment that has an explanation in QT also has one in RQT.*

Proof Since the dimension of systems is experimentally impossible to certify [9], every experiment which has a QT model can be alternatively explained by its corresponding RQT model of 2^n times the matrix size given by proposition 10. □

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Appendices

A Supplementary Discussion: On the impossibility to certify a Kronecker-product state in RQT

The conceptual step outlined in this article concerns how the state of independent systems must be represented in the model. For the model to accurately represent the experiment, it is not sufficient to require it to have a given mathematical shape that suggests independence; it must be supported by an operational heuristic that the reader is ready to accept as independence.

Our main proof –the impossibility to experimentally distinguish between QT and RQT without observing a violation of the predictions of QT– crucially hinges on this distinction; it holds essentially because the state of two independent sources can be taken to be of the form $\rho_{\mathrm{X}} \otimes_{\mathrm{R}} \rho_{\mathrm{Y}}$ instead of $\rho_{\mathrm{X}} \otimes_{K} \rho_{\mathrm{Y}}$. We have argued that these two states obey the same heuristic we use to operationally define independence. Therefore, the reason why only the states of the form $\rho_{\mathrm{X}} \otimes_{K} \rho_{\mathrm{Y}}$ have been considered to represent independent sources in previous works on RQT [2, 11, 15] comes from their mathematical shape being the same as that of independent sources in QT. To do so implicitly assumes the correctness of (complex) QT at the level of preparation, and then use it as an assumption to argue against RQT.

The operational viewpoint avoids this pitfall: Only the differences that manifest at the level of observable statistics from experimentally implementable procedures can be used to distinguish between competing physical theories. If the RQT states $\rho_{\mathrm{X}} \otimes_{\mathrm{R}} \rho_{\mathrm{Y}}$ and $\rho_{\mathrm{X}} \otimes_{K} \rho_{\mathrm{Y}}$ both obey the operational definition of independence (definition 2), it should be admitted that both can represent the state of independent sources. This ensures that comparisons between theories are fair and physically meaningful.

In this section, we shall discuss in greater detail how no local procedure can be used to certify that the sources have been prepared in a state in Kronecker product form within RQT. We will provide more details on how proposition 9 implies that any two experimenters who think that they have locally prepared two sources such that their joint state has a Kronecker-product form can be 'spoofed': no matter which RQT state $\rho_{\mathrm{X}} \otimes_{K} \rho_{\mathrm{Y}}$ they think they have obtained, in reality, it cannot be ruled out that they have prepared the corresponding non-Kronecker-product state $\rho_{\mathrm{X}} \otimes_{\mathrm{R}} \rho_{\mathrm{Y}}$ instead.

Assuming that two local parties X and Y have their system described by a joint state ρ_{XY}, how can they certify that they are in the presence of $\rho_{\mathrm{XY}} = \rho_{\mathrm{X}} \otimes_K \rho_{\mathrm{Y}}$? And, in particular, how can they certify that they have not been spoofed by the state $\rho'_{\mathrm{XY}} = \rho_{\mathrm{X}} \otimes_{\mathrm{R}} \rho_{\mathrm{Y}}$, knowing that these two states are locally indistinguishable. That is, for every RQT POVMs $\{X_\alpha\}_\alpha$ and $\{Y_\beta\}_\beta$ of, respectively, X and Y we have that

$$\mathrm{tr}\left[(\rho_{\mathrm{X}} \otimes_K \rho_{\mathrm{Y}})(X_\alpha \otimes_K Y_\beta)\right] = \mathrm{tr}\left[(\rho_{\mathrm{X}} \otimes_{\mathrm{R}} \rho_{\mathrm{Y}})(X_\alpha \otimes_K Y_\beta)\right] \ , \tag{18}$$

because both sides of the equation are equivalent to $\mathrm{tr}\left[\rho_{\mathrm{X}} X_\alpha\right] \mathrm{tr}\left[\rho_{\mathrm{Y}} Y_\beta\right]$. In RQT, full-state tomography can be performed only globally [3]. So this eventuality is directly ruled out as it would break the locality assumption (and bring the situation back to the multipartite local case, in which case a weaker form of our theorem applies; see proposition 8).

One may argue that the certified states are subsequently sent to spacelike-separated labs, but this reintroduces the loophole, as nothing guarantees that the state of the systems will not evolve into another locally indistinguishable state as soon as they exit the global lab, especially when random experimental noise is taken into account. Preventing this would indeed require studying the evolution equation, yet this evolution would not be local if it were to prevent the state from freely evolving into locally indistinguishable states —except, of course, if it is required by postulate, which amounts to reintroducing the product-state independence assumption. Similarly, one may argue that the parties can always perform a discarding, then a re-preparation of their systems, for example, through a pair of entanglement-breaking maps. But again, how are these procedures certified to be local? This only shifts the product-state assumption from the description of the joint state to the description of the joint measurement.

More generally, any such argument can be reduced to assuming that the parties can locally perform general operations –that is, CPTP maps; see Ref. [26] for their definition and properties in RQT– on their share of the state such that the overall operation is itself described by a pair of CPTP maps in Kronecker-product form. Suppose a CPTP map is applied on each subsystem before the parties try to certify their state ρ_{XY}, and suppose the local application of these maps is indeed in Kronecker product. That is, suppose the map $\mathcal{M}_{\mathrm{X}} \otimes_K \mathcal{N}_{\mathrm{Y}}$ is applied to the state before the parties apply their local POVMs. Then, the joint distribution of the local outcome observed by each parties is $p(\alpha, \beta) = \mathrm{tr}\left[(\mathcal{M}_{\mathrm{X}} \otimes_K \mathcal{N}_{\mathrm{Y}})\{\rho_{\mathrm{XY}}\}(X_\alpha \otimes_K Y_\beta)\right]$. However, the local evolution of the state can be seen as a local evolution of the effects (in other words, we can go to the Heisenberg picture), i.e.,

$$\mathrm{tr}\left[(\mathcal{M}_{\mathrm{X}} \otimes_K \mathcal{N}_{\mathrm{Y}})\{\rho_{\mathrm{XY}}\}(X_\alpha \otimes_K Y_\beta)\right] = \mathrm{tr}\left[\rho_{\mathrm{XY}}(\mathcal{M}^*_{\mathrm{X}}\{X_\alpha\} \otimes_K \mathcal{N}^*_{\mathrm{Y}}\{Y_\beta\})\right] \ , \tag{19}$$

where $\mathcal{M}^*_{\mathrm{X}}$ and $\mathcal{N}^*_{\mathrm{Y}}$ are the adjoint maps to $\mathcal{M}_{\mathrm{X}}$ and $\mathcal{N}_{\mathrm{Y}}$ (recall that, like the complex case, the above trace expression is a Hilbert-Schmidt inner product, that the states and effects are self-adjoint, and that a quantum evolution commutes with the adjoint in both the Heisenberg and Schrödinger pictures). The local evolution of an effect always yield a valid effect, therefore there exist two new POVMs $\{\widetilde{X}_{\alpha'}\}_{\alpha'} := \{\mathcal{M}^*_{\mathrm{X}}\{X_\alpha\}\}_{\alpha'}$

and $\{\widetilde{Y}_{\beta'}\}_{\beta'} := \{\mathcal{M}_{\mathrm{Y}}^*\{Y_\beta\}\}_{\beta'}$ such that

$$\mathrm{tr}\left[(\mathcal{M}_{\mathrm{X}} \otimes_K \mathcal{N}_{\mathrm{Y}})\{\rho_{\mathrm{XY}}\}(X_\alpha \otimes_K Y_\beta)\right] = \mathrm{tr}\left[\rho_{\mathrm{XY}}(\widetilde{X}_{\alpha'} \otimes_K \widetilde{Y}_{\beta'})\right] , \tag{20}$$

and we are back to our starting point: for every possible pair of POVMs, an expression like $p(\alpha', \beta') = \mathrm{tr}\left[\rho_{\mathrm{XY}}(\widetilde{X}_{\alpha'} \otimes_K \widetilde{Y}_{\beta'})\right]$ will produce the same outcome distribution regardless of whether $\rho_{\mathrm{XY}} = \rho_{\mathrm{X}} \otimes_K \rho_{\mathrm{Y}}$ or $\rho_{\mathrm{XY}} = \rho_{\mathrm{X}} \otimes_{\mathrm{R}} \rho_{\mathrm{Y}}$. No matter the amount of local evolutions and of pre- and post-treatment, if the starting state of the experiment, say the electrical noise in the lasers used to produce the systems in each local lab or the state of the universe after the Big Bang, was not assumed a priori to be a Kronecker-product state, there is no way for the experimentalists to declare with certainty that the joint state of their systems will eventually have this form.

Hence, it does not really matter whether the local operations represent the subsystems locally evolving while being taken into spacelike-separated regions, a party replacing her share of the system with another system in her local lab, or anything else. Without assuming that the initial state of the system was a Kronecker-product state, no subsequent local operation on it, from re-preparation to discarding, can certify that its new state will be of the Kronecker-product form; any further attempt to complicate further the certification scheme, for example by using entanglement-assisted operations, by heralding the sources, or by involving complicated network scenarios involving multiple intermediate parties, will always fail; that is to say, any such experiment with a valid QT explanation, will also have a valid RQT explanation, thus forbidding the possibility to falsify RQT.[9]

What to make of an experiment like that of Renou et al. [11] then? In regard to our main theorem, in the case of RQT, their result can be seen as an example of a bilocal game that predicts a difference in winning statistics depending on whether the initial state of the sources was of the form $\rho_{\mathrm{X}} \otimes_K \rho_{\mathrm{Y}}$ or $\rho_{\mathrm{X}} \otimes_{\mathrm{R}} \rho_{\mathrm{Y}}$. This difference can be witnessed by a Bell-like inequality: the maximum violation achievable with a $\otimes_K$-product state is strictly lower than the one achievable with an $\otimes_{\mathrm{R}}$-product state (which is equivalent to the violation one can observe in QT). If the lower bound had been observed, this experiment would have not only been a falsification of quantum theory, but it would have shown that Nature, in its RQT description, only allows Kronecker-product states. Of course, this is assuming that one could rule-out that the measurements were suboptimal. This hypothetical argument is analogous to considering the hypothetical situation in the original Bell experiments: had we never witnessed a violation of the Bell inequality, and we were sure it was not due to noise or suboptimal measurements, then we would be sure that a local hidden variable model (LHVM) can explain the observed results. There arguments are always weaker than those one can make when one sees a violation of the inequalities, since these violations can never be explained by the 'other model' (LHVM in the original Bell tests, or RQT in the case considered in this manuscript).

[9] Not to mention that any new resource thrown at the issue will face it as well: is the addition of a resource on X's side, say an entangled state, described by an extra operator in $\otimes_K$- or $\otimes_{\mathrm{R}}$-product with the state of the subsystem in Y's side?

But since it is the quantum value that has been observed [13, 14], none of this holds true. What can instead be inferred from the the experimental results when interpreted in a RQT description, is that the state was of the form $\rho_{\mathrm{X}} \otimes_{\mathrm{R}} \rho_{\mathrm{Y}}$ rather than $\rho_{\mathrm{X}} \otimes_{K} \rho_{\mathrm{Y}}$. Requiring that this state does not correspond to a pair of independent systems locally prepared in spacelike-separated labs remains an uncertifiable assumption.

B Real representation of quantum theory

The proofs of the main statements of this paper rely on the fact that models assuming real quantum theory (RQT) and systems of dimension $2d$ will always contain a sub-model (i.e., a subset of states and effects) that is isometrically isomorphic to a model assuming quantum theory (QT) and systems of at most dimension d. These QT-isomorphic sub-models within RQT can be seen as models assuming an alternative theory isomorphic to QT and that we call **real number quantum theory (RNQT)**. We developed it in a previous work [18].

To simplify the proofs in this work, we will not use the RNQT model of the larger possible dimension. Rather, for an n-partite RQT model, we will fix the dimension of its system to be $2d_A 2d_B 2d_C ... = 2^n d_A d_B d_C ...$ and we will focus on the RNQT sub-model isomorphic to a QT model with systems of dimension $d_A d_B d_C ...$, rather than the 'biggest' possible RNQT sub-model (i.e., the one isomorphic to a QT model with systems of dimension $2^{n-1} d_A d_B d_C ...$). The reason for considering the 'small' model is that it is better behaved w.r.t. subsystems, making subsystem-wise proofs easier.

B.1 Single-partite models and the standard mapping

We begin by recalling what is the 'standard' mapping between complex and real matrices, and review its relevant properties. This is the mapping upon which the isomorphism between RNQT and RQT will be built.

Definition B.1 (Standard mapping) *Let I and J be the matrices in $\mathbb{R}^{2\times 2}$ defined as $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Then, we refer to the following as the **standard (complex-to-real) mapping**:*

$$\begin{aligned} \Gamma : \mathbb{C}^{d\times d} &\to \mathbb{R}^{2d\times 2d} \,, \\ A \mapsto \Gamma\left\{A\right\} &:= I \otimes_K \mathrm{Re}\left(A\right) + J \otimes_K \mathrm{Im}\left(A\right) \,. \end{aligned} \tag{21}$$

Remark that the standard mapping is very common in the mathematics literature, so we skip the proofs of its properties (the reader can find them in our previous work if needed [18]).

