Extended Abstract: Full nonlocality for non-maximally entangled states

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(Dated: March 7, 2026)

It is a well-established fact that some quantum correlations are nonlocal, meaning they cannot be described by a local hidden variable model. Certain quantum correlations are so strong that they cannot be reproduced even if one allows only a small fraction of the rounds to admit a local hidden variable description. These correlations are called fully nonlocal and they lead to Bell inequalities in which the quantum bound saturates the non-signaling bound. A famous and well-known example is the Peres-Mermin square in which the underlying state is a four dimensional maximally entangled state. Surprisingly, examples of full nonlocality are so-far only known for maximally entangled states. In this work, we show that in every local Hilbert space dimension $d \geq 3$ there are non-maximally entangled states that are fully nonlocal. In fact, we derive simple criteria to ensure full nonlocality that are only based on the smallest and largest Schmidt coefficient. Furthermore, we show that all pure entangled states can be activated to show full nonlocality in the many-copy scenario.

I. INTRODUCTION

Bell nonlocality is a striking feature of quantum theory and challenges our classical understanding of nature [1, 2]. Typically, this is certified by violating a Bell inequality, where a quantum strategy exceeds what can be explained by a local hidden variable model. However, in most Bell experiments, the bound that can be achieved by quantum theory is smaller than the bound that could be achieved by general non-signalling theories. A famous example is Tsirelson's bound of the CHSH inequality, in which non-signaling correlations introduced by Popescu and Rohrlich can achieve the maximal value [3–5]. This naturally leads to the question of whether there exist Bell inequalities, in which the highest possible value can be achieved by a quantum strategy.

Two famous examples are the Greenberger, Horne and Zeilinger [6, 7] argument for many parties and the Peres-Mermin (or magic) square [8, 9] for two parties. In both cases, these quantum strategies allow to win a certain game perfectly (with probability one) while all classical strategies perform worse.

This effect, that we call full nonlocality in this work (and which is also referred to as quantum pseudo-telepathy or all-versus-nothing proofs of nonlocality [10]), is arguably the strongest form of nonlocality. They provide a stronger refutation of local hidden variable models compared to standard Bell tests, where only statistical averages cannot be reproduced by a local model [6]. The interest in full nonlocality extends beyond purely foundational considerations. Instances of full nonlocality have been proven to be a resource for near-term quantum computers for non-oracular problems [11, 12], enabling a quantum advantage in shallow quantum circuits compared to constant-depth classical circuits. In addition, nonlocality is established as an important resource for many quantum information problems, such as quantum cryptography [13, 14] or randomness [15]. Hence, fully nonlocal correlations, as being a stronger form of nonlocal correlations, are expected to be useful for the same tasks. In particular, full nonlocality has applications in several protocols for device-independent quantum key distribution (DI-QKD) [16, 17]. Moreover, these techniques enable the construction of robust randomness certification schemes [18]. Given these advantages, deciding which states exhibit full nonlocality is a pivotal objective, for both theoretical interest and practical implementation.

Despite the notion of full nonlocality being introduced already in the '80s ([7–9, 19–22]), determining exactly which quantum states possess full nonlocality remains an open question. It is known that maximally entangled states are fully nonlocal [23–27]. Beyond these examples, which employ maximally entangled states, it is generally unknown whether other states exhibit fully nonlocal correlations in the bipartite settings.

In this work, we show that full nonlocality extends also to a wide range of non-maximally entangled states. We give sufficient criteria to decide which states can and which states cannot lead to fully nonlocal correlations. In order to show that, we establish a deep connection between full nonlocality and the notion of quantum-state antidistinguishability. The latter was initially introduced by Chaves, Fuchs and Schack [28] and is the basis of the well-known Pusey-Barrett-Rudolph (PBR) theorem [29].

II. PRELIMINARIES

Quantum correlations arise when two distant observers perform measurements on a shared quantum state ρ_{AB}:

$$p_Q(a, b|x, y) = \mathrm{Tr}\big[(A_{a|x} \otimes B_{b|y})\rho_{AB}\big]\,, \tag{1}$$

where the measurements $\{A_x\}_x$ and $\{B_y\}_y$ are given by positive operator-valued measures (POVMs) $A_x = \{A_{a|x}\}_a$ and $B_y = \{B_{b|y}\}_b$, whose elements are positive

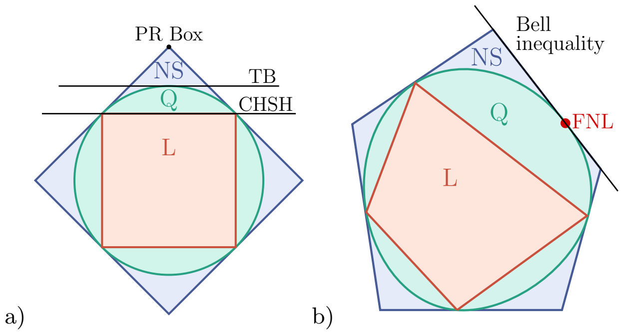

FIG. 1. Graphical representations of the local (L), quantum (Q), and non-signalling (NS) set. a) A section of the set of quantum correlations in the two inputs two outputs scenario. The local polytope (L) is bounded by the CHSH inequality [30]. The Tsirelson's bound (TB) identifies the maximal violation of the CHSH inequality allowed by quantum correlations. The Popescu-Rohrlich (PR) box provides the maximal violation of the CHSH inequality via non-signaling correlations. b) A schematic representation of a fully nonlocal (FNL) quantum correlation. It is a point on the boundary of the non-signaling set that does not contain a vertex of the local polytope.

semidefinite operators $A_{a|x} \succeq 0$ that sum to the identity $\sum_a A_{a|x} = \mathbb{1}$. We will assume that the number of measurement settings and outputs is finite (but not necessarily bounded). These quantum correlations can be written as:

$$p_Q(a,b|x,y) = q \cdot p_L(a,b|x,y) + (1-q) \cdot p_{NL}(a,b|x,y) \quad (2)$$

where $p_{NL}(a,b|x,y)$ is an arbitrary probability distribution, in particular $p_{NL}(a,b|x,y) \geq 0$ for all a,b,x,y, and $p_L(a,b|x,y)$ defines a local strategy:

$$p_L(a,b|x,y) = \sum_\lambda p(\lambda) \cdot p_A(a|x,\lambda) \cdot p_B(b|y,\lambda)\,, \quad (3)$$

with $p(\lambda) \geq 0$ and $\sum_\lambda p(\lambda) = 1$. The local content of some given quantum correlation is defined as the maximum value of q in any such decomposition of p_Q. In this work, we are interested in correlations, in which q is zero in every possible decomposition of p_Q. In that case, we say that the correlations (and the state $|\Psi\rangle_{AB}$) have no local content, or obeys full nonlocality.

It is worth mentioning, that if a state is fully nonlocal, one can construct a Bell inequality, in which the fully nonlocal correlations achieve the highest possible value while all local strategies perform worse, see Ref. [10]. At the same time, if certain quantum correlations have a local content strictly larger than zero, such a Bell inequality cannot exist. Intuitively speaking, one can simulate quantum correlations with a local content by using only shared randomness in some rounds and in those rounds one cannot achieve the highest score. This implies that fully nonlocal correlations lie at the boundary of the non-signalling set, see Fig. 1.

We establish a connection between full nonlocality and the concept of antidistinguishability. A set of states $\{\rho_1, \dots, \rho_n\} = \{\rho_i\}_{i\in[n]}$ is called antidistinguishable if there exists a POVM $\{M_i\}_{i\in[n]}$ (with $\sum_i M_i = \mathbb{1}$ and $M_i \succeq 0$) such that $\mathrm{Tr}[M_i\rho_i] = 0$ holds for all i [28, 31, 32]. Hence, the one who performs the antidistinguishing measurement may not be able to identify the prepared state, but is able to exclude with certainty one of the n preparations, since the outcome M_i implies that the prepared state was not ρ_i. This is also called state exclusion.

A simple set of states $\{|\Psi_0\rangle, |\Psi_1\rangle, |\Psi_2\rangle\}$ that are antidistinguishable are states of the following form:

$$|\Psi_0\rangle = \alpha_1 |1\rangle + \alpha_2 |2\rangle \quad (4)$$
$$|\Psi_1\rangle = \beta_1 |0\rangle + \beta_2 |2\rangle \quad (5)$$
$$|\Psi_2\rangle = \gamma_1 |0\rangle + \gamma_2 |1\rangle\,. \quad (6)$$

Here, the coefficients $\alpha_i, \beta_i, \gamma_i \in \mathbb{C}$ are arbitrary complex numbers (up to normalization of the states). Clearly, a measurement in the computational basis $M_0 = |0\rangle\langle 0|$, $M_1 = |1\rangle\langle 1|$, $M_2 = |2\rangle\langle 2|$ antidistinguishes the three states $\rho_i = |\Psi_i\rangle\langle\Psi_i|$.

III. MAIN THEOREM AND RESULTS

We establish a connection between full nonlocality and antidistinguishability. Denote by $\rho^B_{a|x}$ the post-measurement state on Bob's side when Alice performs the measurement $x \in \{1,\dots,n\} = [n]$ on the bipartite state ρ_{AB} and gets outcome $a \in \{1,\dots,d^A_x\} = [d^A_x]$, formally given as:

$$\rho^B_{a|x} = \frac{\mathrm{Tr}_A[(A_{a|x}\otimes \mathbb{1})\rho_{AB}]}{\mathrm{Tr}[(A_{a|x}\otimes \mathbb{1})\rho_{AB}]}\,. \quad (7)$$

Let us denote by $\boldsymbol{\alpha} = [\alpha_1,\dots,\alpha_n]$, where $\alpha_x \in [d^A_x]$, the collection of outcomes for every input $x \in [n]$.

Theorem III.1. *A bipartite state is fully nonlocal if and only if there exist measurements for Alice such that all the sets of post-measurement states* $\rho^B_{\boldsymbol{\alpha}} := \{\rho^B_{a=\alpha_1|x=1}, \dots, \rho^B_{a=\alpha_n|x=n}\}$ *are antidistinguishable.*

The idea to prove this theorem is that if the set of post-measurement states $\rho^B_{\boldsymbol{\alpha}} := \{\rho^B_{a=\alpha_1|x=1}, \dots, \rho^B_{a=\alpha_n|x=n}\}$ are antidistinguishable and Bob performs the antidistinguishing measurement, then all local strategies in which Alice outputs according to the strategy $\boldsymbol{\alpha} = [\alpha_1,\dots,\alpha_n]$ cannot appear in a decomposition as in Eq. (2). If this is the case for all of Alices strategies, the correlations cannot have a local content and are therefore fully nonlocal.

There are certain sufficient criteria for antidistinguishability that are useful for our work. For instance, it is known that three pure states $\{|\Psi_1\rangle, |\Psi_2\rangle, |\Psi_3\rangle\}$ are antidistinguishable, if all pairs of states fulfill $|\langle\Psi_i|\Psi_j\rangle| \leq 1/2$. This result was first shown by Caves, Fuchs and Schack [28]. Recently, this criterion was further generalized by Johnston, Russo, and Sikora [33], where the following theorem was established:

Theorem III.2 ([33, Cor. 5.3])**.** *A set of n pure quantum states $\{|\Psi_1\rangle, |\Psi_2\rangle, \dots, |\Psi_n\rangle\}$ is antidistinguishable if*

$$\|G\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} |\langle\Psi_i|\Psi_j\rangle|^2} \leq \frac{n}{\sqrt{2}}\,. \tag{8}$$

Here, $G_{ij} = \langle\Psi_i|\Psi_j\rangle$ is the Gram matrix associated with the states $\{|\Psi_i\rangle\}_{i\in[n]}$, and $\|G\|_F$ its Frobenius norm. In particular, the set is antidistinguishable if $|\langle\Psi_i|\Psi_j\rangle| \leq \sqrt{(n-2)/(2n-2)}$ for all pairs $i \neq j$.

Combining these two theorems, we find general classes of states that are fully nonlocal. In fact, we consider bipartite pure states $\rho_{AB} = |\Psi_{AB}\rangle\langle\Psi_{AB}|$ which can be written in the Schmidt basis as

$$|\Psi\rangle_{AB} = \sum_{i=0}^{d-1} \sqrt{\lambda_i}\,|ii\rangle\,. \tag{9}$$

In addition, Alice measures her system along one of n mutually unbiased bases (MUBs), i.e., the POVM elements of Alice's measurements $\{A_x\}_{x\in[n]}$ which are of the form $A_{a|x} = |A_{a|x}\rangle\langle A_{a|x}|$ and satisfy, for every a and a' and every $x \neq x'$:

$$|\langle A_{a|x}|A_{a'|x'}\rangle| = \frac{1}{\sqrt{d}}\,. \tag{10}$$

If Alice and Bob share a maximally entangled state ($\lambda_i = 1/d$ for all i), the post-measurement state on Bob's side conditioned on Alice obtaining the outcome a for the measurement x is given by $|\Psi^B_{a|x}\rangle = |A_{a|x}\rangle^*$, and thus

$$|\langle\Psi^B_{a|x}|\Psi^B_{a'|x'}\rangle| = \frac{1}{\sqrt{d}} \quad (\text{if } x \neq x')\,. \tag{11}$$

Using Theorems III.1 and III.2, this directly proves that all maximally entangled states in dimension $d \geq 4$ exhibit full nonlocality. If we instead have a non-maximally entangled state that is close to being maximally entangled (i.e., all the Schmidt coefficients λ_i are close to $1/d$), then we intuitively expect that the overlaps $|\langle\Psi^B_{a|x}|\Psi^B_{a'|x'}\rangle|$ are close to $1/\sqrt{d}$. As long as these overlaps are below a given bound (as given by Theorem III.2) they are still guaranteed to be antidistinguishable and therefore the state is fully nonlocal. This intuition can be formalized and allows us to prove the following theorem:

Theorem III.3. *Suppose n is the number of MUBs in a given dimension d. A state $|\Psi\rangle_{AB} = \sum_{i=0}^{d-1}\sqrt{\lambda_i}\,|ii\rangle$ is fully nonlocal if*

$$\frac{2}{n}\lambda_{max} + \frac{n-2}{n}\left[1-(d-\sqrt{d})\lambda_{min}\right]^2 \leq \frac{n-2}{2n-2}\,. \tag{12}$$

In particular, this is true if

$$\max\left(\sqrt{\lambda_{max}},\, 1-(d-\sqrt{d})\lambda_{min}\right) \leq \sqrt{\frac{n-2}{2n-2}}\,. \tag{13}$$

Here, λ_{max} and λ_{min} are the largest and smallest Schmidt coefficient of $|\Psi\rangle_{AB}$.

This criterion is simple to apply and allows us to show the following statements:

Result. *Non-maximally entangled states with full nonlocality exist in all dimensions $d \geq 4$.*

Result. *For every pure entangled state $|\Psi\rangle_{AB}$ there exists a number of copies k such that $|\Psi\rangle_{AB}^{\otimes k}$ is fully nonlocal.*

Furthermore, we can also construct an example of partially entangled states in $d = 3$ by using a slightly different approach (in fact, a different set of measurements).

Finally, we also ask which states do not exhibit full nonlocality. We can show that some states with a relatively large Schmidt coefficient have a decomposition as in Eq. 2 and we prove the following theorem:

Theorem III.4. *A d-dimensional entangled state $|\Psi\rangle = \sum_{i=0}^{d-1}\sqrt{\lambda_i}\,|ii\rangle$, where $\lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_{d-1}$ and $\lambda_0 > \lambda_1(d-1)^2$, has a nonzero local content q larger than*

$$q \geq \left[\frac{\sqrt{\lambda_0} - \sqrt{\lambda_1}(d-1)}{d}\right]^2, \tag{14}$$

if Alice and Bob perform projective measurements.

We are also able to show a related theorem regarding the most general class of measurements, namely POVMs.

To conclude, prior to this work the effect of full nonlocality was known for maximally entangled states. Using the connection with antidistinguishability and the Theorems by Caves, Fuchs and Schack [28] as well as Johnston, Russo and Sikora [33], we can show that this effect extends to many pure entangled quantum states. In fact, they exist in all dimensions $d \geq 3$ and can be activated in the many-copy scenario. We are also able to show that in every dimension there are pure entangled states that have a local content and are therefore not fully nonlocal. This was known before only for qubits ($d = 2$) [23, 34–37] and qutrits ($d = 3$) [34]. For future research, it would be interesting to determine exactly which states are fully nonlocal and which states are not, or, extend this notion to the multipartite setting or to mixed quantum states.

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I. INTRODUCTION

Bell nonlocality is a striking feature of quantum theory and challenges our classical understanding of nature [1, 2]. In a typical Bell experiment, the parties perform local measurements on a shared entangled quantum state. If the states and measurements are chosen suitably, one can then verify that the resulting statistics cannot be explained by a local hidden variable model. Typically, this is certified by violating a Bell inequality, where a quantum strategy exceeds what can be explained by a local hidden variable model. However, in most Bell experiments, the bound that can be achieved by quantum theory is smaller than the bound that could be achieved by general non-signalling theories. A famous example is Tsirelson's bound of the CHSH inequality, in which non-signaling correlations introduced by Popescu and Rohrlich can achieve the maximal value [3–5]. This naturally leads to the question of whether there exist Bell inequalities, in which the highest possible value can be achieved by a quantum strategy.

A famous example was introduced by Greenberger, Horne and Zeilinger [6], in which the highest possible score for a three party Bell-inequality can be achieved by the three-qubit GHZ state [7]. Another well-known example is the Peres-Mermin (or magic) square [8, 9], which introduces a two-player Bell inequality in which a perfect quantum strategy exists. These Bell inequalities can also be phrased as a game that can be won perfectly with a quantum strategy, while any classical strategy will fail in some cases. Since the winning probability cannot exceed 1, Tsirelson's bound is equal to the non-signaling bound for these Bell inequalities. In this sense, these quantum correlations allow for a stronger refutation of a local hidden variable description of quantum theory than standard Bell tests, where only statistical averages cannot be reproduced by a local model [6].

This effect, that we call full nonlocality in this work (and which is also referred to as quantum pseudo-telepathy or all-versus-nothing proofs of nonlocality [10]), is arguably the strongest form of nonlocality. The interest in full nonlocality extends beyond purely foundational considerations. It has been proven to be a resource for near-term quantum computers for non-oracular problems [11–13], enabling a quantum advantage in shallow quantum circuits compared to constant-depth classical circuits. Moreover, these techniques enable the construction of robust randomness certification schemes [14]. Full nonlocality has further applications in several protocols for device-independent quantum key distribution (DI-QKD) [15, 16]. Given these significant advantages, deciding which states exhibit full nonlocality is a pivotal objective, for both theoretical interest and practical implementation.

Despite the notion of full nonlocality being introduced already in the '80s ([7–9, 17–21]), determining exactly which quantum states possess full nonlocality remains an open question. In the bipartite case, it is known that all pure non-maximally entangled qubit states cannot exhibit full nonlocality [22–26]. Conversely, all maximally entangled states exhibit full nonlocality: via infinite measurements for qubits [27, 28], and finite Kochen-Specker sets for $d \geq 3$ [28–33]. In the multipartite scenario, beyond full nonlocality as described in [6, 34], the notion of genuine multipartite full nonlocality can be further introduced [35–37]. Beyond maximally entangled states, it is generally unknown whether other states exhibit fully nonlocal correlations in the bipartite setting [38].

In this work, we show that full nonlocality also extends to a wide range of non-maximally entangled states. To show this, we establish a deep connection between full nonlocality and the notion of quantum-state antidistinguishability. The latter was initially introduced by Chaves,

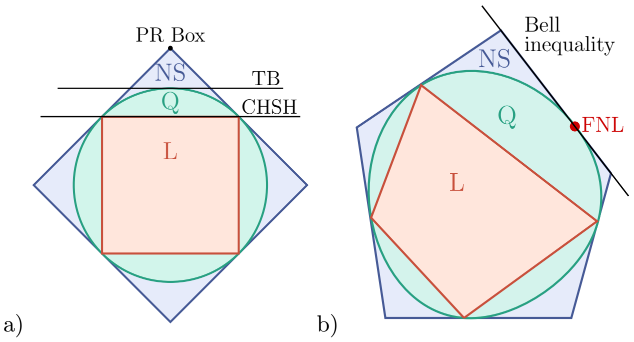

FIG. 1. Graphical representations of the local (L), quantum (Q), and non-signalling (NS) set. a) A section of the set of quantum correlations in the two inputs two outputs scenario. The local polytope (L) is bounded by the CHSH inequality [51]. The Tsirelson's bound (TB) identifies the maximal violation of the CHSH inequality allowed by quantum correlations. The Popescu-Rohrlich (PR) box provides the maximal violation of the CHSH inequality via non-signaling correlations. b) A schematic representation of a fully nonlocal (FNL) quantum correlation. It is a point on the boundary of the non-signaling set that does not contain a vertex of the local polytope.

Fuchs and Schack [39] and is the basis of the well-known Pusey-Barrett-Rudolph (PBR) theorem [40] which challenges the epistemic view of quantum states. Beyond its foundational relevance, the concept of antidistinguishability has found many applications demonstrating quantum advantages in communication [41–43], and in recent years, general theorems have been proven regarding the antidistinguishability of quantum states [44–48]. Finally, the relationship between antidistinguishability and contextuality was studied [49, 50].