Proposition B.2 (Properties of the standard mapping) *The map Γ is a *-algebra homomorphism, i.e., $\forall A, B \in \mathbb{C}^{d\times d}$, $\forall a, b \in \mathbb{R}$*

$$\Gamma\left\{aA + bB\right\} = a\Gamma\left\{A\right\} + b\Gamma\left\{B\right\} \;; \tag{22a}$$

$$\Gamma\left\{AB\right\} = \Gamma\left\{A\right\}\Gamma\left\{B\right\} \;; \tag{22b}$$

$$\Gamma\left\{A^\dagger\right\} = \Gamma\left\{A\right\}^T \;. \tag{22c}$$

In addition, when the vector spaces are endowed with their standard Frobenius inner products and the domain of Γ is restricted to the self-adjoint matrices, then Γ is

1. *positive-(semi-)definitness-preserving,*

$$A \geq 0 \quad \Rightarrow \quad \Gamma\left\{A\right\} \geq 0\,; \tag{23a}$$

2. *unital,*

$$\Gamma\left\{\mathbb{1}_d\right\} = \mathbb{1}_{2d}\,; \tag{23b}$$

3. *trace-rescaling,*

$$\mathrm{tr}\left[A\right] = \frac{1}{2}\mathrm{tr}\left[\Gamma\left\{A\right\}\right]\,; \tag{23c}$$

4. *and isometric up to a constant, i.e.,*

$$\mathrm{tr}\left[A^\dagger\, B\right] = \frac{1}{2}\mathrm{tr}\left[\Gamma\left\{A\right\}^T\;\Gamma\left\{B\right\}\right]\,. \tag{23d}$$

Proposition B.3 *The image of the real-linear subspace of Hermitian matrices* $\mathsf{Herm}_n\left(\mathbb{C}\right)$ *through the standard mapping is the subspace of real matrices characterised by the following:*

$$A \in \mathbb{C}^{d\times d}: \quad A^\dagger = A \iff \Gamma\left\{A\right\} = \Gamma\left\{A\right\}^T \;\wedge\; J_d\Gamma\left\{A\right\} = \Gamma\left\{A\right\}J_d\,, \tag{24}$$

where we used the notation $J_d = J \otimes_K \mathbb{1}_d$ *.*

In this case, J_d is called a complex structure in $\mathbb{R}^{2d\times 2d}$ since it plays the role of the scalar multiplication by i, $\Gamma\left\{i\mathbb{1}_d\right\} = J_d$. For a real matrix to be in the image of the complex theory, it must be well-behaved with respect to the complex structure, meaning it should commute with it. We call these the 'tetradionic' or 'special' matrices, and, in particular, we defined [18]

Definition B.4 (Special Symmetric matrices) *The subspace of symmetric matrices in* $\mathbb{R}^{2d\times 2d}$ *which commute with* J_d *as in proposition B.3 is called* ***special symmetric*** *and noted* $\mathsf{SY}_d(\mathbb{R})$.

This subspace is the image through Γ of the complex Hermitian matrices, $\Gamma\left\{\mathsf{Herm}_d\left(\mathbb{C}\right)\right\} = \mathsf{SY}_d(\mathbb{R})$. It is straightforward to check that both are d^2 real vector spaces and thus in bijection. Because of the above, this bijection is furthermore an isometry of inner product spaces (w.r.t. the Frobenius inner product), so their positive (semi-)definite and/or trace-normalised subsets will also be in bijection.

More generally, we call the elements of RQT that are well-behaved with the complex structure its **QT-image**. The QT-image of an RQT model with systems of dimension $2d$ is therefore its RNQT sub-model isomorphic to a QT of dimension d. Formally, the standard mapping provides a direct way to map between the RNQT sub-model of RQT, i.e., its QT-image, and QT.

Proposition 6 (Single-partite QT models are sub-models of single-partite RQT models.) *A single-partite QT model involving a single source producing a state* ρ *measured by a POVM* $\{A_{a|x}\}_a$ *can always be embedded into an RQT model of twice the matrix size.*

That is, there exists a pair of real-linear injective maps $(M, E) : \mathbb{C}^{d_A\times d_A} \to \mathbb{R}^{2d_A\times 2d_A}$ *such that for any QT state* ρ*,* $M(\rho)$ *is a valid RQT state, such that for any QT POVM* $\{A_{a|x}\}_a$*,* $E(A_{a|x})$ *is a valid RQT POVM effect and* $\{E(A_{a|x})\}_a$ *a valid RQT POVM, and such that every distribution of the QT model is mapped to one of the RQT model as:*

$$\forall a: \quad \mathrm{tr}\left[\rho A_{a|x}\right] = \mathrm{tr}\left[M(\rho)E(A_{a|x})\right]\,. \tag{13}$$

Proof Let the pair of mappings be defined from the standard mapping as

$$M_A = \frac{1}{2}\Gamma\,; \tag{25a}$$

$$E_A = \Gamma\,. \tag{25b}$$

Then, by proposition B.3, the QT-image is made of all the special symmetric states and effects, i.e. the states and effects that commute with the matrix $J_{d_A} = J \otimes_K \mathbb{1}_{d_A}$. From proposition B.2 M_A preserves the trace and positive semi-definiteness, making it a bijection between the QT-image subset of the RQT states and the corresponding set of QT states. Similarly, E_A is a bijection between the QT-image subset of the RQT effects and the set of QT effects because it is positive semi-definiteness-preserving. It furthermore maps RQT POVMs to QT POVMs since $E_A(\sum_a A_a) = \sum_a E_A(A_a)$ follows from the map being an algebra homomorphism and $E_A(\mathbb{1}_{2d_A}) = \mathbb{1}_{d_A}$ follows by unitality. Finally, it also follows from proposition B.2 that the pair of bijections define an isometry:

$$\mathrm{tr}\left[M_A(\rho)E_A(A_{a|x})\right] = \mathrm{tr}\left[\rho\, A_{a|x}\right]\,. \tag{26}$$

□

Simply put, the single-partite RNQT can be seen as restricting the RQT to its QT-image, i.e., the image of states and effects through $(\frac{1}{2}\Gamma, \Gamma)$.

B.2 Local models and the n-fold standard mapping

We now extend the proof by showing that the local RNQT models are always embedded in local RQT models. By 'local models', we mean the models representing network scenarios that assume a single source and several local (i.e. spacelike) measurements. The source produces a multipartite system whose parts are shared between the locally measuring parties.

B.2.1 The n-fold standard mapping

When dealing with multipartite systems, the bijection from quantum theory to its real image provided by the standard mapping may cause trouble. At first glance, Γ requires an extra 2-by-2 matrix to encode the phase of a system, which one can physically interpret as a fiducial system [2]. But what happens when several parties are involved? For example, the state of a system shared by Alice and Bob will live in $\Gamma\left\{\mathbb{C}_A^{n\times n} \otimes \mathbb{C}_B^{m\times m}\right\} = \mathbb{R}^{2\times 2} \otimes \mathbb{R}_A^{n\times n} \otimes \mathbb{R}_B^{m\times m}$, raising the question of to which party the extra system $\mathbb{R}^{2\times 2}$ should be associated? Of course, this is purely a problem of representation, which can be circumvented. Recall that the fiducial system simply encodes the (complex) phase freedom of quantum theory, so one should not attach too much ontology to it. Actually, interpreting it as a physical system shared by all parties would be equivalent to interpreting the phase freedom of (the pure-state representation of) quantum theory (using Hilbert spaces) as a physical phenomenon.

So one can instead make the phase-carrying fiducial system into a globally shared state [2]. To do so, instead of using a single 2×2 matrix for the representation of $(1, i)$, we pick a larger dimensional representation, say a 4-dimensional representation $(\overline{I}^{(2)}, \overline{J}^{(2)}) \subset \mathbb{R}^{4\times 4}$ such that the extra system can be split evenly between Alice and Bob, $\mathbb{R}^{4\times 4} \cong \mathbb{R}_A^{2\times 2} \otimes \mathbb{R}_B^{2\times 2}$. The important property that must be required from this representation is that it is global, in the sense explained in the previous section: suppose the parties share such a bipartite element, e.g. $\overline{J}^{(2)}_{AB}$, then the action of an element of a local 2-d representation acting on it from Alice's side, e.g. $(J_A \otimes_K I_B)\overline{J}^{(2)}_{AB}$, should be equivalent to the situation in which it is acting on Bob's side, but also if it were acting globally, e.g. $(J_A \otimes_K I_B)\overline{J}^{(2)} = (I_A \otimes_K J_B)\overline{J}^{(2)}_{AB} = \overline{J}^{(2)}_{AB}\overline{J}^{(2)}_{AB}$. This stabiliser-state-like representation is precisely the kind of representation presented in the Appendix C.

Definition B.5 (n-fold standard mapping) *Let $\overline{I}^{(n)}$ and $\overline{J}^{(n)}$ be defined as in definition C.2. Then, the following is called the* ***n-fold standard (complex-to-real) mapping****:*

$$\begin{aligned} \overline{\Gamma}^{(n)} : \bigotimes_{i=1}^{n} \mathbb{C}^{d_{A_i} \times d_{A_i}} \to \bigotimes_{i=1}^{n} \mathbb{R}^{2d_{A_i} \times 2d_{A_i}} \ , \\ A \mapsto \overline{\Gamma}^{(n)}\{A\} :\cong \overline{I}^{(n)} \otimes_K \mathrm{Re}\,(A) + \overline{J}^{(n)} \otimes_K \mathrm{Im}\,(A) \ . \end{aligned} \tag{27}$$

We use $:\cong$ instead of $:=$ in the definition because there has been an implicit reordering of the tensor factors from $\bigotimes_{i=1}^{n} \mathbb{R}^{2d_{A_i} \times 2d_{A_i}}$ to $\mathbb{R}^{2^n \times 2^n} \otimes (\bigotimes_{i=1}^{n} \mathbb{R}^{d_{A_i} \times d_{A_i}})$. Specifically, every phase-carrying $\mathbb{R}^{2\times 2}$ subspace has been moved to the left of the tensor factorisation in order to obtain a concise expression that is reminiscent of definition B.1.

With the implicit reordering and because the matrices $\overline{I}^{(n)}$ and $\overline{J}^{(n)}$ are isomorphic to I and J, it should be clear that the n-fold standard mapping is isomorphic to the standard mapping. Therefore, all the properties proven in Proposition B.2 apply to it as well. For example, $\mathrm{tr}\left[\overline{\Gamma}^{(n)}\{A\}\right] = 2\mathrm{tr}\,[A]$. The only difference between the standard mapping and its n-fold version is that the matrices I and J have been 'inflated' into the larger matrices $\overline{I}^{(n)}$ and $\overline{J}^{(n)}$, like in an encoding of a stabiliser state. They nonetheless represent the same thing: the complex structure $(1, i)$. In that sense, the matrices $\overline{\Gamma}^{(n)}\{A\}$ will not only be compatible with a *global* complex structure $\overline{J}^{(n)}_{d_{A_1} d_{A_2} \dots d_{A_n}}$, it will also be compatible with the *local* complex structure of each party, i.e. the matrices $J_{A_1, 2_{A_2} \dots 2_{A_n} d_{A_1} d_{A_2} \dots d_{A_n}}, J_{A_2, 2_{A_1} 2_{A_3} \dots 2_{A_n} d_{A_1} d_{A_2} \dots d_{A_n}}, \dots$ (recall that the notation $J_{A_1, 2_{A_2} \dots 2_{A_n} d_{A_1} d_{A_2} \dots d_{A_n}}$ means $J_{A_1} \otimes_K I_{A_2} \otimes_K \dots I_{A_n} \otimes_K \mathbb{1}_{d_{A_1}} \otimes_K \mathbb{1}_{d_{A_2}} \otimes_K \dots \otimes_K \mathbb{1}_{d_{A_n}}$).

That way, we can define the image of complex matrices representing multipartite systems while keeping each system locally accessible. All single-fold results and definitions then readily generalise to the n-fold case:

Definition B.6 (n-partite Special Symmetric matrices) *In the n-fold tensor product space $\bigotimes_{i=1}^{n} \mathbb{R}^{2d_{A_i} \times 2d_{A_i}}$, the subspace of symmetric matrices commuting with the matrix $\overline{J}^{(n)}_{d_{A_1} d_{A_2} \dots d_{A_n}} \cong \overline{J}^{(n)} \otimes_K \mathbb{1}_{d_{A_1}} \otimes_K \mathbb{1} d_{A_2} \otimes_K \dots \otimes_K \mathbb{1} d_{A_n}$ is called n-partite special symmetric.*

Proposition B.7 (Properties of the n-fold standard mapping) *The map* $\overline{\Gamma}^{(n)}$ *is a* *-*algebra homomorphism. In addition, when the vector spaces are endowed with their standard Frobenius inner products, then for any* $A, B \in \bigotimes_{i=1}^{n} \mathbb{C}^{d_i \times d_i}$, $\overline{\Gamma}^{(n)}$ *is*

1. *positive-semi-definitness-preserving,*

$$A \geq 0 \quad \Rightarrow \quad \overline{\Gamma}^{(n)}\{A\} \geq 0\,; \tag{28a}$$

2. *isomorphic to a unital map* ($d = \prod_i d_i$),

$$\overline{\Gamma}^{(n)}\{\mathbb{1}_d\} \;\cong\; \overline{I}^{(n)} \otimes_K \mathbb{1}_d\,; \tag{28b}$$

3. *trace-rescaling,*

$$\mathrm{tr}\,[A] = \frac{1}{2}\mathrm{tr}\left[\overline{\Gamma}^{(n)}\{A\}\right]\;; \tag{28c}$$

4. *and isometric up to a constant, i.e.,*

$$\mathrm{tr}\left[A^\dagger\, B\right] = \frac{1}{2}\mathrm{tr}\left[\overline{\Gamma}^{(n)}\{A\}^T\, \overline{\Gamma}^{(n)}\{B\}\right]\,. \tag{28d}$$

We shall now see how the n-fold mapping is better suited for multipartite systems. To do so, we need to address the tensor-product decomposition of the real image of an n-partite complex matrix.

B.2.2 Real representation of the complex Kronecker product

The n-fold standard mapping is a way to apply the standard mapping to multipartite systems such that the decomposition of the space into tensor factors is respected. In the following, it is accordingly assumed that every space has a preferred tensor factorization, so that any matrix $A \in \bigotimes_{i=1}^{n} \mathbb{K}^{d_{A_i} \times d_{A_i}}$ is known to be n-partite and the dimension d_{A_i} of each of its n factors is known. Also, each locally-defined space of matrices over the reals will be assumed to contain an image of a complex space through the standard mapping, therefore every space of real matrices will be assumed to be of even dimension. There is no loss of generality in doing so, as no pre-requisite but finiteness was taken on the dimensions of the real spaces. Limiting our analysis to even-dimensional spaces, and, a fortiori, to only the image of complex spaces in their corresponding real spaces, is enough to prove everything we need.

In the following, we will encounter expressions involving the Kronecker product of k-fold standard mappings with j-fold ones. As we already did in the previous section, in order to avoid clustered equations with tensor-factor swaps everywhere, we will assume that all such swaps are implicit. In particular, we will take as a convention to always gather the $\mathbb{R}^{2\times 2}$ subsystems encoding the phase onto the left of expressions.

For example, an equation like

$$
\begin{aligned}
&\overline{\Gamma}^{(j)}\{A\} \otimes_K \overline{\Gamma}^{(k)}\{B\} \\
&= \Big((\overline{I}^{(k)} \otimes_K \mathrm{Re}\,(A) + \overline{J}^{(k)} \otimes_K \mathrm{Im}\,(A)) \otimes_K (\overline{I}^{(j)} \otimes_K \mathrm{Re}\,(B) + \overline{J}^{(j)} \otimes_K \mathrm{Im}\,(B) \Big) \\
&= \big(\overline{I}^{(k)} \otimes_K \overline{I}^{(j)} \otimes_K \mathrm{Re}\,(A) \otimes_K \mathrm{Re}\,(B) + \overline{I}^{(k)} \otimes_K \overline{J}^{(j)} \otimes_K \mathrm{Re}\,(A) \otimes_K \mathrm{Im}\,(B) \\
&\quad + \overline{J}^{(k)} \otimes_K \overline{I}^{(j)} \otimes_K \mathrm{Im}\,(A) \otimes_K \mathrm{Re}\,(B) + \overline{J}^{(k)} \otimes_K \overline{J}^{(j)} \otimes_K \mathrm{Im}\,(A) \otimes_K \mathrm{Im}\,(B) \big)
\end{aligned}
\tag{29}
$$

will be assumed to hold, although the second equality is only true up to the permutation of the tensor factors that allowed us to bring the $\overline{I}^{(j)}$ and $\overline{J}^{(j)}$ matrices to the left.

In our previous work [18], we remarked that since $(\overline{I}^{(n)})^2 = \overline{I}^{(n)}$, it is a projector. This is precisely the projector sending to the quotient space of matrices global with respect to $(\overline{I}^{(n)}, \overline{J}^{(n)})$. In other words, since $\overline{I}^{(n)}$ is global w.r.t. any $(\overline{I}^{(k)}, \overline{J}^{(k)})$, $k \leq n$, then $\overline{I}^{(n)} X \overline{I}^{(n)}$ will also be global; by the adjoint action of $\overline{I}^{(n)}$, the matrix X has been projected into the space of all matrices global w.r.t. local representation of $(1, i)$. For n-fold special symmetric matrices, this globality of the representation is encoded by the following lemma:

Lemma B.8 *Let $\widetilde{A} \in \bigotimes_{i=1}^{n} \mathbb{R}^{2d_{A_i} \times 2d_{A_i}} \cong \mathbb{R}^{2^n d_A \times 2^n d_A}, d_A = \prod_i d_{A_i}$ be a n-fold special symmetric matrix, then the following holds (up to tensor-factor permutations):*

1.
$$\widetilde{A} = \overline{I}^{(n)}_{d_A}\, \widetilde{A} = \widetilde{A}\, \overline{I}^{(n)}_{d_A} = \overline{I}^{(n)}_{d_A}\, \widetilde{A}\, \overline{I}^{(n)}_{d_A}\,; \tag{30a}$$

2. $\forall k \leq n$,
$$(\overline{J}^{(k)} \otimes_K I^{\otimes(n-k)})_{d_A} \widetilde{A} = \overline{J}^{(n)}_{d_A}\, \widetilde{A}\,. \tag{30b}$$

Proof Item 1 follows from proposition B.7: since the n-fold special symmetric matrix in $\mathbb{R}^{2^n d_A \times 2^n d_A}$ are isomorphic to the Hermitian matrices in $\mathbb{C}^{d_A \times d_A}$, it must have the form (once again, up to tensor-factor permutations)

$$\widetilde{A} = \overline{I}^{(n)} \otimes_K \mathrm{Re}\,(A) + \overline{J}^{(n)} \otimes_K \mathrm{Im}\,(A)\ , \tag{31}$$

for some Hermitian A in $\mathbb{C}^{d_A \times d_A}$. Then,

$$
\begin{aligned}
\overline{I}^{(n)}_{d_A}\, \widetilde{A}\, \overline{I}^{(n)}_{d_A} &= (\overline{I}^{(n)} \otimes_K \mathbb{1}_{d_A})(\overline{I}^{(n)} \otimes_K \mathrm{Re}\,(A) + \overline{J}^{(n)} \otimes_K \mathrm{Im}\,(A))(\overline{I}^{(n)} \otimes_K \mathbb{1}_{d_A}) \\
&= (\overline{I}^{(n)})^3 \otimes_K \mathrm{Re}\,(A) + (\overline{I}^{(n)} \overline{J}^{(n)} \overline{I}^{(n)}) \otimes_K \mathrm{Im}\,(A)) \\
&= \overline{I}^{(n)} \otimes_K \mathrm{Re}\,(A) + \overline{J}^{(n)} \otimes_K \mathrm{Im}\,(A) \\
&= \widetilde{A}\,,
\end{aligned}
\tag{32}
$$

and we simply used the algebraic relations between $\overline{I}^{(n)}$ and $\overline{J}^{(n)}$. The second item is proven similarly, but this time using lemma C.7 □

Such projection can be used to make the real Kronecker product well-behaved with respect to all local irreps of the complex structure. That way, the local phase freedom present in complex quantum theory is preserved when going to the real representation. This was one of the main points of this paper, as well as Ref. [17], although both these works focused on the 'small' version of the projection, i.e. the one reducing the number of subsystems encoding the phase to only one. This composition, respectively denoted $\otimes_r$ and $\otimes_F$ in the aforementioned works, was favoured for compactness reasons as well as for its property of being the image of the Kronecker product through the standard mapping, $\Gamma\{A \otimes_K B\} = \Gamma\{A\} \otimes_r \Gamma\{B\}$.

Differently here, the composition rule we considered will be called the 'R-product' and denoted $\otimes_R$.

Definition B.9 (R-product) *Let $\widetilde{A} \in \bigotimes_{i=1}^{j} \mathbb{R}^{2d_{A_i} \times 2d_{A_i}} \cong \mathbb{R}^{2^j d_A \times 2^j d_A}$ and $\widetilde{B} \in \bigotimes_{l=1}^{k} \mathbb{R}^{2d_{B_l} \times 2d_{B_l}} \cong \mathbb{R}^{2^k d_B \times 2^k d_B}$ be two real matrices, where $d_A := \prod_i d_{A_i}$ and $d_B := \prod_l d_{B_l}$, and let $\overline{I}^{(j+k)}_{d_A d_B} = \overline{I}^{(j+k)} \otimes_K \mathbb{1}_{d_A} \otimes_K \mathbb{1}_{d_B}$ with $\overline{I}^{(j+k)}$ be defined as in Definition C.2. Then, the $\otimes_R$-product of $\widetilde{A}$ and $\widetilde{B}$ is the matrix $\widetilde{A} \otimes_R \widetilde{B} \in \left(\left(\bigotimes_{i=1}^{j} \mathbb{R}^{2d_{A_i} \times 2d_{A_i}}\right) \otimes \left(\bigotimes_{l=1}^{k} \mathbb{R}^{2d_{B_l} \times 2d_{B_l}}\right)\right) \cong \mathbb{R}^{2^{j+k} d_A d_B \times 2^{j+k} d_A d_B}$ defined as*

$$\begin{gathered} \forall \widetilde{A} \in \bigotimes_{i=1}^{j} \mathbb{R}^{2d_{A_i} \times 2d_{A_i}},\ \forall \widetilde{B} \in \bigotimes_{l=1}^{k} \mathbb{R}^{2d_{B_l} \times 2d_{B_l}}, \\ (\widetilde{A} \otimes_R \widetilde{B}) := \overline{I}^{(j+k)}_{d_A d_B} (\widetilde{A} \otimes_K \widetilde{B}) \overline{I}^{(j+k)}_{d_A d_B} . \end{gathered} \tag{33}$$

It should be evident that, like the $\otimes_K$ and the $\otimes_r$, this is a real-bilinear and associative composition rule. Similarly to how the $\otimes_r$-product is the image of the complex Kronecker product through the standard mapping Γ, we can show that the $\otimes_R$-product is its image through the n-fold standard mapping. Indeed, the $\otimes_R$ reduces to $\otimes_r$ when $n = 1$ and notice that its definition implies

$$\overline{I}^{(j+k)}_{d_A d_B} = \overline{I}^{(j)}_{d_A} \otimes_R \overline{I}^{(k)}_{d_B} = -\overline{J}^{(j)}_{d_A} \otimes_R \overline{J}^{(k)}_{d_B} ; \tag{34a}$$

$$\overline{J}^{(j+k)}_{d_A d_B} = \overline{I}^{(j)}_{d_A} \otimes_R \overline{J}^{(k)}_{d_B} = \overline{J}^{(j)}_{d_A} \otimes_R \overline{I}^{(k)}_{d_B} . \tag{34b}$$

From there, it is direct to show that the R-product is the image through the standard mapping of the complex Kronecker product.

Proposition B.10 *The R-product is the image of the complex Kronecker product through the $\overline{\Gamma}^{(n)}$ map. In symbols:*

$$\begin{gathered} \forall j, k, \quad \forall A \in \bigotimes_{i=1}^{j} \mathbb{C}^{d_{A_i} \times d_{A_i}},\ \forall B \in \bigotimes_{l=1}^{k} \mathbb{C}^{d_{B_l} \times d_{B_l}}, \\ \overline{\Gamma}^{(j+k)}\{A \otimes_K B\} \cong \overline{\Gamma}^{(j)}\{A\} \otimes_R \overline{\Gamma}^{(k)}\{B\} . \end{gathered} \tag{35}$$

Proof Let $d_A = \prod_i d_{A_i}$ and $d_B = \prod_i d_{B_i}$, then:

$$
\begin{aligned}
&\overline{\Gamma}^{(j)}\{A\} \otimes_R \overline{\Gamma}^{(k)}\{B\} = \overline{I}^{(j+k)}_{d_A d_B} \left(\overline{\Gamma}^{(j)}\{A\} \otimes_K \overline{\Gamma}^{(k)}\{B\}\right) \overline{I}^{(j+k)}_{d_A d_B} \\
&= \overline{I}^{(j+k)}_{d_A d_B} \left((\overline{I}^{(k)} \otimes_K \mathrm{Re}\,(A) + \overline{J}^{(k)} \otimes_K \mathrm{Im}\,(A)) \otimes_K (\overline{I}^{(j)} \otimes_K \mathrm{Re}\,(B) + \overline{J}^{(j)} \otimes_K \mathrm{Im}\,(B)\right) \overline{I}^{(j+k)}_{d_A d_B} \\
&= \overline{I}^{(j+k)}_{d_A d_B} \big((\overline{I}^{(k)} \otimes_K \mathrm{Re}\,(A) \otimes_K \overline{I}^{(j)} \otimes_K \mathrm{Re}\,(B) + \overline{J}^{(k)} \otimes_K \mathrm{Im}\,(A) \otimes_K \overline{J}^{(j)} \otimes_K \mathrm{Im}\,(B)) \\
&+ (\overline{I}^{(k)} \otimes_K \mathrm{Re}\,(A) \otimes_K \overline{J}^{(j)} \otimes_K \mathrm{Im}\,(B) + \overline{J}^{(k)} \otimes_K \mathrm{Im}\,(A) \otimes_K \overline{I}^{(j)} \otimes_K \mathrm{Re}\,(B))\big) \overline{I}^{(j+k)}_{d_A d_B} \\
&= \big(\overline{I}^{(j+k)} \otimes_K (\mathrm{Re}\,(A) \otimes_K \mathrm{Re}\,(B) - \mathrm{Im}\,(A) \otimes_K \mathrm{Im}\,(B))\big) \overline{I}^{(j+k)}_{d_A d_B} \\
&+ \big(\overline{J}^{(j+k)} \otimes_K (\mathrm{Re}\,(A) \otimes_K \mathrm{Im}\,(B) + \mathrm{Im}\,(A) \otimes_K \mathrm{Re}\,(B))\big) \overline{I}^{(j+k)}_{d_A d_B} \\
&= \overline{I}^{(j+k)} \otimes_K \mathrm{Re}\,(A \otimes_K B) + \overline{J}^{(j+k)} \otimes_K \mathrm{Im}\,(A \otimes_K B) \\
&= \overline{\Gamma}^{(j+k)}\{A \otimes_K B\}\,. \qquad (36)
\end{aligned}
$$

In the above, some re-ordering of factors was necessary to pass from one step to the next. □

As first shown in Ref. [18], it entails that every property of the Kronecker product is true for the R-product when restricted to real matrices that have a preimage through the standard mapping, and most of its properties remain true in the general case. The following corollary summarises the ones we need.