Elaborating on the relation between full nonlocality and antidistinguishability, we are able to show full nonlocality for a broad class of pure entangled quantum states. We derive a simple criterion (see Theorem V.1) and use it to show that non-maximally entangled fully nonlocal states exist in every dimension $d \geq 4$. Furthermore, we are also able to provide examples of fully nonlocal partially entangled states even in dimension $d = 3$, thus proving that full nonlocality for non-maximally entangled states exists apart from the case $d = 2$.

We further prove that full nonlocality can be activated for every pure entangled state in the many-copy scenario. To be more precise, given a pure state, there exists a sufficient number of copies for which the system can generate fully nonlocal correlations. Finally, we will show that, for all dimensions, there exist pure states which do not exhibit full nonlocality, even with respect to the most general class of quantum measurements. Specifically, this occurs when the state's maximal Schmidt coefficient is too large relative to the other Schmidt coefficients.

II. PRELIMINARIES

In this work, we consider bipartite pure states $\rho_{AB} = |\Psi_{AB}\rangle\langle\Psi_{AB}|$ which can be written in the Schmidt basis as

$$|\Psi\rangle_{AB} = \sum_{i=0}^{d-1} \sqrt{\lambda_i}\,|ii\rangle\ . \tag{1}$$

Quantum correlations arise when two distant observers perform measurements on a shared quantum state ρ_{AB}:

$$p_Q(a,b|x,y) = \mathrm{Tr}\big[(A_{a|x}\otimes B_{b|y})\rho_{AB}\big]\,, \tag{2}$$

where the measurements $\{A_x\}_x$ and $\{B_y\}_y$ are given by positive operator-valued measures (POVMs) $A_x = \{A_{a|x}\}_a$ and $B_y = \{B_{b|y}\}_b$, whose elements are positive semidefinite operators $A_{a|x} \succeq 0$ that sum to the identity $\sum_a A_{a|x} = \mathbb{1}$. We will assume that the number of measurement settings and outputs is finite (but not necessarily bounded). These quantum correlations can be written as:

$$p_Q(a,b|x,y) = q\cdot p_L(a,b|x,y) + (1-q)\cdot p_{NL}(a,b|x,y) \tag{3}$$

where $p_{NL}(a,b|x,y)$ is an arbitrary probability distribution, in particular $p_{NL}(a,b|x,y) \geq 0$ for all a,b,x,y[1], and $p_L(a,b|x,y)$ defines a local strategy:

$$p_L(a,b|x,y) = \sum_\lambda p(\lambda)\cdot p_A(a|x,\lambda)\cdot p_B(b|y,\lambda)\,, \tag{4}$$

with $p(\lambda) \geq 0$ and $\sum_\lambda p(\lambda) = 1$. The local content of some given quantum correlation is defined as the maximum value of q in any such decomposition of p_Q. If there exists a decomposition, where $q = 1$, the correlation are said to be local. If $q < 1$ in any such decomposition, the correlations are said to be nonlocal. This can be, for instance, certified by the violation of a Bell inequality [1, 2]. In this work, we are interested in correlations, in which q is zero in every possible decomposition of p_Q. If this is the case, we say that the correlation has no local content, or obeys full nonlocality. Similarly, we will say that a state ρ_{AB} is fully nonlocal if it can be employed to generate fully nonlocal correlations. While in this work we mostly consider scenarios with finitely many inputs and outputs, it is worth pointing out that some quantum states, for instance the maximally entangled two-qubit state, can show full nonlocality only if an infinite set of measurements is considered [30].

By observing that local correlations as in Eq. (4) can be decomposed into local deterministic strategies, one obtains a convenient and standard way to check full nonlocality (see Appendix A for further details). If none of

[1] It can be shown that p_{NL} is non-signaling, since it is the difference of two non-signaling distributions p_Q and p_L.

the local deterministic strategies can appear in the correlations of $p_Q(a,b|x,y)$, the correlations are fully nonlocal. See Table I for an illustration of the Popescu-Rohrlich correlations, which represent the simplest fully nonlocal (but *not* quantum) correlations.

$p_{PR}(ab|xy) =$

	y	0		1	
x	a \ b	0	1	0	1
0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0
	1	0	$\frac{1}{2}$	0	$\frac{1}{2}$
1	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$
	1	0	$\frac{1}{2}$	$\frac{1}{2}$	0

$p_L(ab|xy) =$

	y	0		1	
x	a \ b	0	1	0	1
0	0	1	0	0	1
	1	0	0	0	0
1	0	1	0	0	1
	1	0	0	0	0

TABLE I. Left: Correlations $p_{PR}(a,b|x,y)$ for the Popescu-Rohrlich Box, the simplest fully nonlocal correlations (they cannot be achieved in quantum theory). Right: A local deterministic strategy, where Alice outputs $a = 0$ for both inputs and Bob outputs $b = y$. p_{PR} is fully nonlocal, since any of the 16 local deterministic strategies cannot appear in $p_{PR}(a,b|x,y)$. For the local strategy shown here, the contradiction arises since $p_{PR}(a=0,b=1|x=0,y=1) = 0$ but $p_L(a=0,b=1|x=0,y=1) = 1$.

In addition, a Bell inequality is defined by $I_{abxy} \in \mathbb{R}$ and the value w for given correlations $p(a,b|x,y)$ as

$$w = \sum_{a,b,x,y} I_{abxy}\, p(a,b|x,y)\,. \tag{5}$$

We denote as w_L, w_Q, w_{NS} the largest score that can be achieved with local, quantum, and non-signalling distributions. It is worth mentioning, that if a state is fully nonlocal, one can construct a Bell inequality, in which the fully nonlocal correlations achieve the highest possible value while all local strategies perform worse ($w_L < w_Q = w_{NS}$). While the details for this connection are made explicit in Ref. [10], it can be understood in the following way. Define the Bell inequality such that it assigns a value of one to every event that occurs with a nonzero probability in $p_Q(a,b|x,y)$ and a value of zero to any event that occurs with zero probability in $p_Q(a,b|x,y)$. It is clear that the quantum value for this inequality is the largest possible one and no non-signalling correlations can achieve a higher value. At the same time, any local deterministic strategy cannot avoid all zeros in the Bell inequality when $p_Q(a,b|x,y)$ is fully nonlocal. Hence, the local bound is strictly less than the quantum (or non-signalling) bound. This motivates the picture in Fig. 1.

At the same time, if certain quantum correlations have a local content strictly larger than zero, such a Bell inequality with $w_L < w_Q = w_{NS}$ cannot exist. Since the Bell inequality is a linear function, a decomposition of the form in Eq. (3) implies that $w_Q \leq q\, w_L + (1-q)\, w_{NS}$. Intuitively speaking, one can simulate quantum correlations that have a local content with local resources in some rounds and in those rounds the performance is bounded by the local bound.

We will establish a connection between full nonlocality and the concept of antidistinguishability. A set of states $\{\rho_1, \ldots, \rho_n\} = \{\rho_i\}_{i\in[n]}$ is called antidistinguishable if there exists a POVM $\{M_i\}_{i\in[n]}$ (with $\sum_i M_i = \mathbb{1}$ and $M_i \succeq 0$) such that $\mathrm{Tr}[M_i\rho_i] = 0$ holds for all i [39, 45, 52]. Hence, the one who performs the antidistinguishing measurement may not be able to identify the prepared state, but is able to exclude with certainty one of the n preparations, since the outcome M_i implies that the prepared state was not ρ_i. This is also called state exclusion and it is worth mentioning that it can be solved via semidefinite programming (SDP) [44, 48]:

$$\begin{aligned} &\min_{\{M_i\}_i} \sum_i \mathrm{Tr}[M_i\rho_i] \\ &\text{subject to}: \sum_i M_i = \mathbb{1}\,,\ M_i \succeq 0. \end{aligned} \tag{6}$$

The set is antidistinguishable if that minimum is zero, since $\mathrm{Tr}[M_i\rho_i] \geq 0$ implies that $\mathrm{Tr}[M_i\rho_i] = 0$ for all i. Note that the measurement that minimizes this expression does not have to be unique (see App. D for an example). It is also possible that $M_i = 0$ for some i. In particular, if a subset $\{\rho_i\}_{i\in\mathcal{S}\subseteq[n]}$ of n states is antidistinguishable, then the whole set $\{\rho_i\}_{i\in[n]}$ is antidistinguishable by defining $M_i = 0$ for all $i \in [n]\backslash\mathcal{S}$.[2] Furthermore, we say a measurement $\{M_j\}_{j\in[k]}$ is an antidistinguishing measurement for $\{\rho_j\}_{j\in[n]}$ if and only if it satisfies

$$\forall M \in \{M_j\}_{j\in[k]}\,, \exists \rho \in \{\rho_j\}_{j\in[n]} \text{ s.t. } \mathrm{Tr}[M\rho] = 0\,. \tag{7}$$

Hence, for every outcome j of the antidistinguishing measurement, there is always a state $\rho \in \{\rho_j\}_{j\in[n]}$ such that the outcome j never occurs. In Appendix A, we show that the existence of such a measurement is equivalent to the antidistinguishability of $\{\rho_j\}_{j\in[n]}$, see Obs. 1.

A simple set of states $\{|\Psi_0\rangle, |\Psi_1\rangle, |\Psi_2\rangle\}$ that are antidistinguishable are states of the following form:

$$|\Psi_0\rangle = \alpha_1 |1\rangle + \alpha_2 |2\rangle \tag{8}$$
$$|\Psi_1\rangle = \beta_1 |0\rangle + \beta_2 |2\rangle \tag{9}$$
$$|\Psi_2\rangle = \gamma_1 |0\rangle + \gamma_2 |1\rangle\,. \tag{10}$$

Here, the coefficients $\alpha_i, \beta_i, \gamma_i \in \mathbb{C}$ are arbitrary complex numbers (up to normalization). Clearly, a measurement in the computational basis $M_0 = |0\rangle\!\langle 0|$, $M_1 = |1\rangle\!\langle 1|$, $M_2 = |2\rangle\!\langle 2|$ antidistinguishes the three states $\rho_i = |\Psi_i\rangle\!\langle\Psi_i|$.

III. CONNECTING FULL NONLOCALITY AND ANTIDISTINGUISHABILITY

In this section, we establish a connection between full nonlocality and antidistinguishability. Denote by $\rho^B_{a|x}$

[2] Note that there are (slightly) different notions of antidistinguishability in the literature. The one that is relevant for this work, it is allowed that $M_i = 0$ for some i. This is sometimes called weak antidistinguishability, as for instance in Ref. [50].

the post-measurement state on Bob's side when Alice performs the measurement $x \in \{1,\dots,n\} = [n]$ on the bipartite state ρ_{AB} and gets outcome $a \in \{1,\dots,d_x^A\} = [d_x^A]$, formally given as:

$$\rho^B_{a|x} = \frac{\mathrm{Tr}_A[(A_{a|x} \otimes \mathbb{1})\rho_{AB}]}{\mathrm{Tr}[(A_{a|x} \otimes \mathbb{1})\rho_{AB}]}\,. \tag{11}$$

Let us denote by $\boldsymbol{\alpha} = [\alpha_1,\dots,\alpha_n]$, where $\alpha_x \in [d_x^A]$, the collection of outcomes for every input $x \in [n]$. Suppose now that for some fixed $\boldsymbol{\alpha}$ the post-measurement states $\rho^B_{\boldsymbol{\alpha}} := \{\rho^B_{a=\alpha_1|x=1},\dots,\rho^B_{a=\alpha_n|x=n}\}$ are antidistinguishable. Further suppose that Bob, by choosing a measurement labeled as $y = \boldsymbol{\alpha}$, performs the antidistinguishing measurement $\{B_{b|y=\boldsymbol{\alpha}}\}_{b\in[n]}$. Then, it follows from the definition of antidistinguishability that $\mathrm{Tr}[\rho^B_{a=\alpha_j|x=j}B_{b=j|y=\boldsymbol{\alpha}}] = 0$ for all values of $j \in [n]$. Therefore, conditioned on Alice giving the outcome $a = \alpha_j$ for $x = j$, the probability that Bob gives the outcome $b = j$ for $y = \boldsymbol{\alpha}$ is 0, i.e.,

$$\begin{aligned} p_Q(b = j|x = j, y = \boldsymbol{\alpha}, a = \alpha_j) \\ = \mathrm{Tr}[\rho^B_{a=\alpha_j|x=j}B_{b=j|y=\boldsymbol{\alpha}}] = 0\,, \end{aligned} \tag{12}$$

which further implies, using Bayes' rule $p_Q(a,b|x,y) = p_Q(b|x,y,a)p_Q(a|x,y)$, that

$$p_Q(a = \alpha_j, b = j|x = j, y = \boldsymbol{\alpha}) = 0\,. \tag{13}$$

The equation above implies that in any decomposition of p_Q as in Eq. (3), the local strategy p_L must satisfy $p_L(a = \alpha_j, b = j|x = j, y = \boldsymbol{\alpha}) = 0$ for all values of $j \in [n]$. It is now easy to see that any deterministic local strategy p_L in which Alice's device outputs the outcomes according to the strategy given by $\boldsymbol{\alpha}$ cannot appear in the decomposition of Eq. (3). Indeed, for any such local deterministic strategy, Bob must assign some outcome $b = j$ for the measurement choice $y = \boldsymbol{\alpha}$, and hence $p_L(a = \alpha_j, b = j|x = j, y = \boldsymbol{\alpha}) = 1 > 0$ for some $j \in [n]$ (see also Tab. II).

This directly establishes a link– if for all choices of $\boldsymbol{\alpha}$ (i.e., for all possible local deterministic strategies of Alice), the set of post-measurement states $\rho^B_{\boldsymbol{\alpha}}$ are antidistinguishable, then no local deterministic strategy can appear in the decomposition of Eq. (3), which is enough to guarantee that the resulting correlations have zero local content (also see Appendix A for a more explicit proof).

A converse statement also holds: If for a given state and some measurements for Alice there is at least one $\boldsymbol{\alpha}$ such that the post-measurement states $\rho^B_{\boldsymbol{\alpha}}$ are *not* antidistinguishable, then the resulting correlations always have a local content, irrespective of the measurements performed by Bob (see Appendix A for proof). This leads us to the following theorem:

Theorem III.1. *A bipartite state is fully nonlocal if and only if there exist measurements for Alice such that all the sets* $\rho^B_{\boldsymbol{\alpha}} := \{\rho^B_{a=\alpha_1|x=1},\dots,\rho^B_{a=\alpha_n|x=n}\}$ *of post-measurement states are antidistinguishable.*

Note that the number of measurement settings required for Bob can decrease (sometimes significantly) if the same measurement can antidistinguish several sets of post-measurement states $\rho^B_{\boldsymbol{\alpha}}$ as it happens, for instance, in the Peres-Mermin square (see Appendix D).

		...	$y = (a_1 = 2, a_2 = 4, a_3 = 1)$			...
		...	$b = 1$	$b = 2$	$b = 3$	...
$x = 1$	$a_1 = 1$	...	×	×	×	
	$a_1 = 2$	...	0	×	×	
	$a_1 = 3$	...	×	×	×	
	$a_1 = 4$	...	×	×	×	
$x = 2$	$a_2 = 1$	...	×	×	×	
	$a_2 = 2$	...	×	×	×	
	$a_2 = 3$	...	×	×	×	
	$a_2 = 4$	...	×	0	×	
$x = 3$	$a_3 = 1$	...	×	×	0	
	$a_3 = 2$	...	×	×	×	
	$a_3 = 3$	...	×	×	×	
	$a_3 = 4$	...	×	×	×	

TABLE II. Alice has three sets of measurements labeled by x with four outcomes a_x each. Suppose the three post-measurement states on Bob's side for the outcomes $(a_1 = 2, a_2 = 4, a_3 = 1)$ (hence $\rho^B_{2|1}$, $\rho^B_{4|2}$, and $\rho^B_{1|3}$) are antidistinguishable. Then Bob can perform a measurement that we label with $y = (a_1 = 2, a_2 = 4, a_3 = 1)$ such that $p(a_1, b = 1|x,y) = p(a_2, b = 2|x,y) = p(a_3, b = 3|x,y) = 0$ (marked in red). The other values denoted by "×" are arbitrary and do not matter for our argument. If such a measurement exists, it shows directly that every strategy in which Alice outputs $a_1 = 2$, $a_2 = 4$, $a_3 = 1$ must have zero weight; If Bob wants to output $b = 1$, it might be that Alice has the first measurement setting and the joint outcome will be $a_1 = 2$ and $b = 1$. However, this is prohibited by the 0 in the probability table. Similarly, Bob cannot output $b = 2$ since Alice might have the second measurement setting $x = 2$ and so on. If such an antidistinguishing measurement can be found for all combinations $(a_1, a_2, ..., a_n)$, the behaviour has zero local content.

Now, we can use some general statement regarding the antidistinguishability of quantum states. For instance, it is known that three pure states $\{|\Psi_1\rangle, |\Psi_2\rangle, |\Psi_3\rangle\}$ are antidistinguishable, if all pairs of states fulfill $|\langle\Psi_i|\Psi_j\rangle| \leq 1/2$. This result was first shown by Caves, Fuchs and Schack [39]. Recently, this criterion was further generalized by Johnston, Russo, and Sikora [48], where the following theorem was established:

Theorem III.2 ([48, Cor. 5.3])**.** *A set of n pure quantum states $\{|\Psi_1\rangle, |\Psi_2\rangle,\dots,|\Psi_n\rangle\}$ is antidistinguishable if*

$$\|G\|_F = \sqrt{\sum_{i=1}^n \sum_{j=1}^n |\langle\Psi_i|\Psi_j\rangle|^2} \leq \frac{n}{\sqrt{2}}\,. \tag{14}$$

Here, $G_{ij} = \langle\Psi_i|\Psi_j\rangle$ is the Gram matrix associated with the states $\{|\Psi_i\rangle\}_{i\in[n]}$, and $\|G\|_F$ its Frobenius norm. In particular, the set is antidistinguishable if $|\langle\Psi_i|\Psi_j\rangle| \leq \sqrt{(n-2)/(2n-2)}$ for all pairs $i \neq j$.

In what follows, we will use the relation between full nonlocality and antidistinguishability, together with the aforementioned theorem, to find general classes of states that are fully nonlocal. Note that, in order to apply the theorem, the states need to be pure (and normalized). This is always the case when Alice applies sharp measurements, and the shared state $|\Psi\rangle_{AB}$ is pure.

IV. ACTIVATION OF PURE ENTANGLED QUBIT PAIRS

As a simple first example, we will show that all pure entangled qubit pairs are fully nonlocal in the multi-copy scenario using a finite number of measurements. In contrast, it is known that with a single copy only the maximally entangled two-quibt state is fully nonlocal and further requires an infinite number of measurements settings [22–24]. Consider first a general two-qubit state (we assume $p \geq 1/2$ without loss of generality)

$$|\Psi\rangle_{AB} = \sqrt{p}\,|00\rangle + \sqrt{1-p}\,|11\rangle \ , \tag{15}$$

and suppose Alice has three measurement settings corresponding to the qubit observables X, Y, and Z. The post-measurement states on Bob's side conditioned on Alice obtaining the outcome $a \in \{0,1\}$ for the measurement choice $x \in \{X,Y,Z\}$ are given by $|\Psi^B_{a|Z}\rangle = |a\rangle$, $|\Psi^B_{a|X}\rangle = \sqrt{p}\,|0\rangle + (-1)^a\sqrt{1-p}\,|1\rangle$, and $|\Psi^B_{a|Y}\rangle = \sqrt{p}\,|0\rangle + (-1)^a\,\mathrm{i}\sqrt{1-p}\,|1\rangle$. Calculating the inner products of these post-measurement states and using $\sqrt{p^2+(1-p)^2} \leq \sqrt{p}$ for $p \geq 1/2$, we obtain, for $x, x' \in \{X,Y,Z\}$ and $a, a' \in \{0,1\}$,

$$|\langle \Psi^B_{a|x}|\Psi^B_{a'|x'}\rangle| \leq \sqrt{p} \quad (x \neq x')\,. \tag{16}$$

Suppose now that Alice and Bob share k copies of this two-qubit state $|\Psi\rangle_{AB}$ and once again Alice has three measurement settings corresponding to performing one of X, Y, or Z basis measurement on all the individual qubits. That is, Alice measures along the same basis on all of her k qubits, and obtains the k-tuple outcome $a = (a_1, \ldots, a_k) \in \{0,1\}^k$. The post-measurement states on Bob's side are then the tensor products of single copy post-measurement states, i.e.,

$$\begin{aligned}
|\Psi^B_{a|Z}\rangle &= |a\rangle = |a_1 \ldots a_k\rangle \ , \\
|\Psi^B_{a|X}\rangle &= \bigotimes_{i=1}^{k} \left(\sqrt{p}\,|0\rangle + (-1)^{a_i}\sqrt{1-p}\,|1\rangle\right) \ , \\
|\Psi^B_{a|Y}\rangle &= \bigotimes_{i=1}^{k} \left(\sqrt{p}\,|0\rangle + \mathrm{i}(-1)^{a_i}\sqrt{1-p}\,|1\rangle\right) \ ,
\end{aligned} \tag{17}$$

and satisfy, for all $x, x' \in \{X,Y,Z\}$,

$$|\langle \Psi^B_{a|x}|\Psi^B_{a'|x'}\rangle| \leq \sqrt{p}^{\,k} \quad (x \neq x')\,, \tag{18}$$

due to the tensor product structure. If $\sqrt{p}^{\,k} \leq 1/2$, then all the sets of post-measurement states $\{|\Psi^B_{a|X}\rangle, |\Psi^B_{a'|Y}\rangle, |\Psi^B_{a''|Z}\rangle\}$ are antidistinguishable (Theorem. III.2) and hence the state $(\sqrt{p}\,|00\rangle + \sqrt{1-p}\,|11\rangle)^{\otimes k}$ demonstrates full nonlocality (Theorem. III.1).