Corollary B.11 *When acting on n-fold special symmetric matrices, the R-product obeys the mixed-product rule (or interchange law) and commutes with the trace up to a factor of 2:*

$$\forall j,k \leq n:\ j+k=n, \quad \forall A, A' \in \bigotimes_{i=1}^{j} \mathbb{C}^{d_{A_i} \times d_{A_i}},\ \forall B, B' \in \bigotimes_{l=1}^{k} \mathbb{C}^{d_{B_l} \times d_{B_l}},$$

$$
\begin{aligned}
&(\overline{\Gamma}^{(j)}\{A\} \otimes_R \overline{\Gamma}^{(k)}\{B\})(\overline{\Gamma}^{(j)}\{A'\} \otimes_R \overline{\Gamma}^{(k)}\{B'\}) \\
&\quad = (\overline{\Gamma}^{(j)}\{A\}\overline{\Gamma}^{(j)}\{A'\}) \otimes_R (\overline{\Gamma}^{(k)}\{B\}\overline{\Gamma}^{(k)}\{B'\})\,;
\end{aligned} \tag{37a}
$$

$$\mathrm{tr}\left[\overline{\Gamma}^{(j)}\{A\} \otimes_R \overline{\Gamma}^{(k)}\{B\}\right] = \frac{1}{2}\mathrm{tr}\left[\overline{\Gamma}^{(j)}\{A\}\right] \mathrm{tr}\left[\overline{\Gamma}^{(k)}\{B\}\right]\,. \tag{37b}$$

B.2.3 From QT to RNQT models

With the introduction of the R-product, we can characterise the image of an n-partite QT model through the n-fold standard mapping.

Proposition 7 (QT models are RNQT models.) *A n-partite QT model can always be embedded into an RNQT model of 2^n times the matrix size.*

That is, there exists a pair of real-linear injective maps $(M_{\mathrm{AB}\dots}, E_{\mathrm{AB}\dots})$: $\mathbb{C}^{d_A d_B \dots \times d_A d_B \dots} \to \mathbb{R}^{2d_A 2d_B \dots \times 2d_A 2d_B \dots}$ that respectively map any valid QT state and effect to a valid RQT state and effect that commute with a given J matrix, such that every distribution predicted by the QT model has a real-matrix counterpart and such that the system-composition rules of QT are mapped to those of RNQT. I.e.,

$$M_{\mathrm{XY}}(\rho_{\mathrm{X}} \otimes_K \sigma_{\mathrm{Y}}) = M_{\mathrm{X}}(\rho_{\mathrm{X}}) \otimes_{\mathrm{R}} M_{\mathrm{Y}}(\sigma_{\mathrm{Y}})\,; \tag{14a}$$

$$E_{\mathrm{XY}}(X_{\alpha|\mu} \otimes_K Y_{\beta|\nu}) = E_{\mathrm{X}}(X_{\alpha|\mu}) \otimes_{\mathrm{R}^*} E_{\mathrm{Y}}(Y_{\beta|\nu})\,; \tag{14b}$$

where X *and* Y *are any two subsets of the parties* AB…, *ρ_{X} and σ_{Y} (respectively, $X_{\alpha|\mu}$ and $Y_{\beta|\nu}$) are valid QT states (resp. effetcs) of* X *and* Y, *and $(M_{\mathrm{X}}, E_{\mathrm{X}})$ denotes a local application of the maps on the* X *subset.*

Proof Using the pair of maps $(M, E) = (\frac{1}{2}\overline{\Gamma}^{(n)}, \overline{\Gamma}^{(n)})$ and the system-composition rules that fit with this representation is

$$\cdot \otimes_{\mathrm{R}} \cdot := 2(\cdot \otimes_R \cdot)\,; \tag{38a}$$

$$\cdot \otimes_{\mathrm{R}^*} \cdot := \cdot \otimes_R \cdot\,; \tag{38b}$$

the result follows from applying the properties propositions B.2, B.7 and B.10. □

With this proposition, what remains to show is that the RNQT composition rules can be safely substituted by the RQT composition rules. That way, the RNQT becomes the QT-image of the RQT, and the inclusion of a sub-model isomorphic to QT within RQT is proven.

B.2.4 Local RNQT models are embedded in local RQT models

Before moving on to the general case, let us focus on the image of an n-partite, single-source, local QT model through the n-fold standard mapping.

Corollary B.12 *The image of the states and of the effects of the n-partite, single-source, local QT model of matrix dimension $d_A d_B \dots$ through the pair of maps $(\frac{1}{2}\overline{\Gamma}^{(n)}, \overline{\Gamma}^{(n)})$ is a sub-model of the single-partite RQT model of matrix dimension $2d_A 2d_B \dots$ that has underwent the following modification:*

- *The effect composition rule $\otimes_K$ has been replaced by $\cdot \otimes_{\mathrm{R}^*} \cdot := \cdot \otimes_R \cdot$.*

The QT image with this modified effect composition rule will be referred to as the local RNQT model. With the results of the previous sections, we already have enough material to show that the local RNQT model can be seen as the "n-fold QT-image" of the RQT model. We now show that every multipartite, single-source, local RQT model has a sub-model equivalent to a local QT model of $(1/2)^n$-th the matrix dimension. As there is only one source, the only modification that should be addressed to pass from the RNQT defined by corollary B.12 to the QT-image of RQT is the different composition rule for effects.

The next proposition does that by proving that the $\otimes_R$- and $\otimes_K$-products of effects are operationally equivalent for single-source RQ(N)T models. As it turns out, the n-fold QT-image is a subspace which is invariant under the projection $\overline{I}^{(n)}_{d_A d_B \dots}$, and the projection $\overline{I}^{(n)}_{d_A d_B \dots}$ 'swallows smaller projections' like e.g. $\overline{I}^{(n)}_{d_A d_B \dots}(\overline{I}^{(1)}_{d_A} \otimes_K \overline{I}^{(n-1)}_{d_B \dots}) \cong \overline{I}^{(n)}_{d_A d_B \dots}$. This implies that in an inner product of a multipartite state in the n-fold QT-image with the Kronecker product of any real POVMs, the Kronecker products will be turned into R-products.

Proposition 8 (Local QT models are sub-models of local RQT models.) *A multipartite QT model with a single source producing a state $\rho_{\mathrm{AB}\dots}$ and n local measurements represented by the POVMs $\{A_{a|x}\}_a$, $\{B_{b|y}\}_b$, ..., can always be embedded into an RQT model of 2^n the matrix size.*

That is, there exists a pair of real-linear injective maps $(M_{\mathrm{AB}\dots}, E_{\mathrm{A}} \otimes_K E_{\mathrm{B}} \otimes_K \dots)$ from $\mathbb{C}^{d_A d_B \dots \times d_A d_B \dots}$ to $\mathbb{R}^{2d_A 2d_B \dots \times 2d_A 2d_B \dots}$ such that $M_{\mathrm{AB}\dots}(\rho_{\mathrm{AB}\dots})$ is a valid RQT state of *the multipartite system; such that* $E_{\mathrm{X}}(X_{\alpha|\mu}) \in \mathbb{R}^{2d_{\mathrm{X}}\times 2d_{\mathrm{X}}}$ *is a valid RQT effect on the local system associated with party* X *and* $\{E_{\mathrm{X}}(X_{\alpha|\mu})\}_\mu$ *is a valid RQT POVM for every party* X*; and such that every distribution of the QT model is mapped to one of the RQT model as:*

$$\forall a, \forall b, ... : \mathrm{tr}\left[\rho_{\mathrm{AB}...}(A_{a|x} \otimes_K B_{b|y} \otimes_K ...)\right] = \mathrm{tr}\left[M_{\mathrm{AB}...}(\rho_{\mathrm{AB}...})(E_{\mathrm{A}}(A_{a|x}) \otimes_K E_{\mathrm{B}}(B_{b|y}) \otimes_K ...)\right] . \quad (15)$$

Proof Similar to the RNQT case, set $M_{\mathrm{AB}...} = 2\overline{\Gamma}^{(n)}$. By proposition B.7, $M_{\mathrm{AB}...}(\rho_{\mathrm{AB}...})$ is a valid RQT state. On the other hand, set $E_{\mathrm{X}}(X_{\alpha|\mu}) = \Gamma\{X_{\alpha|\mu}\}$ for every local party X; by proposition B.2, this transforms the QT POVMs into RQT POVMs.

Then, it remains to show that the inner products are preserved. For every $a, b, \ldots$:

$$\begin{aligned}
\mathrm{tr}&\left[M_{\mathrm{AB}...}(\rho_{\mathrm{AB}...})(E_{\mathrm{A}}(A_{a|x}) \otimes_K E_{\mathrm{B}}(B_{b|y}) \otimes_K ...)\right] \\
&= 2\mathrm{tr}\left[\overline{\Gamma}^{(n)}\{\rho_{\mathrm{AB}...}\}\Big(\Gamma\{A_{a|x}\} \otimes_K \Gamma\{B_{b|y}\} \otimes_K ...\Big)\right] \\
&= 2\mathrm{tr}\left[\overline{I}^{(n)}_{d_{\mathrm{A}}d_{\mathrm{B}}...}\overline{\Gamma}^{(n)}\{\rho_{\mathrm{AB}...}\}\overline{I}^{(n)}_{d_{\mathrm{A}}d_{\mathrm{B}}...}\Big(\Gamma\{A_{a|x}\} \otimes_K \Gamma\{B_{b|y}\} \otimes_K ...\Big)\right] \\
&= 2\mathrm{tr}\left[\overline{\Gamma}^{(n)}\{\rho_{\mathrm{AB}...}\}\overline{I}^{(n)}_{d_{\mathrm{A}}d_{\mathrm{B}}...}\Big(\Gamma\{A_{a|x}\} \otimes_K \Gamma\{B_{b|y}\} \otimes_K ...\Big)\overline{I}^{(n)}_{d_{\mathrm{A}}d_{\mathrm{B}}...}\right] \\
&= 2\mathrm{tr}\left[\overline{\Gamma}^{(n)}\{\rho_{\mathrm{AB}...}\}\Big(\Gamma\{A_{a|x}\} \otimes_R \Gamma\{B_{b|y}\} \otimes_R ...\Big)\right] \\
&= 2\mathrm{tr}\left[\overline{\Gamma}^{(n)}\{\rho_{\mathrm{AB}...}\}\overline{\Gamma}^{(n)}\{A_{a|x} \otimes_K B_{b|y} \otimes_K ...\}\right] \\
&= \mathrm{tr}\left[\rho_{\mathrm{AB}...}\ (A_{a|x} \otimes_K B_{b|y} \otimes_K ...)\right] \quad (39)
\end{aligned}$$

where we first substituted the mappings by their expression using the standard mappings, then we successively used lemma B.8, cyclicity of the trace, definition B.9, proposition B.10, and proposition B.7 to pass from each line to the next. □

It remains to show that the proof generalises for any network structure. If the situation involves multiple sources each measured locally, it reduces to applying proposition 8 over each source. So the only remaining cases are those with joint measurements across several sources.

B.3 Multilocal models and relaxing the model-dependency of RQT

The multilocal model generalises the local model by allowing for several sources. These models represent the network scenarios that assume multiple local measurements from multiple independent sources. The sources can produce multipartite systems that are shared among the measuring parties.

B.3.1 RNQT models and operationally independent sources

As we did in the previous section, the first step is to apply the n-fold standard mapping to the multilocal QT model, and then to split the mapping into n single-fold standard mappings $\overline{\Gamma}^{(n)} = \Gamma \otimes_R \Gamma \otimes_R \cdots \otimes_R \Gamma$ via proposition B.10 to obtain a representation-local image of QT as real matrices, i.e., a RNQT model as in proposition 7.

Corollary B.13 *The image of the states and of the effects of the n-partite, multiple-source, multilocal QT model of matrix dimension $d_A d_B \ldots$ through the pair of maps $(\frac{1}{2}\overline{\Gamma}^{(n)}, \overline{\Gamma}^{(n)})$ form a sub-model in an n-partite multilocal RQT model of matrix dimension $2d_A 2d_B \ldots$ that has underwent the following modifications:*

- *The state composition rule $\otimes_K$ has been replaced by $\cdot \otimes_{\mathrm{R}} \cdot := 2(\cdot \otimes_R \cdot)$;*
- *The effect composition rule $\otimes_K$ has been replaced by $\cdot \otimes_{\mathrm{R}^*} \cdot := \cdot \otimes_R \cdot$.*

The next step would be to show that the modifications can be removed while preserving all predicted distributions. Nevertheless, there is no hope that this can be done directly: as it was shown by Renou et al., if the joint sources are represented by a state in Kronecker product form, then there is no possibility to represent all QT models within RQT. Their proof [11] presented a counterexample in the bilocal scenario (two sources, three measurements, among which one measurement is jointly applied on one subsystem from each source).

But this only means there is no way to represent all QT models within the naive RQT model, which assumes independent sources to be represented as (Kronecker-)product states. As argued in the main text, representing the joint state of independent sources via a state in a pure Kronecker product state is operationally hard to justify: operationally independent states are result in independent distributions (i.e., totally uncorrelated outcomes) for all local measurements, so they seem to be the right candidate to represent the joint state of several systems prepared by independent procedures. Yet, in RQT proposition 5 shows that the pure Kronecker-product states are only a subset of these states. Assuming that independent sources are represented by product states, therefore, amounts to forbidding a large number of independent preparation procedures.

By consequence, the multilocal RQT model with product-state independence will be weakened into the multilocal RQT model with operational independence, which is a better representation of the experiment in the sense that it no longer arbitrarily forbids many preparation procedures. The goal of this section is to show that the multilocal RNQT models obtained through corollary B.13 are equivalent to multilocal RQT models with operationally independent sources.

B.3.2 The 'contamination' property of the R-product

We first need to generalise one of the key elements in the proof that RNQT embeds into RQT in the local case: the property that the R-product 'contaminates' Kronecker products when applied to matrices that have a preimage through the (n-fold) standard mapping; equations involving mixtures of the R-product and the Kronecker product are equivalent to the same equations with all the $\otimes_K$-products replaced by

$\otimes_R$-products:

$$\begin{aligned}
&\forall j,k \leq n : j+k=n, \quad \forall A, A' \in \bigotimes_{i=1}^{j} \mathbb{C}^{d_{A_i}\times d_{A_i}}, \ \forall B, B' \in \bigotimes_{l=1}^{k} \mathbb{C}^{d_{B_l}\times d_{B_l}}, \\
&\qquad (\overline{\Gamma}^{(j)}\{A\} \otimes_R \overline{\Gamma}^{(k)}\{B\})(\overline{\Gamma}^{(j)}\{A'\} \otimes_K \overline{\Gamma}^{(k)}\{B'\}) \\
&\qquad = (\overline{\Gamma}^{(j)}\{A\} \otimes_K \overline{\Gamma}^{(k)}\{B\})(\overline{\Gamma}^{(j)}\{A'\} \otimes_R \overline{\Gamma}^{(k)}\{B'\}) \\
&\qquad = (\overline{\Gamma}^{(j)}\{A\} \otimes_R \overline{\Gamma}^{(k)}\{B\})(\overline{\Gamma}^{(j)}\{A'\} \otimes_R \overline{\Gamma}^{(k)}\{B'\})\,.
\end{aligned} \tag{40}$$

This happens because the Kronecker product between any $j-$ and $k-$fold special symmetric matrices $\overline{\Gamma}^{(j)}\{A\}$ and $\overline{\Gamma}^{(k)}\{B\}$ commutes with the projector $\overline{I}^{(j+k)}_{d_A d_B}$ (where $d_A = \prod_i^j d_{A_i}$, $d_B = \prod_l^k d_{B_l}$),

$$\overline{I}^{(j+k)}_{d_A d_B}(\overline{\Gamma}^{(j)}\{A\} \otimes_K \overline{\Gamma}^{(k)}\{B\}) = (\overline{\Gamma}^{(j)}\{A\} \otimes_K \overline{\Gamma}^{(k)}\{B\})\overline{I}^{(j+k)}_{d_A d_B}\,. \tag{41}$$

This, in turn, implies that any of the two sides of the above defines an R-product, for instance

$$\overline{I}^{(j+k)}_{d_A d_B}(\overline{\Gamma}^{(j)}\{A\} \otimes_K \overline{\Gamma}^{(k)}\{B\}) = \overline{\Gamma}^{(j)}\{A\} \otimes_R \overline{\Gamma}^{(k)}\{B\}\,. \tag{42}$$

Indeed since $\overline{\Gamma}^{(j)}\{A\}\overline{J}^{(j)}_{d_A} = \overline{J}^{(j)}_{d_A}\overline{\Gamma}^{(j)}\{A\}$ and $\overline{\Gamma}^{(k)}\{B\}\overline{J}^{(k)}_{d_B} = \overline{J}^{(k)}_{d_B}\overline{\Gamma}^{(k)}\{B\})$ and using lemma C.7 and corollary C.6 gives

$$\begin{aligned}
&\overline{\Gamma}^{(j)}\{A\} \otimes_R \overline{\Gamma}^{(k)}\{B\} = \overline{I}^{(j+k)}_{d_A d_B}(\overline{\Gamma}^{(j)}\{A\} \otimes_K \overline{\Gamma}^{(k)}\{B\})\overline{I}^{(j+k)}_{d_A d_B} \\
&= \frac{1}{2}(\overline{I}^{(k)}_{d_A} \otimes_K \overline{I}^{(j)}_{d_B} - \overline{J}^{(k)}_{d_A} \otimes_K \overline{J}^{(j)}_{d_B})(\overline{\Gamma}^{(j)}\{A\} \otimes_K \overline{\Gamma}^{(k)}\{B\})\frac{1}{2}(\overline{I}^{(k)}_{d_A} \otimes_K \overline{I}^{(j)}_{d_B} - \overline{J}^{(k)}_{d_A} \otimes_K \overline{J}^{(j)}_{d_B}) \\
&= \frac{1}{4}(\overline{I}^{(k)}_{d_A} \otimes_K \overline{I}^{(j)}_{d_B} - \overline{J}^{(k)}_{d_A} \otimes_K \overline{J}^{(j)}_{d_B})(\overline{I}^{(k)}_{d_A} \otimes_K \overline{I}^{(j)}_{d_B} - \overline{J}^{(k)}_{d_A} \otimes_K \overline{J}^{(j)}_{d_B})(\overline{\Gamma}^{(j)}\{A\} \otimes_K \overline{\Gamma}^{(k)}\{B\}) \\
&\qquad\qquad = (\overline{I}^{(j+k)}_{d_A d_B})^2(\overline{\Gamma}^{(j)}\{A\} \otimes_K \overline{\Gamma}^{(k)}\{B\}) \\
&\qquad\qquad\qquad = \overline{I}^{(j+k)}_{d_A d_B}(\overline{\Gamma}^{(j)}\{A\} \otimes_K \overline{\Gamma}^{(k)}\{B\})\,.
\end{aligned} \tag{43}$$

When the matrices do not have a preimage as complex matrices, this relation no longer holds in general. Similarly to how the properties (37a) and (37b) are also not true for all real matrices.

Yet, these hold in sufficiently many cases for our proof techniques to apply. First, Eq. (37b) holds as soon as one of the matrices involved is symmetric, i.e. $\forall A \in \bigotimes_i^j \mathbb{R}^{2d_{A_i}\times 2d_{A_i}}, \forall B \in \bigotimes_l^k \mathbb{R}^{2d_{B_l}\times 2d_{B_l}}$,

$$A = A^T \vee B = B^T \Longrightarrow \mathrm{tr}\left[A \otimes_R B\right] = \frac{1}{2}\mathrm{tr}\left[A \otimes_K B\right] = \frac{1}{2}\mathrm{tr}\left[A\right]\mathrm{tr}\left[B\right]\,. \tag{44}$$

This can be obtained from the definition by a direct computation

$$\mathrm{tr}\left[A \otimes_R B\right] = \frac{1}{2}(\mathrm{tr}\left[A\right]\mathrm{tr}\left[B\right] - \mathrm{tr}\left[\overline{J}^{(j)}_{d_A}A\right]\mathrm{tr}\left[\overline{J}^{(k)}_{d_B}B\right])\,, \tag{45}$$

and using that $\mathrm{tr}\left[\overline{J}^{(j)}_{d_A} A\right] = 0$ if $A^T = A$ since $\overline{J}^{(j)}_{d_A}$ is an antisymmetric matrix.

As for Eq. (37a), it still holds if an overall trace is taken and one of the sides is globally special symmetric. This result, phrased as a lemma below, is the main ingredient to prove every statement of the main text. As is the case with every proof concerning the R-product to some extent, the proof of this lemma is but a consequence of the $\overline{I}^{(n)}$ matrix being a projector.

Lemma B.14 (Equivalence of traces of R-products with traces of Kronecker products) *Let $A, L \in \bigotimes_i \mathbb{R}^{2d_{A_i}\times 2d_{A_i}}$ and $B, R \in \bigotimes_l \mathbb{R}^{2d_{B_l}\times 2d_{B_l}}$ be four matrices. Then, consider the trace of the $\otimes_R$-product of A and B multiplied by the Kronecker product of L and R, $\mathrm{tr}\left[(A \otimes_R B)(L \otimes_K R)\right]$. It has the following properties:*

1. *It is equivalent to the trace of two $\otimes_R$-products,*

$$\mathrm{tr}\left[(A \otimes_R B)(L \otimes_K R)\right] = \mathrm{tr}\left[(A \otimes_R B)(L \otimes_R R)\right] \ . \tag{46}$$

2. *The interchange law can be used if A and B are special symmetric,*

$$\begin{gathered}\forall A \in \mathsf{SY}_{d_A}(\mathbb{R}),\ \forall B \in \mathsf{SY}_{d_B}(\mathbb{R})\ , \\ \mathrm{tr}\left[(A \otimes_R B)(L \otimes_K R)\right] = \mathrm{tr}\left[(AL) \otimes_R (BR)\right]\ ,\end{gathered} \tag{47}$$

3. *If R and L are furthermore symmetric, then the trace commutes with the $\otimes_R$-product,*

$$\begin{gathered}\forall A \in \mathsf{SY}_{d_A}(\mathbb{R}),\ \forall B \in \mathsf{SY}_{d_B}(\mathbb{R}),\ \forall L = L^T,\ \forall R = R^T, \\ \mathrm{tr}\left[(A \otimes_R B)(L \otimes_K R)\right] = \frac{1}{2}\mathrm{tr}\left[AL\right]\mathrm{tr}\left[BR\right]\ .\end{gathered} \tag{48}$$

In this last case, the trace of an $\otimes_R$-product times a Kronecker product is equivalent to the trace of two Kronecker products (up to a factor of 2),

$$\begin{gathered}\forall A \in \mathsf{SY}_{d_A}(\mathbb{R}),\ \forall B \in \mathsf{SY}_{d_B}(\mathbb{R}),\ \forall L = L^T,\ \forall R = R^T, \\ \mathrm{tr}\left[(A \otimes_R B)(L \otimes_K R)\right] = \frac{1}{2}\mathrm{tr}\left[(A \otimes_K B)(L \otimes_K R)\right]\ .\end{gathered} \tag{49}$$

Proof Suppose A is j-partite and B is k-partite, the first equation is a consequence of $\overline{I}^{(j+k)}_{d_A d_B}$ being a projector,

$$\begin{aligned}\mathrm{tr}\left[(A \otimes_R B)(L \otimes_K R)\right] &= \mathrm{tr}\left[\overline{I}^{(j+k)}_{d_A d_B}(A \otimes_K B)\overline{I}^{(j+k)}_{d_A d_B}(L \otimes_K R)\right] \\ &= \mathrm{tr}\left[(\overline{I}^{(j+k)}_{d_A d_B})^2(A \otimes_K B)(\overline{I}^{(j+k)}_{d_A d_B})^2(L \otimes_K R)\right] \\ &= \mathrm{tr}\left[\overline{I}^{(j+k)}_{d_A d_B}(A \otimes_K B)\overline{I}^{(j+k)}_{d_A d_B}\overline{I}^{(j+k)}_{d_A d_B}(L \otimes_K R)\overline{I}^{(j+k)}_{d_A d_B}\right] \\ &= \mathrm{tr}\left[(A \otimes_R B)(L \otimes_R R)\right]\ .\end{aligned} \tag{50}$$

The second equation (47) is proven by using that $\overline{I}^{(j+k)}_{d_A d_B} = \frac{1}{2}(\overline{I}^{(k)}_{d_A} \otimes_K \overline{I}^{(j)}_{d_B} - \overline{J}^{(k)}_{d_A} \otimes_K \overline{J}^{(j)}_{d_B})$. The right-hand side extends into

$$\mathrm{tr}\left[(AL)\otimes_R(BR)\right]=\mathrm{tr}\left[\overline{I}^{(j+k)}_{d_Ad_B}((AL)\otimes_K(BR))\overline{I}^{(j+k)}_{d_Ad_B}\right]$$
$$=\frac{1}{2}\mathrm{tr}\left[((AL)\otimes_K(BR))-((\overline{J}^{(j)}_{d_A}AL)\otimes_K(\overline{J}^{(k)}_{d_B}BR))\right]. \quad (51)$$

Using the first equation (46), the left-hand side extends into

$$\begin{aligned}\mathrm{tr}\left[(A\otimes_R B)(L\otimes_K R)\right]&=\mathrm{tr}\left[(A\otimes_R B)(L\otimes_R R)\right]\\ &=\mathrm{tr}\left[\overline{I}^{(j+k)}_{d_Ad_B}(A\otimes_K B)\overline{I}^{(j+k)}_{d_Ad_B}(L\otimes_K R)\overline{I}^{(j+k)}_{d_Ad_B}\right]\\ &=\frac{1}{4}\mathrm{tr}\big[(AL)\otimes_K(BR)-(A\overline{J}^{(j)}_{d_A}L)\otimes_K(B\overline{J}^{(k)}_{d_B}R)\\ &\quad-(\overline{J}^{(j)}_{d_A}AL)\otimes_K(\overline{J}^{(k)}_{d_B}BR)+(\overline{J}^{(j)}_{d_A}A\overline{J}^{(j)}_{d_A}L)\otimes_K(\overline{J}^{(k)}_{d_B}B\overline{J}^{(k)}_{d_B}R)\big]. \end{aligned} \quad (52)$$

From there, one can see that if A and B commute with, respectively, $\overline{J}^{(j)}_{d_A}$ and $\overline{J}^{(k)}_{d_B}$ then

$$\mathrm{tr}\left[(A\otimes_R B)(L\otimes_K R)\right]=\frac{1}{4}\mathrm{tr}\left[2((AL)\otimes_K(BR))-2((\overline{J}^{(j)}_{d_A}AL)\otimes_K(\overline{J}^{(k)}_{d_B}BR))\right], \quad (53)$$

and so the two sides become equivalent.