For the maximally entangled state ($p = 1/2$) two copies ($k = 2$) are enough to demonstrate full nonlocality with a finite number of measurement settings but even very weakly entangled states can be activated if sufficiently many copies are available, since there always exists a value of k such that $\sqrt{p}^{\,k} \leq 1/2$ for $p \leq 1$. It is worth noting that we have used only three separable measurements for Alice. Later on, we will use similar ideas to extend this result to arbitrary d-dimensional pure entangled states.

V. SUFFICIENT CONDITIONS FOR FULL NONLOCALITY FROM MUTUALLY UNBIASED BASES

In this section, we will derive some simple sufficient criteria for full nonlocality. Recall that a general pure state has the following form in the Schmidt basis

$$|\Psi\rangle_{AB} = \sum_{i=0}^{d-1} \sqrt{\lambda_i}\,|ii\rangle \ . \tag{19}$$

Suppose Alice measures this state along one of n mutually unbiased bases (MUBs), i.e., the POVM elements (projectors) of Alice's n measurements $\{A_x\}_{x\in[n]}$ with d outcomes each are of the form $A_{a|x} = |A_{a|x}\rangle\langle A_{a|x}|$ and satisfy, for $x, x' \in [n]$ and $a, a' \in \{0, \ldots, d-1\}$,

$$|\langle A_{a|x}|A_{a'|x'}\rangle| = \frac{1}{\sqrt{d}} \quad (x \neq x')\,. \tag{20}$$

It is known that at least three MUBs exist in every dimension d, while the exact number for a given dimension is not known. If Alice and Bob share a maximally entangled state ($\lambda_i = 1/d$ for all i), the post-measurement state on Bob's side conditioned on Alice obtaining the outcome a for the measurement x is given by $|\Psi^B_{a|x}\rangle = |A_{a|x}\rangle$, and thus

$$|\langle \Psi^B_{a|x}|\Psi^B_{a'|x'}\rangle| = \frac{1}{\sqrt{d}} \quad (x \neq x')\,. \tag{21}$$

Using Theorems III.1 and III.2, this directly proves that all maximally entangled states in dimension $d \geq 4$ exhibit full nonlocality, which also follows from Ref. [27] by using the connection to Kochen-Specker sets. If we instead have a nonmaximally entangled state that is close to being maximally entangled (i.e., all the Schmidt coefficients λ_i are close to $1/d$), then we intuitively expect that the overlaps $|\langle \Psi^B_{a|x}|\Psi^B_{a'|x'}\rangle|$ above are close to $1/\sqrt{d}$. As long as these overlaps are below a given bound (e.g., $1/2$ for three measurements) then we have full nonlocality due to Theorems III.1 and III.2.

More precisely, let us fix Alice's first measurement to be along the computational basis, i.e., $|A_{a|x=1}\rangle = |a\rangle$.

Due to Eq. (21), the other measurements are then of the form

$$\left|A_{a|x}\right\rangle = \sum_{j=0}^{d-1} \langle j|A_{a|x}\rangle\, |j\rangle = \sqrt{\frac{1}{d}} \sum_{j=0}^{d-1} \mathrm{e}^{\mathrm{i}\phi^j_{a|x}}\, |j\rangle \quad (x \neq 1), \tag{22}$$

where we have used $\langle j|A_{a|x}\rangle = \langle A_{j|x=1}|A_{a|x}\rangle = e^{\mathrm{i}\phi^j_{a|x}}/\sqrt{d}$ for some phase $e^{\mathrm{i}\phi^j_{a|x}}$ whose precise value is not relevant for the discussion. If Alice measures along these MUBs, the post-measurement states on Bob's side conditioned on Alice obtaining the outcome a for measurement choice x are given by

$$|\Psi^B_{a|x}\rangle = \begin{cases} |a\rangle & \text{if} \quad x = 1\,, \\ \sum_{j=0}^{d-1} \sqrt{\lambda_j}\; \mathrm{e}^{-\mathrm{i}\phi^j_{a|x}}\, |j\rangle & \text{if} \quad x \neq 1\,. \end{cases} \tag{23}$$

As shown in Appendix B, we can bound the overlaps of these post-measurement states as follows

$$\forall x < x'\,,$$
$$|\langle\Psi^B_{a|x}|\Psi^B_{a'|x'}\rangle| \leq \begin{cases} \sqrt{\lambda_{max}} & \text{if} \quad x = 1\,, \\ 1 - (d - \sqrt{d})\lambda_{min} & \text{if} \quad x \neq 1\,, \end{cases} \tag{24}$$

which lets us apply Theorems III.1 and III.2 to arrive at the following theorem.

Theorem V.1. *Let n be the number of MUBs in a given dimension d. A state $|\Psi\rangle_{AB} = \sum_{i=0}^{d-1} \sqrt{\lambda_i}\, |ii\rangle$ is fully nonlocal if*

$$\frac{2}{n}\lambda_{max} + \frac{n-2}{n}\left[1 - (d - \sqrt{d})\lambda_{min}\right]^2 \leq \frac{n-2}{2n-2}\,. \tag{25}$$

In particular, this is true if

$$\max\left(\sqrt{\lambda_{max}},\, 1 - (d - \sqrt{d})\lambda_{min}\right) \leq \sqrt{\frac{n-2}{2n-2}}\,. \tag{26}$$

We refer the reader to Appendix C for further details and remarks. There we can show a stronger version of this theorem that is more general but also more difficult to state and to prove. However, this version here is already sufficient to establish our main results.

Result 1. *Non-maximally entangled states with full nonlocality exist in all dimensions $d \geq 4$.*

In dimension $d = 3$ it is known that the number of MUBs is $n = 4$, and the bounds in Theorem V.1 are satisfied if and only if $\lambda_{max} = \lambda_{min} = \bar{\lambda} = 1/3$. This demonstrates that the maximally entangled state is fully nonlocal, but tells us nothing about partially entangled states. However, for $d \geq 4$, we can show that there are always nonmaximally entangled states which are fully nonlocal. Indeed, for $d = 4$ there are exactly 5 MUBs and the condition in Eq. (26) for full nonlocality becomes

$$\max\left(\sqrt{\lambda_{max}},\, 1 - 2\lambda_{min}\right) \leq \sqrt{\frac{3}{8}}\,, \tag{27}$$

which is satisfied whenever $\lambda_{max} \leq 3/8 = 0.375$ and $\lambda_{min} \geq (1/2) - \sqrt{3/32} \approx 0.194$.

For $d \geq 5$, we note that at least three MUBs [53] exist, and the condition in Eq. (26) can be simplified to

$$\max\left(\sqrt{\lambda_{max}},\, 1 - (d - \sqrt{d})\lambda_{min}\right) \leq \frac{1}{2}\,, \tag{28}$$

which is satisfied whenever $\lambda_{max} \leq 1/4$ and $\lambda_{min} \geq 1/(2(d - \sqrt{d}))$. It is simple to check that there are nontrivial values of λ_{min} and λ_{max} within these bounds such that $\lambda_{min} < \lambda_{max}$ whenever $d \geq 5$. Hence, there are partially entangled states that demonstrate full nonlocality also in every dimensions $d \geq 5$.

Corollary. *There are states with a relatively large coefficient $\lambda_{max} > 1/2$ that demonstrate full nonlocality.*

These states are interesting because they cannot be deterministically transformed to any maximally entangled state by local operations and classical communication (LOCC) [54]. Hence, the full nonlocality of these states cannot be understood in terms of the full nonlocality of the maximally entangled state, even when we allow for arbitrary preprocessing via LOCC. To construct such an example, choose a large prime power dimension, say $d = 101$, where it is known that $n = d + 1 = 102$ MUBs exist. Eq. (25) then shows that the state $|\Psi\rangle_{AB} = \sqrt{0.67}\,|00\rangle + \sum_{j=1}^{101} \sqrt{0.0033}\,|jj\rangle$ is fully nonlocal.

Result 2. *All pure entangled states are fully nonlocal if sufficiently many copies are available.*

Proof. This activation of full nonlocality is a simple extension of the qubit case in Sec. IV. It follows from Eq. (24) that when Alice and Bob share a (single copy) pure entangled state of the form Eq. (19) and Alice measures along the n MUBs (at least three MUBs exist in every dimension), the overlaps of the post-measurement states obey, for $x \neq x'$,

$$|\langle\Psi^B_{a|x}|\Psi^B_{a'|x'}\rangle| \leq \max\left(\sqrt{\lambda_{max}},\, 1 - (d - \sqrt{d})\lambda_{min}\right)\,. \tag{29}$$

The right hand side in the equation above may not be sufficient to certify full nonlocality of a single copy of the entangled state. However, for every pure entangled state ($\lambda_{max} < 1$ and $\lambda_{min} > 0$) we have $c := \max(\sqrt{\lambda_{max}}, 1 - (d - \sqrt{d})\lambda_{min}) < 1$ whenever $d \geq 2$. Choose an integer k that is large enough to satisfy $c^k \leq \sqrt{(n-2)/(2n-2)}$. If Alice and Bob share k copies of the entangled state and Alice performs one of the n MUB measurements on each copy like in the case for qubits above, the inner product of the resulting post-measurement states is upper bounded by $c^k \leq \sqrt{(n-2)/(2n-2)}$ (note $n \geq 3$ since three MUBs always exist). Theorem. III.2 then guarantees that all the resulting sets of post-measurement states that appear in Theorem. III.1 are antidistinguishable, and hence the correlation is fully nonlocal. □

Note that the strategy above to generate fully nonlocal correlations is not optimal in the number of copies required. For instance, consider two copies of the two qubit state $|\Psi\rangle_{AB}^{\otimes 2} = (\sqrt{p}\,|00\rangle + \sqrt{1-p}\,|11\rangle)^{\otimes 2}$ with $p \geq 1/2$, which can also be seen as a single copy state in dimension $d = 4$ with $\lambda_{max} = p^2$ and $\lambda_{min} = (1-p)^2$. Using Eq. (26), this state is fully nonlocal if $\lambda_{max} = p^2 \leq 3/8$ and $\lambda_{min} = (1-p)^2 \geq 1/2 - \sqrt{3/32}$ which is true when $1/2 \leq p \leq 1 - \sqrt{1/2 - \sqrt{3/32}} \approx 0.560$. Hence, some nonmaximally entangled two qubit states only need two copies to show full nonlocality. However, if Alice performs separable qubit MUBs on the two copies as in the strategy described above (there are exactly three MUBs for $d = 2$ which are precisely the X, Y, and Z qubit observables used in Sec. IV), only the two qubit maximally entangled state can be shown to be fully nonlocal.