Finally, to prove the third and fourth equations, Eqs. (48) and (49), we apply Eq. (44). □

B.3.3 Allowing operationally independent sources

The interpretation of each of the two equations of the last item of the previous lemma has two important consequences.

Corollary B.15 *A n-partite RQT state in R-product form is operationally independent.*

Corollary B.16 *A n-partite RQT state in Kronecker product form is locally indistinguishable from the corresponding RQT state in R-product form.*

With these, it is straightforward to weaken the multilocal RNQT model into a 'less modified' submodel of RQT with operationally independent sources.

Lemma B.17 *The image of the states and of the effects of the n-partite multilocal QT model of matrix dimension $d_Ad_B\ldots$ through the pair of maps $(\frac{1}{2}\overline{\Gamma}^{(n)},\overline{\Gamma}^{(n)})$ form a sub-model in an n-partite multilocal RQT model of matrix dimension $2d_A2d_B\ldots$ that has underwent the following modifications:*

- *The state of independent sources is represented by operationally independent states.*
- *The k-partite joint effects on subsystems $\mathrm{XY}\ldots$ sum up to $\overline{I}^{(k)}_{d_Xd_Y\ldots}$ instead of $\mathbb{1}_{2^kd_Xd_Y\ldots}$.*

Proof We start with the states. From corollary B.13, any QT product state on the n-parties, say $\rho^{\mathbb{C}}_{\mathrm{X}}\otimes_K\sigma^{\mathbb{C}}_{\mathrm{Y}}$ with X encompassing the first $k\le n$ subsystems, is mapped to an R-product RQT state via

$$\frac{1}{2}\overline{\Gamma}^{(n)}\{\rho^{\mathbb{C}}_{\mathrm{X}}\otimes_K\sigma^{\mathbb{C}}_{\mathrm{Y}}\}=2\left(\frac{1}{2}\overline{\Gamma}^{(k)}\{\rho^{\mathbb{C}}_{\mathrm{X}}\}\otimes_R\frac{1}{2}\overline{\Gamma}^{(n-k)}\{\sigma^{\mathbb{C}}_{\mathrm{Y}}\}\right), \quad (54)$$

where $\frac{1}{2}\overline{\Gamma}^{(k)}\{\rho_{\rm X}^{\mathbb{C}}\} := \rho_{\rm X}$ is a valid RQT state of the subsystem shared by the first k parties; $\frac{1}{2}\overline{\Gamma}^{(n-k)}\{\sigma_{\rm Y}^{\mathbb{C}}\} := \sigma_{\rm Y}$, one of the following $n-k$ parties; and $2\,(\cdot \otimes_R \cdot) := \cdot \otimes_{\rm R} \cdot$ is the image of the QT state-composition rule.

Because of lemma B.14, these R-product states are always operationally independent. Hence, the first modification of corollary B.13 can be weakened to simply requiring the state of independent sources to be operationally independent states.

We now turn our attention on the effects. By lemma corollary B.13 again, any QT product effect, say $X_x^{\mathbb{C}} \otimes_K Y_y^{\mathbb{C}}$, is mapped to a R-product of real matrices via

$$\overline{\Gamma}^{(n)}\{X_x^{\mathbb{C}} \otimes_K Y_y^{\mathbb{C}}\} = \overline{\Gamma}^{(k)}\{X_x^{\mathbb{C}}\} \otimes_R \overline{\Gamma}^{(n-k)}\{Y_y^{\mathbb{C}}\}\,, \tag{55}$$

where $\overline{\Gamma}^{(k)}\{X_x^{\mathbb{C}}\} := X_x$ is a positive semi-definite matrix part of a set $\{X_x\}_x$ of positive semi-definite real matrices summing up to $\sum_x X_x = \overline{I}^{(k)} \otimes_K \mathbb{1}_{d_X} = \overline{I}^{(k)}_{d_X}$; and $\overline{\Gamma}^{(n-k)}\{Y_y^{\mathbb{C}}\} := Y_y$ is a positive semi-definitite matrix part of a set summing up to $\sum_y Y_y = \overline{I}^{(n-k)} \otimes_K \mathbb{1}_{d_Y}$.

Finally we can use the same proof as proposition 8 –namely making the projector $\overline{I}^{(n)}_{d_X d_Y}$ appear on the state side of any inner product, i.e, $\rho_{\rm X} \otimes_{\rm R} \sigma_{\rm Y} = \overline{I}^{(n)}_{d_X d_Y}(\rho_{\rm X} \otimes_{\rm R} \sigma_{\rm Y})\overline{I}^{(n)}_{d_X d_Y}$, and then passing this projector on the effect side via the cyclicity of the trace in order to turn the $\otimes_K$ of effects into an $\otimes_R$– to show that the $\otimes_R$-product of effects can be replaced by a Kronecker product without altering any predicted probability distribution. $\square$

By the corollaries of lemma B.14, an RQT state of the form $\rho_{\rm X} \otimes_{\rm R} \sigma_{\rm Y}$ is operationally independent (as well as locally indistinguishable from the state $\rho_{\rm X} \otimes_K \sigma_{\rm Y}$), which explains the first modification of the model. This modification can also be thought of as 'operational independence allows using $\rho_{\rm X} \otimes_{\rm R} \sigma_{\rm Y}$ as the image of the QT state $\rho_{\rm X}^{\mathbb{C}} \otimes_{\rm R} \sigma_{\rm Y}^{\mathbb{C}}$ in lieu of $\rho_{\rm X} \otimes_K \sigma_{\rm Y}$' to a certain extent (namely, within Born rules involving local measurements, but keep in mind that the R-product and Kronecker-product states still technically correspond to different independent preparation procedures that a global measurement could distinguish).

The other difference with the local case is that the effects must also be modified. To understand why it is the case, see how one passes from RNQT to RQT in a local scenario, for example the Bell one:

$$\begin{aligned}
&\mathrm{tr}\left[\rho_{AB}^{\mathbb{C}}(A_a^{\mathbb{C}} \otimes_K B_b^{\mathbb{C}})\right] \\
&\overset{(\text{Lem. } B.12)}{=} \frac{1}{2}\mathrm{tr}\left[\overline{\Gamma}^{(2)}\{\rho_{AB}^{\mathbb{C}}\}(\Gamma\{A_a^{\mathbb{C}}\} \otimes_{\mathrm{R}^*} \Gamma\{B_b^{\mathbb{C}}\})\right] \\
&\overset{(\text{Prop. B.10})}{=} \frac{1}{2}\mathrm{tr}\left[\overline{\Gamma}^{(2)}\{\rho_{AB}^{\mathbb{C}}\}(\Gamma\{A_a^{\mathbb{C}}\} \otimes_K \Gamma\{B_b^{\mathbb{C}}\})\right] \\
&\overset{(\text{Prop. B.2, B.7})}{=} \mathrm{tr}\left[\rho_{AB}^{\mathbb{R}}(A_a^{\mathbb{R}} \otimes_K B_b^{\mathbb{R}})\right]\,,
\end{aligned} \tag{56}$$

such that $\rho_{AB}^{\mathbb{R}} = \frac{1}{2}\overline{\Gamma}^{(2)}\{\rho_{AB}^{\mathbb{C}}\}$, $A_a^{\mathbb{R}} = \Gamma\{A_a^{\mathbb{C}}\}$, and $B_b^{\mathbb{R}} = \Gamma\{B_b^{\mathbb{C}}\}$ are valid elements of RQT. This succession of steps is essentially the proof of proposition 6. Now compare this to a multilocal scenario, for example the bilocal one:

$$\begin{aligned}
&\mathrm{tr}\left[\rho^{\mathbb{C}}_{AB_1}\otimes_K\sigma^{\mathbb{C}}_{B_2C})(A^{\mathbb{C}}_a\otimes_K B^{\mathbb{C}}_b\otimes_K C^{\mathbb{C}}_z)\right]\\
&\overset{(\text{Lem. }B.13)}{=}\frac{1}{4}\mathrm{tr}\left[(\overline{\Gamma}^{(2)}\{\rho^{\mathbb{C}}_{AB_1}\}\otimes_R\overline{\Gamma}^{(2)}\{\sigma^{\mathbb{C}}_{B_2C}\})(\Gamma\{A^{\mathbb{C}}_a\}\otimes_R\overline{\Gamma}^{(2)}\{B^{\mathbb{C}}_b\}\otimes_R\Gamma\{C^{\mathbb{C}}_z\})\right]\\
&\overset{(\text{Lem. }B.17)}{=}\frac{1}{4}\mathrm{tr}\left[(\overline{\Gamma}^{(2)}\{\rho^{\mathbb{C}}_{AB_1}\}\otimes_R\overline{\Gamma}^{(2)}\{\sigma^{\mathbb{C}}_{B_2C}\})(\Gamma\{A^{\mathbb{C}}_a\}\otimes_K\overline{\Gamma}^{(2)}\{B^{\mathbb{C}}_b\}\otimes_K\Gamma\{C^{\mathbb{C}}_z\})\right],
\end{aligned} \tag{57}$$

We would like to apply propositions B.2 and B.7 to finish the argument, but this cannot be done since the joint effects $\{B_b=\overline{\Gamma}^{(2)}\{B^{\mathbb{C}}_b\}\}_b$ are not part of a POVM: they do not sum up to the identity matrix.

As we have argued that operational independence is a justified relaxation of the state space of multilocal RQT models, it only remains to show that this second modification is not essential to the model. That is, it remains to show that the RNQT effects of the form $\{B_b=\overline{\Gamma}^{(2)}\{B^{\mathbb{C}}_b\}\}_b$ can be freely replaced by genuine RQT effects.

Now, the reason why the joint effects do not sum to the identity matrix in the multilocal case is that they act on parts of independent states. As a result, their subsystems of definition cannot be bundled into a single global system. Thus, we must use the k-fold standard mapping $\overline{\Gamma}^{(k)}$ instead of the one-fold one Γ in order for the complex phase factor to be properly represented across the local R(N)QT states and effects.

B.3.4 Relocalising the n-fold standard mapping

As a last step for the proof, we thus need to find a way to reduce the k-fold standard mappings to a mapping akin to the (single-fold) standard mapping, but in a way that does not interfere with the real representation of the complex phase.

This is not as big as an issue as it may appear: $\overline{I}^{(n)}\otimes_K\mathbb{1}_{d_A}\in\mathbb{R}^{2^nd_A\times 2^nd_A}$ is isomorphic to $\mathbb{1}_{2d_A}$; we only went into this larger matrix space because the operators behave nicely w.r.t. the tensor factorisation. Recall from previous works [2, 17, 18] that the $n-1$ extra $\mathbb{R}^{2\times 2}$ spaces that the $\overline{\Gamma}^{(n)}$ map requires when passing from the complex to the real are there such that each subsystem has its own $\mathbb{R}^{2\times 2}$ 'phase-encoding' extra part. Contrastingly, the Γ map only requires one such 'phase-encoding' extra part that is shared among all subsystems. The matrices $\overline{I}^{(n)}$ and $\overline{J}^{(n)}$ are then but non-local representations of 1 and i that act on every 'phase-encoding' part of each subsystem so to emulate the global complex factors. As for the R-product, it is the correct composition rule for this emulation, as it ensures that the real representation of two local phases is combined into a global one, unlike the Kronecker product [17, 18].

This is fundamentally why the R-product contaminates the Kronecker product: in a multiplication of several operators, it is sufficient that only one operator treats the phase properly (i.e. can be interpreted as an R-product) for the whole multiplication to treat the phase properly, as the phase will just commute with every other operator until reaching the 'good' one. Take for example the Bell scenario Eq. (56), the effects are in Kronecker product, so we have

$$\Gamma\{A_a^{\mathbb{C}}\} \otimes_K \Gamma\{e^{i\theta} B_b^{\mathbb{C}}\} \neq \Gamma\{e^{i\theta} A_a^{\mathbb{C}}\} \otimes_K \Gamma\{B_b^{\mathbb{C}}\}\,. \tag{58}$$

This has somehow 'recluded' the phase on one side of the tensor factorisation. On the other hand, since the state is global, it can be seen as having been applied the 2-fold mapping, and thus all its phase factors can move freely from one side to the other. The contamination property, then, is just the fact that the 'more global representations' (those with an interpretation as the highest-fold standard mapping) can be used as intermediaries to 'pass the phase around'. In the Bell case, it is the joint state that is used as the intermediary:

$$\begin{aligned}
&\operatorname{tr}\left[\overline{\Gamma}^{(2)}\{\rho_{AB}^{\mathbb{C}}\}(\Gamma\{A_a^{\mathbb{C}}\} \otimes_K \Gamma\{e^{i\theta} B_b^{\mathbb{C}}\})\right] \\
&\qquad = \operatorname{tr}\left[\overline{\Gamma}^{(2)}\{e^{i\theta}\rho_{AB}^{\mathbb{C}}\}(\Gamma\{A_a^{\mathbb{C}}\} \otimes_K \Gamma\{B_b^{\mathbb{C}}\})\right] \\
&\qquad\qquad = \operatorname{tr}\left[\overline{\Gamma}^{(2)}\{\rho_{AB}^{\mathbb{C}}\}(\Gamma\{e^{i\theta} A_a^{\mathbb{C}}\} \otimes_K \Gamma\{B_b^{\mathbb{C}}\})\right]\,,
\end{aligned} \tag{59}$$

explaining why the effects do not need to be in R-product form for the phase to be passed around properly.