Note the measurements that we have used to activate full nonlocality do not depend on the specific entangled state. We might find better bounds and more examples when adapting the measurements according to the state. Note also that we assume very little about the structure of the MUBs. It might be possible to derive better bounds by using the explicit form of the MUBs (i.e., the explicit phases $\phi^j_{a|x}$) or other measurements. In fact, this allows us to find some partially entangled two-qutrit states in the next section.

VI. SUFFICIENT CONDITIONS FOR FULL NONLOCALITY OF QUTRITS

We have shown in Result 1 that fully nonlocal partially entangled states exist in all dimensions $d \geq 4$. However, with MUBs we were only able to certify full nonlocality of the partially entangled state. While MUBs are easy to work with, they are ultimately limited in the number of outcomes and in $d = 3$ there are exactly 4 MUBs. Here, we can surpass this limitation by considering a set of 5 measurements that was used in Ref. [33] to show full nonlocality of a maximally entangled two-qutrit state. Up to normalization, Alice's measurements $|A_{a|x}\rangle$ correspond to the following vectors ($\omega = \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}$):

$$x = 1 \qquad (1,0,0), (0,1,0), (0,0,1) \tag{30}$$
$$x = 2 \qquad (1,\omega,\omega^2), (1,1,1), (\omega^2,\omega,1) \tag{31}$$
$$x = 3 \qquad (-1,\omega,\omega^2), (-1,1,1), (-\omega^2,\omega,1) \tag{32}$$
$$x = 4 \qquad (1,-\omega,\omega^2), (1,-1,1), (\omega^2,-\omega,1) \tag{33}$$
$$x = 5 \qquad (1,\omega,-\omega^2), (1,1,-1), (\omega^2,\omega,-1)\,. \tag{34}$$

The idea is to increase the number of bases while allowing a larger scalar product for some vectors. In total, the resulting bounds become looser. In particular, the first basis is mutually unbiased with respect to the other bases

$$|\langle A_{a|x=1}|A_{a'|x'\neq 1}\rangle| = \frac{1}{\sqrt{3}}, \tag{35}$$

while the scalar product between vectors of different bases has a richer structure (see Lemma 3 for details):

$$\left|\langle A_{a|x\neq 1}|A_{a'|x'\neq 1}\rangle\right| = \frac{\delta_{aa'}}{3} + 2\frac{1-\delta_{aa'}}{3},\ (x \neq x'). \tag{36}$$

Suppose now that Alice and Bob share a pure partially entangled state $|\Psi\rangle\, AB = \sum_{j=0}^{2} \sqrt{\lambda_j}\,|jj\rangle$ Following a similar calculation as in the case of the MUBs, we can calculate the post-measurement states $|\Psi^B_{a|x}\rangle$ and, as before in Eq. (24), show an upper bound, which reads:

$$\forall x < x',$$
$$|\langle\Psi^B_{a|x}|\Psi^B_{a'|x'}\rangle| \leq \begin{cases} \sqrt{\lambda_{max}} & \text{if} \quad x = 1\,, \\ 1 - 2\lambda_{min} & \text{if} \quad x \neq 1\,, a = a'\,, \\ 1 - \lambda_{min} & \text{if} \quad x \neq 1\,, a \neq a'\,. \end{cases} \tag{37}$$

The proof of this formula can be found in Appendix E, see Lemma 4. Together with Theorem III.2 this allows us to obtain a sufficient condition to the state $|\Psi\rangle_{AB}$ to be fully nonlocal. The only complication relative to the case of MUBs is that the scalar product has different bounds when $a \neq a'$ and when $a = a'$.

Result 3. *A bipartite qutrit state* $|\Psi\rangle_{AB} = \sum_{i=0}^{2} \sqrt{\lambda_i}\,|ii\rangle$ *is fully nonlocal if*

$$0.328 \approx \frac{22 - \sqrt{259}}{18} \leq \lambda_{min} \leq \frac{1}{3}\,. \tag{38}$$

Proof. see Appendix E ☐

Although, the range of allowed λ_{min} is extremely narrow, it proves that full nonlocality of partially entangled states exists in dimension $d = 3$, which, together with Result 1 implies they exist in all dimensions $d \geq 3$.

VII. LOCAL CONTENT FOR SOME PARTIALLY ENTANGLED STATES

So far, we asked which states exhibit full nonlocality; now we want to ask which states do not. Consider an arbitrary finite dimensional pure entangled state $|\Psi\rangle = \sum_{i=0}^{d-1} \sqrt{\lambda_i}\,|ii\rangle$. We will show that if the maximal eigenvalue λ_{max} is large enough, then the state must have a nonzero local content. For simplicity, we will assume $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{d-1}$ without loss of generality, so that $\lambda_{max} = \lambda_0$.

Recall that if there exists a set of outcomes $\boldsymbol{\alpha} = [\alpha_x]_x$ for every input x of Alice and a set of outcomes $\boldsymbol{\beta} = [\beta_y]_y$ for every input y of Bob such that

$$\forall x, y, \quad p_Q(a = \alpha_x, b = \beta_y|x, y) \geq q, \tag{39}$$

then the correlations p_Q have local content at least q (see equation Eqs. (A4)–(A6) in the appendix). Therefore, to show that a state ρ_{AB} has a nonzero local content, it is enough to prove that for every extremal measurement

$\{A_a\}_a$ of Alice there is always an outcome a, and for every measurement of Bob $\{B_b\}_b$ there is always an outcome b such that

$$\mathrm{Tr}[A_a \otimes B_b\ \rho_{AB}] = p_Q(a,b|x,y) > 0\,. \tag{40}$$

We will use the following lemma, which is proved in App. F, to lower bound the local content of partially entangled states.

Lemma 1. *Consider a d-dimensional entangled state $|\Psi\rangle = \sum_{i=0}^{d-1}\sqrt{\lambda_i}\,|ii\rangle$, where $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{d-1}$. If Alice and Bob perform rank-1 measurements with at most k_A and k_B outcomes respectively on this state, then, for every measurement choice of Alice and Bob, there exists a POVM element $A = |A\rangle\langle A|$ for Alice and $B = |B\rangle\langle B|$ for Bob such that*

$$|\langle\Psi|A\otimes B\rangle| \geq \frac{\sqrt{\lambda_0}-\sqrt{\lambda_1}\sqrt{(k_A-1)(k_B-1)}}{\sqrt{k_A k_B}}\,. \tag{41}$$

The above inequality is nontrivial when $\lambda_0 > \lambda_1(k_A-1)(k_B-1)$ and demonstrates a nonzero local content for the correlations generated since $|\langle\Psi|A\otimes B\rangle|^2 = \mathrm{Tr}[|\Psi\rangle\langle\Psi|\,A\otimes B] > 0$. We use this theorem now to prove our results. We first consider the case when Alice and Bob perform projective measurements where we can obtain the following lower bound on the local content.

Theorem VII.1. *A d-dimensional entangled state $|\Psi\rangle = \sum_{i=0}^{d-1}\sqrt{\lambda_i}\,|ii\rangle$, where $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{d-1}$ and $\lambda_0 > \lambda_1(d-1)^2$, has a nonzero local content q larger than*

$$q \geq \left[\frac{\sqrt{\lambda_0}-\sqrt{\lambda_1}(d-1)}{d}\right]^2, \tag{42}$$

if Alice and Bob perform projective measurements.

Proof. The proof is a straightforward application of Lemma 1 to projective measurements. We consider first rank-1 projective measurements. Since there are exactly d outcomes for a rank-1 projective measurement, we can substitute $k_A = k_B = d$ in the inequality (41) to obtain

$$\begin{aligned}\mathrm{Tr}[A\otimes B\,|\Psi\rangle\langle\Psi|] &= |\langle\Psi|A\otimes B\rangle|^2 \\ &\geq \left[\frac{\sqrt{\lambda_0}-\sqrt{\lambda_1}(d-1)}{d}\right]^2,\end{aligned} \tag{43}$$

which must hold for some projector A of Alice and B of Bob for every measurement choice.

Furthermore, we can always split a rank-k projector into k rank-1 projectors, increasing the number of outcomes associated with the projector from 1 to k. The new correlation generated by such splitting of projectors can be used to obtain the original correlation by classically binning the k outcomes into a single outcome. Such local classical post-processing cannot decrease the local content, and hence the new correlation (obtained by splitting the projectors) has a local content that is less than or equal to that of the original correlation. Therefore, proving a lower bound on the local content for rank-1 projective measurements is equivalent to proving a lower bound for all projective measurements. $\square$

For $d = 2$, this directly shows that all partially entangled states (i.e., $\lambda_0 > \lambda_1$) have a nonzero local content with projective measurements, which was known before [22–26]. Further, for the special class of states in which $\lambda_1 = \cdots = \lambda_{d-1}$, we have $\lambda_0 + \lambda_1(d-1) = 1$, and hence the condition $\lambda_0 > \lambda_1(d-1)^2$ for nonzero local content is equivalent to $\lambda_0 > (d-1)/d$. For $d = 3$, this gives us an analytic proof for the result by Scarani [23] where it was shown numerically that when $\lambda_0 > 2/3$, the state $|\Psi\rangle = \sqrt{\lambda_0}\,|00\rangle + \sqrt{\lambda_1}(|11\rangle + |22\rangle)$ has a nonzero local content.

We next remove the restriction to projective measurements and generalize our result to arbitrary POVMs.

Theorem VII.2. *A d-dimensional entangled state $|\Psi\rangle = \sum_{i=0}^{d-1}\sqrt{\lambda_i}\,|ii\rangle$, where $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{d-1}$ and $\lambda_0 > \lambda_1(d^2-1)^2$, has a nonzero local content q larger than*

$$q \geq \left[\frac{\sqrt{\lambda_0}-\sqrt{\lambda_1}(d^2-1)}{d^2}\right]^2, \tag{44}$$

if Alice and Bob perform general POVMs.

Proof. We can use Naimark's dilation theorem [55], which implies that a d-dimensional POVM can be expressed as a projective measurement in a d^2-dimensional Hilbert space. The bound follows than by using Theorem VII.1 above replacing d with d^2. Alternatively, one can use the fact that in dimension d every non-extremal POVM can be written as a convex mixture of extremal rank-1 POVMs with at most d^2 outcomes [56]. The result then follows by Lemma 1 for $k_A = k_B = d^2$ to obtain

$$\begin{aligned}\mathrm{Tr}[|\Psi\rangle\langle\Psi|\,A\otimes B] &= |\langle\Psi|A\otimes B\rangle|^2 \\ &\geq \left[\frac{\sqrt{\lambda_0}-\sqrt{\lambda_1}(d^2-1)}{d^2}\right]^2,\end{aligned} \tag{45}$$

which must hold for some POVM element A of Alice and B of Bob for every measurement choice. $\square$

At first glance, the results in this section look disconnected from the relation we established between full nonlocality and antidistinguishability. Recall that if for a given state there is a combination of outcomes, one for every measurement of Alice, such that the resulting post-measurement states on Bob's side are not antidistinguishable, the state is not fully nonlocal. However, it turns out that these results are in fact inspired by antidistinguishability. A simple example is a partially entangled two-qubit state $\sqrt{p}\,|00\rangle + \sqrt{1-p}\,|11\rangle$, where $p > 1/2$, on which Alice performs projective measurements. It can be shown that for every projective measurement of Alice, one of the pure post-measurement states $|\psi^B\rangle$ created on Bob's side always satisfies $\langle\psi^B|Z|\psi^B\rangle > (2p-1)$, i.e., the

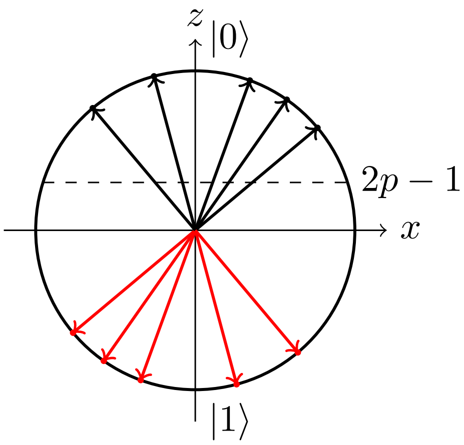

FIG. 2. For a non-maximally entangled two-qubit state $\sqrt{p}\,|00\rangle + \sqrt{1-p}\,|11\rangle$ each projective measurement of Alice has one outcome such that the corresponding post-measurement state is above the dashed line (black arrows). For each of these states there is only a single measurement orthogonal to it (antipodal red arrows). However, these measurements cannot be combined to a basis and therefore the corresponding post-measurement states are not antidistinguishable. This shows that every non-maximally entangled two-qubit state has a local content with projective measurements.

$Z-$component in the Bloch sphere is larger than $2p-1$ (see Fig. 2). It is not difficult to see that such a set of states is not antidistinguishable. Indeed, note that an operator B excluding a state $|\psi^B\rangle$ ($\mathrm{Tr}[\psi^B B] = 0$) corresponds to the unit vector (r_X, r_Y, r_Z) antipidal to $|\psi^B\rangle$ in the Bloch sphere, i.e., $B = (c\mathbb{1} + r_X X + r_Y Y + r_Z Z)/2$ has a Z-component less than $r_Z < -(2p-1)$. These operators however cannot sum to the identity, and thus cannot form a valid measurement (as shown in Prop. 6 of Ref. [45]).

VIII. CONCLUSION

We have established a strong connection between full nonlocality and the concept of antidistinguishability. Using the connection, we have derived some sufficient criteria that ensure an entangled pure state in a given dimension to be fully nonlocal. We found that also non-maximally entangled states can have full nonlocality and examples exist in all dimensions $d \geq 3$. At the same time, we have also shown that in all dimensions there are some states that are not fully nonlocal. Furthermore, we have shown that all pure entangled states show full nonlocality if sufficiently many copies are available. For future research, it is an interesting open problem to find a necessary and sufficient condition for a pure state to show full nonlocality.

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Appendix A: Relation between full nonlocality and antidistinguishability

In this section we prove the statements made in Sec. III that leads to Theorem III.1. Recall that the local content of a quantum correlation p_Q is defined as the maximum value of q in any decomposition of the form

$$p_Q(a,b|x,y) = q \cdot p_L(a,b|x,y) + (1-q) \cdot p_{NS}(a,b|x,y)\,, \tag{A1}$$

where $p_L(a,b|x,y)$ is a local correlation. Since any local correlation can be written as a convex mixture of local deterministic strategies, we can equivalently write the above decomposition as,

$$p_Q(a,b|x,y) = q \cdot \sum_{\boldsymbol{\alpha},\boldsymbol{\beta}} p(\boldsymbol{\alpha},\boldsymbol{\beta})\, D_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a,b|x,y) + (1-q) \cdot p_{NS}(a,b|x,y)\,, \tag{A2}$$

where the tuple $(\boldsymbol{\alpha} = [\alpha_1,\ldots,\alpha_n], \boldsymbol{\beta} = [\beta_1,\ldots,\beta_m])$ corresponds to the deterministic local strategy in which Alice (Bob) outputs $a = \alpha_x$ $(b = \beta_y)$ for the input $x \in [n]$ $(y \in [m])$ so that $D_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a,b|x,y) = \delta_{\alpha_x,a}\,\delta_{\beta_y,b}$.

Note that if $p_Q(a,b|x,y) = 0$ for some fixed value of (a,b,x,y), then any local deterministic correlation that appears in the above decomposition (with weight $q \cdot p(\boldsymbol{\alpha},\boldsymbol{\beta}) > 0$) must have $D_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a,b|x,y) = 0$, i.e.,

$$p_Q(a,b|x,y) = 0 \implies D_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a,b|x,y) = 0. \tag{A3}$$

On the other hand, if there exists a value of $(\boldsymbol{\alpha},\boldsymbol{\beta})$ such that

$$\forall x \in [n], y \in [m]\,, \quad p_Q(a = \alpha_x, b = \beta_y|x,y) \geq q > 0\,, \tag{A4}$$

then the correlation p_Q has local content at least q. Indeed, letting

$$p_{NS}(a,b|x,y) := \frac{p_Q(a,b|x,y) - q \cdot D_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a,b|x,y)}{1-q}\,, \tag{A5}$$

we can write

$$p_Q(a,b|x,y) = q \cdot\, D_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a,b|x,y) + (1-q)\cdot p_{NS}(a,b|x,y)\,. \tag{A6}$$

To see the connection with antidistinguishability, suppose that for some fixed $\boldsymbol{\alpha}$ the post-measurement states $\rho^B_{\boldsymbol{\alpha}} := \{\rho^B_{a=\alpha_1|x=1}, \ldots, \rho^B_{a=\alpha_n|x=n}\}$ are antidistinguishable and Bob, given the measurement choice $y = \boldsymbol{\alpha}$, performs the antidistinguishing measurement $\{B_{b|y=\boldsymbol{\alpha}}\}_{b\in[n]}$, i.e.,

$$\forall x \in [n]\,, \quad \mathrm{Tr}[B_{b=x|y=\boldsymbol{\alpha}}\, \rho^B_{a=\alpha_x|x}] = 0\,. \tag{A7}$$

Therefore,

$$\forall x \in [n]\,, p_Q(a=\alpha_x, b=x|x, y=\boldsymbol{\alpha}) = \mathrm{Tr}\big[(A_{a=\alpha_x|x}\otimes B_{b=x|y=\boldsymbol{\alpha}})\,|\Psi\rangle\!\langle\Psi|_{AB}\big] = k\cdot \mathrm{Tr}[B_{b=x|y=\boldsymbol{\alpha}}\,\rho^B_{a=\alpha_x|x}] = 0\,, \tag{A8}$$

where $k = \mathrm{Tr}\big[(A_{a=\alpha_x|x}\otimes \mathbb{1})\,|\Psi\rangle\!\langle\Psi|_{AB}\big]$. It is now easy to see from Eq. (A3) that any deterministic local correlation $D_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ in which Alice's strategy is given by $\boldsymbol{\alpha}$ has zero weight in the decomposition of Eq. (A2). Indeed, for any such local deterministic strategy, Bob must assign some outcome $b \in \boldsymbol{\alpha}$ for the antidistiguishing measurement $y = \boldsymbol{\alpha}$, and hence, for every strategy $\boldsymbol{\beta}$ of Bob, there exists $x \in [n]$ such that $D_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a=\alpha_x, b=x|x, y=\boldsymbol{\alpha}) = 1 \neq 0$. Furthermore, if for all possible values of $\boldsymbol{\alpha}$, the set of post-measurements states $\rho^B_{\boldsymbol{\alpha}}$ are antidistinguishable, and if Bob performs all the corresponding antidistinguishing measurements, then all the local deterministic strategies have zero weight, and hence the correlation produced is fully nonlocal.

Before proving the converse statement, it will be useful to note the following equivalent characterization of antidistinguishability.

Observation 1. *A set of states $\{\rho_j\}_{j\in[n]}$ is antidistinguishable if and only if there exists a measurement $\{M_j\}_{j\in[k]}$ satisfying*

$$\forall M \in \{M_j\}_{j\in[k]}\,, \exists \rho \in \{\rho_j\}_{j\in[n]}\ \text{s.t.}\ \mathrm{Tr}[M\rho] = 0\,, \tag{A9}$$

that is, an antidistinguishing measurement.

Proof. First, we show that $\{\rho_j\}_{j\in[n]}$ is antidistinguishable if Eq. (A9) is satisfied. Define N_i as the sum of all the operators $M \in \{M_j\}_{j\in[k]}$ such that $\mathrm{Tr}[M\rho_i] = 0$, i.e., $N_i := \sum_{M\in\mathcal{S}_i} M$, where $\mathcal{S}_i = \{M \in \{M_j\}_{j\in[k]} : \mathrm{Tr}[M\rho_i] = 0\}$. If the set $\mathcal{S}_i$ is empty for some $i \in [n]$, then define $N_i := 0$. Clearly, for all $i \in [n]$, $\mathrm{Tr}[N_i\rho_i] = 0$. However, $\{N_i\}_{i\in[n]}$ may not form a valid measurement if some $M \in \{M_j\}_{j\in[k]}$ appears in two or more sets $\mathcal{S}_i$. To rectify the situation, we can pick some set $\mathcal{S}_i$ containing M and delete M from all the other sets $\mathcal{S}_j \neq \mathcal{S}_i$. Thus, we have constructed a valid measurement $\{N_i\}_{i\in[n]}$ that satisfies, for all values of $i \in [n]$, $\mathrm{Tr}[N_i\rho_i] = 0$, which shows that the set $\{\rho_j\}_{j\in[n]}$ is antidistinguishable.