Now the fact is that operationally independent states are always in R-product form, so they can always be used to pass the phase around for the effects. The use of k-fold mappings on the effect side, which is why lemma B.17 has modified effects, is therefore unnecessary. For example, in the bilocal model, we reached

$$\begin{aligned}
&\operatorname{tr}\left[(\rho_{AB_1}^{\mathbb{C}} \otimes_K \sigma_{B_2C}^{\mathbb{C}})(A_a^{\mathbb{C}} \otimes_K B_b^{\mathbb{C}} \otimes_K C_z^{\mathbb{C}})\right] \\
&\quad = \frac{1}{2}\operatorname{tr}\left[(\overline{\Gamma}^{(2)}\{\rho_{AB_1}^{\mathbb{C}}\} \otimes_R \overline{\Gamma}^{(2)}\{\sigma_{B_2C}^{\mathbb{C}}\})(\Gamma\{A_a^{\mathbb{C}}\} \otimes_K \overline{\Gamma}^{(2)}\{B_b^{\mathbb{C}}\} \otimes_K \Gamma\{C_z^{\mathbb{C}}\})\right]\,,
\end{aligned} \tag{60}$$

but the use of $\overline{\Gamma}^{(2)}$ at Bob's effect, $\overline{\Gamma}^{(2)}\{B_b^{\mathbb{C}}\}$, is purely superfluous if it were not for ensuring that the dimensions of states and effects match. Thus, there is no need to delocalize the phase across Bob's subsystems in this situation; we can simply use a different representation that relocalizes Bob's phase within one of his subsystems, and let the state side handle moving it around.

On that account, we need a mapping that 'relocalises' the n-fold mapping without changing its dimension. This mapping is inspired by but slightly different from the mapping $\tau_{\otimes_R}$ introduced in our previous work [18], and that is used to collapse an n-fold standard mapping into a single-fold one.

Definition B.18 (Relocalisation map) *Let $\widetilde{X} \in \mathbb{R}^{2^n d_X \times 2^n d_X}$ be a matrix on a space with a n-fold factorization like $\mathbb{R}^{2^n d_X \times 2^n d_X} \cong \otimes_{i=1}^n \left(\mathbb{R}^{2\times 2} \otimes \mathbb{R}^{d_{X_i} \times d_{X_i}}\right)$ and where $d_X = \prod_i d_{X_i}$. Then, for a given factor X_j, $1 \leq j \leq n$, we say that the map $\tau_{X_j}^{(n)} : \mathbb{R}^{2^n d_X \times 2^n d_X} \to \mathbb{R}^{2^n d_X \times 2^n d_X}$ defined by:*

$$\tau^{(n)}_{X_j}\{\widetilde{X}\} :\cong I^{\otimes_K n} \otimes_K \mathrm{tr}_{2_{X_1}2_{X_2}\cdots 2_{X_n}}\left[\frac{1}{2}\overline{I}^{(n)}_{d_{X_1}d_{X_2}\cdots d_{X_n}}\ \widetilde{X}\right] + (I^{\otimes_K (j-1)} \otimes_K J \otimes_K I^{\otimes_K (n-j)}) \otimes_K \mathrm{tr}_{2_{X_1}2_{X_2}\cdots 2_{X_n}}\left[\frac{1}{2}\overline{J}^{(n)}_{d_{X_1}d_{X_2}\cdots d_{X_n}}\ \widetilde{X}\right], \quad (61)$$

is called the ***relocalization (at system X_j) map****. Here the* $\mathrm{tr}_{2_{X_1}2_{X_2}\cdots 2_{X_n}}[\cdot]$ *symbols stands for taking the trace only over the 2-by-2 systems of each fold.*

Simply put, the relocalisation map replaces the delocalized state of the n 2-by-2 phase-carrying-subsystems by a version that is localised at a chosen subsystem X_j. It does so by replacing the pair $\{\overline{I}^{(n)}, \overline{J}^{(n)}\}$ defined over $\otimes_{i=1}^n \left(\mathbb{R}^{2\times 2}_{X_i}\right)$ by the pair $\{\overline{I}^{(1)}, \overline{J}^{(1)}\}$ at factor X_j and then padding all the other factors with 2×2 identity matrices I.

To lessen clutter, if, in the following, the subscript of the relocalisation map is omitted, it should always be understood as the map that relocalises the phase in the leftmost factor, e.g., $\tau^{(n)} = \tau^{(n)}_{X_1}$ in the above equation.

The relocalisation map is a real-linear map that, when composed with an n-fold mapping, has exactly the properties we need:

Lemma B.19 *The map* $\tau^{(n)} \circ \overline{\Gamma}^{(n)} : \bigotimes_{i=1}^n \mathbb{C}^{d_i\times d_i} \to \bigotimes_{i=1}^n \mathbb{R}^{2d_i\times 2d_i}$ *is a positive-semi-definitness-preserving $*$-algebra homomorphism. In addition, it is unital,*

$$\overline{\Gamma}^{(n)}\{\mathbb{1}_d\} = \mathbb{1}_{2^n d}\,, \quad (62)$$

($d = \prod_{i=1}^n d_i$), and it verifies the following for all $A, B \in \bigotimes_{i=1}^n \mathbb{C}^{d_i\times d_i}$*:*

$$\mathrm{tr}\left[\overline{\Gamma}^{(n)}\{A\}\overline{\Gamma}^{(n)}\{B\}\right] = \mathrm{tr}\left[\overline{\Gamma}^{(n)}\{A\}(\tau^n \circ \overline{\Gamma}^{(n)})\{B\}\right]. \quad (63)$$

Proof Since the relocalisation map $\tau^{(n)}$ is, by definition, a 'trace-and-replace' map that does not act on the subsystems where X is defined, it should be obvious why composing it with $\overline{\Gamma}^{(n)}$ is again a positive-semi-definiteness-preserving $*$-algebra homomorphism. Unitality follows by directly applying the definitions. As for the last property, a direct computation leads to

$$\begin{aligned}
&\mathrm{tr}\left[\overline{\Gamma}^{(n)}\{A\}(\tau^n \circ \overline{\Gamma}^{(n)})\{B\}\right] \\
&= \mathrm{tr}\left[(\overline{I}^{(n)} \otimes_K \mathrm{Re}\,(A) + \overline{J}^{(n)} \otimes_K \mathrm{Im}\,(A))\,\tau^n\{(\overline{I}^{(n)} \otimes_K \mathrm{Re}\,(B) + \overline{J}^{(n)} \otimes_K \mathrm{Im}\,(B)\}\right] \\
&= \mathrm{tr}\Big[(\overline{I}^{(n)} \otimes_K \mathrm{Re}\,(A) + \overline{J}^{(n)} \otimes_K \mathrm{Im}\,(A)) \\
&\qquad \{((I \otimes_K I^{\otimes_K(n-1)}) \otimes_K \mathrm{Re}\,(B) + (J \otimes_K I^{\otimes_K(n-1)}) \otimes_K \mathrm{Im}\,(B)\}\Big] \\
&= \mathrm{tr}\Big[\overline{I}^{(n)} \otimes_K (\mathrm{Re}\,(A)\,\mathrm{Re}\,(B)) + \overline{J}^{(n)} \otimes_K (\mathrm{Re}\,(A)\,\mathrm{Im}\,(B)) \\
&\qquad + \overline{J}^{(n)} \otimes_K (\mathrm{Im}\,(A)\,\mathrm{Re}\,(B)) - \overline{I}^{(n)} \otimes_K \mathrm{Im}\,(A)\,\mathrm{Im}\,(B)\Big] \\
&= \mathrm{tr}\left[\overline{I}^{(n)} \otimes_K \mathrm{Re}\,(AB) + \overline{J}^{(n)} \otimes_K \mathrm{Im}\,(AB)\right] \\
&= \mathrm{tr}\left[\overline{\Gamma}^{(n)}\{AB\}\right] = \mathrm{tr}\left[\overline{\Gamma}^{(n)}\{A\}\overline{\Gamma}^{(n)}\{B\}\right],
\end{aligned} \quad (64)$$

where we used the definitions for the first and second equalities, lemma C.7 for the third, definitions for the fourth and fifth, and proposition B.7 for the last one. □

With this, we are ready to prove that QT embeds into RQT as RNQT also in the multilocal model.

B.3.5 Multilocal RNQT models are embedded in multilocal RQT models

With the introduction of the R-product, we can characterise the image of an n-partite, single-source, local QT model through the n-fold standard mapping.

Lemma B.20 *The image of the states and of the effects of the n-partite multilocal QT model of matrix dimension $d_A d_B \dots$ through the pair of maps $(\frac{1}{2}\overline{\Gamma}^{(n)}, \overline{\Gamma}^{(n)})$ form a sub-model in an n-partite multilocal RQT model of matrix dimension $2d_A 2d_B \dots$ that has underwent the following modification:*

- *The state of independent sources is represented by operationally independent states.*

Proof Starting from lemma B.17, we only have to prove that there is a way to remove the modification on effects. This is exactly what the localisation map has been developed for: by lemma B.19 we can apply the relocalisation map on every effect without changing any predicted distribution[10]. Since this map still preserves sums and positivity, but makes the n-fold standard mapping unital, the effects now satisfy their RQT definition. □

This brings us to the main result for this section.

Proposition 10 (Multilocal QT models are sub-models of multilocal RQT models.) *A multipartite QT model involving n parties, $m \leq n$ measurements, and $p \leq n$ operationally independent sources, can always be embedded into an RQT model of 2^n-th the matrix size.*

Specifically, let $\mathcal{N} := \{A, B, ...\}$ be the set of parties, $|\mathcal{N}| = n$; let $\mathcal{M} = \{M_1, M_2, ..., M_m\}$ be the set of subsets of $\mathcal{N}$ representing how the parties are regrouped to perform the measurements (i.e., $M_i \subseteq \mathcal{N}$ and $\bigcup_{i=1}^{m} M_i = \mathcal{N}$); and let the sources be represented by a joint state $\rho_{\mathrm{AB}\dots}$ operationally independent w.r.t. the given partitionning into p. Then, there exists a pair of map $(M_{\mathrm{AB}\dots}, E_{\mathrm{AB}\dots})$ from $\mathbb{C}^{d_A d_B \dots \times d_A d_B \dots}$ to $\mathbb{R}^{2d_A 2d_B \dots \times 2d_A 2d_B \dots}$ such that $M_{\mathrm{AB}\dots}$ injectively sends the QT state to an RQT state while preserving operational independence; such that $E_{\mathrm{AB}\dots}$ surjectively sends the QT POVMs to RQT POVMs while preserving locality, (i.e., $E_{\mathrm{AB}\dots} = E_{M_1} \otimes_K E_{M_2} \otimes_K \cdots \otimes_K E_{M_m}$), and such that the pair of map preserves all inner products between any states and effects.

[10] In case this was not clear: the choice of party for the relocalisation is totally arbitrary. Pick your favourite.

Proof Lemma B.20 guarantees that the QT-image of the RQT model is in isometric bijection with the QT model. The mappings are explicitly obtained as

$$\begin{aligned} M_{\mathrm{AB}\dots} &= \frac{1}{2}\overline{\Gamma}^{(n)} \ ; \\ \forall i, 1 \leq i \leq m \ : E_{M_i} &= \tau^{(|M_i|)} \circ \overline{\Gamma}^{(|M_i|)} \ . \end{aligned} \tag{65}$$

□

This last proposition exhausts all possible network structures for nonlocal inequalities. Since the standard mapping is an algebra homomorphism, the proof will readily extend to iterative scenarios. We therefore have covered all possible models of nonlocal games.

C Real representation of the complex structure

Definition C.1 (Two-dimensional real representation of $(1, i)$) *The following two matrices in* $\mathbb{R}^{2\times 2}$,

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \tag{66a}$$

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ , \tag{66b}$$

form what we call a two-dimensional real representation of the complex units $(1, i)$*.*

The reason for the name is self-explanatory and is only presented as a definition to condense the representation-theory background; these matrices obey the same algebraic rules as the units of $\mathbb{C}$, i.e., $I^2 = -J^2 = 1$ and $IJ = JI = J$. They are therefore used to represent a complex number as a real 2×2 matrix. This representation is indeed based on seeing the field $\mathbb{C}$ as the two-dimensional real vector space spanned by the real 1 and imaginary i units, and provides an equivalent representation as the two-dimensional subspace of $\mathbb{R}^{2\times 2}$ spanned by I and J.

We will need a global generalisation of this representation. By this, we mean a representation on an n-fold tensor product space which generalises the matrices I and J into their 'stabiliser code' logical counterparts $\overline{I}^{(n)}$ and $\overline{J}^{(n)}$. In a nutshell, we define $(I, J) \in \mathbb{R}^{2\times 2}$ as the single-fold representation $(\overline{I}^{(1)}, \overline{J}^{(1)})$. The n-fold generalization $(\overline{I}^{(n)}, \overline{J}^{(n)}) \in \bigotimes_{k=1}^{n} \mathbb{R}_k^{2\times 2}$ is defined by its globality property, meaning that we want these to be irreducible representations of $(1, i)$ such that they treat any local application (with respect to the real Kronecker product) of a k-fold representation $(\overline{I}^{(k)}, \overline{J}^{(k)})$, $k \leq n$, as if it were the corresponding n-fold representation. For example, we would need $\overline{I}^{(n)}(\overline{J}^{(k)} \otimes_K \underbrace{I \otimes_K I \otimes_K \dots}_{n-k \text{ terms}})$ to be equivalent to $\overline{I}^{(n)}\overline{J}^{(k)}$. A recursive definition of such matrices is given by the following.

Definition C.2 (n-fold two-dimensional real representation of $(1, i)$) *For* $n \in \mathbb{N}_0$*, the matrices* $(\overline{I}^{(n)}, \overline{J}^{(n)})$ *in* $\mathbb{R}^{2n\times 2n}$ *are defined via the following recursive rule:*

$$(1, i) \in \mathbb{C} \times \mathbb{C} \mapsto (\overline{I}^{(n)}, \overline{J}^{(n)}) \in \mathbb{R}^{2n\times 2n} \times \mathbb{R}^{2n\times 2n} \ :$$

$$\overline{I}^{(1)} := I\,, \quad \overline{I}^{(n)} := \frac{1}{2}\left(\overline{I}^{(n-1)} \otimes_K I - \overline{J}^{(n-1)} \otimes_K J\right)\,; \tag{67a}$$

$$\overline{J}^{(1)} := J\,, \quad \overline{J}^{(n)} := \frac{1}{2}\left(\overline{J}^{(n-1)} \otimes_K I + \overline{I}^{(n-1)} \otimes_K J\right)\,; \tag{67b}$$

with $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ *and* $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

Proposition C.3 *The matrices* $(\overline{I}^{(n)}, \overline{J}^{(n)})$ *provide an irreducible representation in* $\mathbb{R}^{2n\times 2n}$ *of the cyclic group* $\mathbb{Z}_2$ *under matrix multiplication, similarly to how* $(1, i)$ *provides one in* $\mathbb{C}$.

Proof The following are straightforward to show:

$$\overline{I}^{(n)} = \left(\overline{I}^{(n)}\right)^2 = -\left(\overline{J}^{(n)}\right)^2\,; \tag{68a}$$

$$\overline{J}^{(n)} = \overline{J}^{(n)}\overline{I}^{(n)} = \overline{I}^{(n)}\overline{J}^{(n)}\,. \tag{68b}$$

The proof of irreducibility can be done via character theory; see our previous work [18]. □

We will need some notation to prove the properties of these matrices relevant to this manuscript. First of all, the notation $A^{\otimes n} := \underbrace{A \otimes A \otimes \ldots \otimes A}_{n}$ will be used for an n-fold application of a given combination rule to the matrix A. For example, instead of writing $\underbrace{I \otimes_K I \otimes_K \ldots \otimes_K I}_{n} \in \underbrace{\mathbb{R}^{2\times 2} \otimes \mathbb{R}^{2\times 2} \otimes \ldots \otimes \mathbb{R}^{2\times 2}}_{n}$, we shall use the more compact $I^{\otimes_K n} \in \bigotimes_{i=1}^{n} \mathbb{R}^{2\times 2}$.

We will also need to address independent tensor factors in a composite space. To do so, we associate each factor with a party like Alice, Bob, Charlie, etc. as a handle to individually address factors. That way, we can quickly refer to the dimension of a factor by using the notation d_A, to mean that Alice is associated with a space of $d_A \times d_A$-dimensional matrices.

Finally, we will need a notation to 'pad' a matrix with a Kronecker product of unit matrices in order for it to reach a certain size (this procedure is sometimes called an *amplification* in the literature); to do so, we will use a subscript with the corresponding dimension: $A_{d_B} := A \otimes_K \mathbb{1}_{d_B}$. Because the Kronecker product is associative and because $\mathbb{1}_{d_B d_C} = \mathbb{1}_{d_B} \otimes_K \mathbb{1}_{d_C}$, several such subscripts can be combined, like in $A_{d_B d_C} = A \otimes_K \mathbb{1}_{d_B} \otimes_K \mathbb{1}_{d_C}$ for instance. By mixing this notation with the previous ones, we shall then write expressions like

$$I_{A,2_B 2_C} = I_{B,2_A 2_C} = I_{C,2_A 2_B} = I_A \otimes_K I_B \otimes_K I_C\,; \tag{69a}$$

$$I_{A,2_B 2_D} \otimes_K J_C = I_A \otimes_K I_B \otimes_K J_C \otimes_K I_D\,. \tag{69b}$$

Also, when there is no risk of confusion, the labels may be dropped. In the first line, we could have written $I_{2_B 2_C}$ or $I^{\otimes_K 3} = I \otimes_K I \otimes_K I$ instead of the leftmost term in each equation.

Proposition C.4 (Characterisation of the n-fold representation)

$$\overline{I}^{(n)} = \sum_{\substack{\vec{i}=(i_1 i_2 \ldots i_n)\in\{0,1\}^n, \\ \sum_k i_k \textit{ is even}}} \frac{1}{2^{n-1}} (-1)^{\frac{\sum_k i_k}{2}} J^{i_1} \otimes_K J^{i_2} \otimes_K \ldots \otimes_K J^{i_n} \;; \tag{70a}$$

$$\overline{J}^{(n)} = \sum_{\substack{\vec{i}=(i_1 i_2 \ldots i_n)\in\{0,1\}^n, \\ \sum_k i_k \textit{ is odd}}} \frac{1}{2^{n-1}} (-1)^{\frac{\sum_k i_k -1}{2}} J^{i_1} \otimes_K J^{i_2} \otimes_K \ldots \otimes_K J^{i_n} \;; \tag{70b}$$

Proof Using definition C.2, the cases $n=1$, i.e. $\overline{I}^{(1)} = I, \overline{J}^{(1)} = J$ and $n=2$, i.e. $\overline{I}^{(2)} = \frac{1}{2}\left(I \otimes_K I - J \otimes_K J\right)$, $\overline{J}^{(2)} = \frac{1}{2}\left(I \otimes_K J + J \otimes_K I\right)$ can be proven to fit Eq. (70) by direct inspection. We shall now use the $n=2$ case to prove the induction on n.

Suppose it holds for $n-1$, then for the case n, we have for $\overline{I}^{(n)}$,

$$\begin{aligned}
\overline{I}^{(n)} = \frac{1}{2}\Big(\Big(&\sum_{\substack{\vec{i}\in\{0,1\}^{n-1}, \\ \sum_k i_k \text{ is even}}} \frac{1}{2^{n-2}} (-1)^{\frac{\sum_k i_k}{2}} J^{i_1} \otimes_K J^{i_2} \otimes_K \ldots \otimes_K J^{i_{n-1}}\Big) \otimes_K I \\
&- \Big(\sum_{\substack{\vec{i}\in\{0,1\}^{n-1}, \\ \sum_k i_k \text{ is odd}}} \frac{1}{2^{n-2}} (-1)^{\frac{\sum_k i_k -1}{2}} J^{i_1} \otimes_K J^{i_2} \otimes_K \ldots \otimes_K J^{i_{n-1}}\Big) \otimes_K J\Big) \\
=\Big(&\sum_{\substack{\vec{i}\in\{0,1\}^{n-1}, \\ \sum_k i_k \text{ is even}}} \frac{1}{2^{n-1}} (-1)^{\frac{\sum_k i_k}{2}} J^{i_1} \otimes_K J^{i_2} \otimes_K \ldots \otimes_K J^{i_{n-1}} \otimes_K I\Big) \\
&- \Big(\sum_{\substack{\vec{i}\in\{0,1\}^{n-1}, \\ \sum_k i_k \text{ is odd}}} \frac{1}{2^{n-1}} (-1)^{\frac{\sum_k i_k -1}{2}} J^{i_1} \otimes_K J^{i_2} \otimes_K \ldots \otimes_K J^{i_{n-1}} \otimes_K J\Big)\,. \\
=\Big(&\sum_{\substack{\vec{j}\in\{0,1\}^{n-1}, j_n=0, \\ \sum_k^{n-1} j_k \text{ is even}}} \frac{1}{2^{n-1}} (-1)^{\frac{\sum_k j_k}{2}} J^{j_1} \otimes_K J^{j_2} \otimes_K \ldots \otimes_K J^{i_{n-1}} \otimes_K J^{j_n}\Big) \\
&- \Big(\sum_{\substack{\vec{j}\in\{0,1\}^{n}, j_n=1 \\ \sum_k^{n-1} j_k \text{ is odd}}} \frac{1}{2^{n-1}} (-1)^{\frac{\sum_k i_k -1}{2}} J^{i_1} \otimes_K J^{i_2} \otimes_K \ldots \otimes_K J^{i_{n-1}} \otimes_K J^{j_n}\Big)\,.
\end{aligned} \tag{71}$$

This is exactly $\overline{I}^{(n)}$ as in Eq. (70a), since the first term is the sum up to n but only for the cases where the last bit is fixed to 0, $\vec{i} = (i_1, i_2, \ldots i_{n-1},\ i_n = 0)$ whereas the second is the one for the cases where it is fixed to 1, $\vec{i} = (i_1, i_2, \ldots i_{n-1},\ i_n = 1)$. Since i_n can only have a value in $\{0,1\}$, this covers all cases proving the induction.

The proof of (70b) is a similar induction. □

This immediately implies

Corollary C.5 *The matrices* $(\overline{I}^{(n)}, \overline{J}^{(n)})$ *are tensor-factor-permutation-independent.*

since the even (resp. odd) words $\vec{i} \in \{0,1\}^n$ are permutation-invariant.

Also, the same inductive techniques as in the proof of Eq. (70) can be used to further regroup the terms on the right as a k-fold representation.

Corollary C.6 *For every $k \in \mathbb{N}_0, k \leq n$, the definition (67) verifies:*

$$\overline{I}^{(n)} = \frac{1}{2}\left(\overline{I}^{(n-k)} \otimes_K \overline{I}^{(k)} - \overline{J}^{(n-k)} \otimes_K \overline{J}^{(k)}\right) ; \tag{72a}$$

$$\overline{J}^{(n)} = \frac{1}{2}\left(\overline{J}^{(n-k)} \otimes_K \overline{I}^{(k)} + \overline{I}^{(n-k)} \otimes_K \overline{J}^{(k)}\right) . \tag{72b}$$

This last property can be used to show that representation on more subsystems acts globally in the following sense:

Lemma C.7 *The matrices $(\overline{I}^{(n)}, \overline{J}^{(n)})$ are a global representation of $(1, i)$ in the sense that they behave properly when multiplying elements of a smaller-dimensional representation $(\overline{I}^{(k)}, \overline{J}^{(k)})$ with $k \leq n$:*

$\forall k \leq n,$

$$(\overline{I}^{(k)} \otimes_K I^{\otimes(n-k)})\overline{I}^{(n)} = \overline{I}^{(n)}(\overline{I}^{(k)} \otimes_K I^{\otimes(n-k)}) = (\overline{I}^{(n)})^2 ; \tag{73a}$$

$$(\overline{J}^{(k)} \otimes_K I^{\otimes(n-k)})\overline{J}^{(n)} = \overline{J}^{(n)}(\overline{J}^{(k)} \otimes_K I^{\otimes(n-k)}) = (\overline{J}^{(n)})^2 ; \tag{73b}$$

$$(\overline{I}^{(k)} \otimes_K I^{\otimes(n-k)})\overline{J}^{(n)} = \overline{J}^{(n)}(\overline{I}^{(k)} \otimes_K I^{\otimes(n-k)}) = \overline{J}^{(n)} ; \tag{73c}$$

$$(\overline{J}^{(k)} \otimes_K I^{\otimes(n-k)})\overline{I}^{(n)} = \overline{I}^{(n)}(\overline{J}^{(k)} \otimes_K I^{\otimes(n-k)}) = \overline{J}^{(n)} . \tag{73d}$$

Proof The proof is the same for every equation and relies on Eq. (72). We prove the first line as an example:

$$\begin{aligned}
\overline{I}^{(n)}(\overline{I}^{(k)} \otimes_K I^{\otimes(n-k)}) &= \frac{1}{2}\left(\overline{I}^{(k)} \otimes_K \overline{I}^{(n-k)} - \overline{J}^{(k)} \otimes_K \overline{J}^{(n-k)}\right)(\overline{I}^{(k)} \otimes_K I^{\otimes(n-k)}) \\
&= \frac{1}{2}\left((\overline{I}^{(k)})^2 \otimes_K \overline{I}^{(n-k)} - (\overline{J}^{(k)}\overline{I}^{(k)}) \otimes_K \overline{J}^{(n-k)}\right) \\
&= (\overline{I}^{(k)} \otimes_K I^{\otimes(n-k)})\frac{1}{2}\left(\overline{I}^{(k)} \otimes_K \overline{I}^{(n-k)} - \overline{J}^{(k)} \otimes_K \overline{J}^{(n-k)}\right) \\
&= (\overline{I}^{(k)} \otimes_K I^{\otimes(n-k)})\overline{I}^{(n)} \\
&= \frac{1}{2}\left((\overline{I}^{(k)})^2 \otimes_K \overline{I}^{(n-k)} - (\overline{J}^{(k)}\overline{I}^{(k)}) \otimes_K \overline{J}^{(n-k)}\right) \\
&= \frac{1}{2}\left(\overline{I}^{(k)} \otimes_K \overline{I}^{(n-k)} - \overline{J}^{(k)} \otimes_K \overline{J}^{(n-k)}\right) \\
&= (\overline{I}^{(n)})^2 \\
&= \overline{I}^{(n)} .
\end{aligned} \tag{74}$$

□

In a more technically accurate language, a global representation is the tensor product of representations of algebras. In this case, $(\overline{I}^{(n)}, \overline{J}^{(n)})$ is the $n-$fold tensor product of the 2-dimensional irreducible representation (I, J) of $\mathbb{C}$ seen as a real algebra. In terms of representation of algebras, the above lemma is a necessary condition for $(\overline{I}^{(n)}, \overline{J}^{(n)})$ to be a tensor product of representations.