The 'only if' statement is trivial, since an antidistinguishable set $\{\rho_j\}_{j\in[n]}$ implies, by definition, that there exists a measurement $\{M_j\}_{j\in[n]}$ satisfying Eq. (A9), i.e., $\forall M_i \in \{M_j\}_{j\in[n]}\,, \exists \rho_i \in \{\rho_j\}_{j\in[n]}\ \text{s.t.}\ \mathrm{Tr}[M_i\rho_i] = 0\,.$ □

We can now prove the converse relationship between full nonlocality and antidistinguishablility: If for a given state and some measurements on Alice's side there is at least one $\boldsymbol{\alpha}$ such that the post-measurement states $\rho^B_{\boldsymbol{\alpha}} := \{\rho^B_{a=\alpha_1|x=1}, \ldots, \rho^B_{a=\alpha_n|x=n}\}$ are *not* antidistinguishable, then the resulting correlations always have a local content, irrespective of the measurements performed by Bob. The proof goes as follows.

Since the states $\rho^B_{\boldsymbol{\alpha}}$ are *not* antidistinguishable, from Observation 1, there must exist, for every measurement y of Bob, at least one outcome b such that

$$\forall x \in [n]\,, \quad \mathrm{Tr}[B_{b|y}\rho^B_{a=\alpha_x|x}] > 0\,. \tag{A10}$$

Pick an outcome b, for every measurement y, that satisfies the above equation and define $\boldsymbol{\beta} := [\beta_1, \ldots, \beta_m]$, where $\beta_y = b$. We then have

$$p_Q(a=\alpha_x, b=\beta_y|x,y) = \mathrm{Tr}\big[(A_{a=\alpha_x|x}\otimes B_{b=\beta_y|y})\,|\Psi\rangle\!\langle\Psi|_{AB}\big] = k\cdot\mathrm{Tr}[B_{b=\beta_y|y}\,\rho^B_{a=\alpha_x|x}] > 0, \tag{A11}$$

where $k = \mathrm{Tr}\big[(A_{a=\alpha_x|x}\otimes \mathbb{1})\,|\Psi\rangle\!\langle\Psi|_{AB}\big]$. Hence, from the discussion surrounding Eq. (A4), the deterministic local strategy $D_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ has a nonzero weight in the decomposition of p_Q.

Appendix B: Bounds on the overlap for the post-measurement states

In this section, we derive bounds on the overlaps of Bob's post-measurement states when Alice performs measurements using mutually unbiased bases on a pure bipartite state. This will allow us to obtain a sufficient condition for full nonlocality of a state, as expressed in Theorem V.1.

Lemma 2. *Alice chooses a mutually unbiased measurement $|\langle A_{a_i|x=i}|A_{a_j|x=j}\rangle| = 1/\sqrt{d}$, where the first setting corresponds to the computational basis $|A_{a|x=1}\rangle = |a\rangle$. If she applies these measurements on a given state $|\Psi\rangle_{AB} = \sum_{i=0}^{d-1}\sqrt{\lambda_i}\,|ii\rangle$, the post measurement states on Bob's side fulfill:*

$$|\langle\Psi^B_{a_1|x=1}|\Psi^B_{a_x|x\neq 1}\rangle| \leq \sqrt{\lambda_{max}} \tag{B1}$$

$$|\langle\Psi^B_{a_i|x=i}|\Psi^B_{a_k|x=k}\rangle| \leq \min_{\bar{\lambda}\in\mathbb{R}_+}\left(\sum_j|\lambda_j-\bar{\lambda}|+\bar{\lambda}\sqrt{d}\right) \leq 1-(d-\sqrt{d})\lambda_{min} \tag{B2}$$

Proof. The proof is a rather straightforward calculation. Since the first measurement basis is the computational basis $|A_{a|x=1}\rangle = |a\rangle$, the other measurements, being mutually unbiased, read:

$$|A_{a|x\neq 1}\rangle = \sqrt{\frac{1}{d}}\sum_j \mathrm{e}^{\mathrm{i}\phi^j_{a|x}}\,|j\rangle\ . \tag{B3}$$

Here, $\phi^j_{a|x}$ are some arbitrary phases whose precise value is not important for now. If Alice applies these MUBs on her system, it is simple to calculate the post-measurement states on Bob's side for $x\neq 1$:

$$|\Psi^B_{a|x\neq 1}\rangle = \sum_j\sqrt{\lambda_j}\ \mathrm{e}^{-\mathrm{i}\phi^j_{a|x}}\,|j\rangle\ . \tag{B4}$$

Note that these states are normalized due to $\sum_j\lambda_j = 1$. For the first measurement, the post measurement states are simply $|\Psi^B_{a|x=1}\rangle = |a\rangle$. Therefore, we can calculate

$$|\langle\Psi^B_{a_1|x=1}|\Psi^B_{a_x|x\neq 1}\rangle| = \sqrt{\lambda_{a_1}} \leq \sqrt{\lambda_{max}}\,. \tag{B5}$$

Furthermore, we find a bound on the inner product between two states of the form $|\Psi^B_{a|x\neq 1}\rangle$ coming from two different bases ($i\neq k$, $i\neq 1$, $k\neq 1$):

$$\langle\Psi^B_{a_i|x=i}|\Psi^B_{a_k|x=k}\rangle = \sum_j\lambda_j\ \mathrm{e}^{\mathrm{i}(\phi^j_{a_i|i}-\phi^j_{a_k|k})} = \sum_j(\lambda_j-\bar{\lambda})\ \mathrm{e}^{\mathrm{i}(\phi^j_{a_i|i}-\phi^j_{a_k|k})} + \sum_j\bar{\lambda}\ \mathrm{e}^{\mathrm{i}(\phi^j_{a_i|i}-\phi^j_{a_k|k})} \tag{B6}$$

Here, $\bar{\lambda}\in\mathbb{R}$ can be chosen arbitrarily. Since $|A_{a_i|x=i}\rangle$ and $|A_{a_k|x=k}\rangle$ are mutually unbiased, we know that:

$$\frac{1}{\sqrt{d}} = |\langle A_{a_k|x=k}|A_{a_i|x=i}\rangle| = \frac{1}{d}\Big|\sum_j\mathrm{e}^{\mathrm{i}(\phi^j_{a_i|i}-\phi^j_{a_k|k})}\Big|\,. \tag{B7}$$

Using this and the fact that $|\sum_i z_i|\leq\sum_i|z_i|$ for complex numbers $z_i\in\mathbb{C}$ we obtain the following upper bound:

$$|\langle\Psi^B_{a_i|x=i}|\Psi^B_{a_k|x=k}\rangle| \leq \sum_j|\lambda_j-\bar{\lambda}| + |\bar{\lambda}|\Big|\sum_j\mathrm{e}^{\mathrm{i}(\phi^j_{a_i|i}-\phi^j_{a_k|k})}\Big| = \sum_j|\lambda_j-\bar{\lambda}|+|\bar{\lambda}|\sqrt{d}\,. \tag{B8}$$

Note that $\bar{\lambda}$ is arbitrary, so we can minimize this expression with respect to $\bar{\lambda}\in\mathbb{R}$ to obtain the best bound. In particular, we can limit to the case where $\bar{\lambda}\geq 0$. One convenient choice for $\bar{\lambda}$ (but not always the best one) is to choose the smallest coefficient $\bar{\lambda}=\lambda_{min}$. This leads to a simple expression:

$$\min_{\bar{\lambda}\in\mathbb{R}_+}\left[\sum_j|\lambda_j-\bar{\lambda}|+\bar{\lambda}\sqrt{d}\right] \leq \sum_j|\lambda_j-\lambda_{min}|+\lambda_{min}\sqrt{d} = \sum_j(\lambda_j-\lambda_{min})+\lambda_{min}\sqrt{d} = 1-(d-\sqrt{d})\lambda_{min}\,. \tag{B9}$$

In the previous equation, we used that $\lambda_j\geq\lambda_{min}$ and $\sum_j\lambda_j = 1$. □

Appendix C: Proof and remarks of Theorem V.1

Theorem C.1. *Let n be the number of MUBs in a given dimension d. A state $|\Psi\rangle_{AB} = \sum_{i=0}^{d-1} \sqrt{\lambda_i}\, |ii\rangle$ is fully nonlocal if:*

$$\frac{2}{n}\lambda_{max} + \frac{n-2}{n} \min_{\bar{\lambda}\in\mathbb{R}_+} \left[\sum_j |\lambda_j - \bar{\lambda}| + \bar{\lambda}\sqrt{d}\right]^2 \le \frac{n-2}{2n-2}\,. \tag{C1}$$

Proof. We have to show that every combination of the post-measurement states (one for each of the n measurements) is antidistinguishable. For that, we can combine Lemma 2 and the Theorem III.2 by Johnston et al [48]. For each of these combinations, we can bound the Frobenius norm of the Gram matrix as follows:

$$(\|G\|_F)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} |\langle \Psi^B_{a_i|x=i}|\Psi^B_{a_j|x=j}\rangle|^2 \le n + (2n-1)\lambda_{max} + (n-1)(n-2) \min_{\bar{\lambda}\in\mathbb{R}_+} \left(\sum_j |\lambda_j - \bar{\lambda}| + \bar{\lambda}\sqrt{d}\right)^2 \tag{C2}$$

The first term corresponds to the diagonal elements ($i = j$), the second to the pairs with $i = 1$ and $j \neq 1$ or $i \neq 1$ and $j = 1$ and the last term to the remaining terms. Here, we used the bounds obtained from Lemma 2. Hence, all of these combinations are antidistinguishable if:

$$n + 2(n-1)\lambda_{max} + (n-1)(n-2) \min_{\bar{\lambda}\in\mathbb{R}_+} \left(\sum_j |\lambda_j - \bar{\lambda}| + \bar{\lambda}\sqrt{d}\right)^2 \le \frac{n^2}{2} \tag{C3}$$

Subtracting n first and dividing by $n(n-1)$ second gives the desired bound. □

Furthermore, note that when we use Eq. (B2) (by choosing $\bar{\lambda} = \lambda_{min}$) we arrive at the theorem stated in the main text. Choosing $\bar{\lambda} = \lambda_{min}$ is a convenient choice for practical purposes. However, we can also explicitly minimize the function. Suppose λ_i is ordered with $\lambda_{d-1} \ge \lambda_{d-2} \ge ... \ge \lambda_0$. Now, we can see that the function:

$$\sum_j |\lambda_j - \bar{\lambda}| + \bar{\lambda}\sqrt{d} = \sum_{\lambda_j \ge \bar{\lambda}} (\lambda_j - \bar{\lambda}) + \sum_{\lambda_j < \bar{\lambda}} (\bar{\lambda} - \lambda_j) + \bar{\lambda}\sqrt{d} \tag{C4}$$

is piecewise linear in $\bar{\lambda}$ and the slope in the interval $[\lambda_j, \lambda_{j+1}]$ is given by $\sqrt{d} + j - (d-j) = 2j + \sqrt{d} - d$. Hence the minimal value is reached when the slope changes sign and becomes positive, which happens at $\bar{\lambda} = \lambda_{\bar{j}}$ for $\bar{j} = \lceil \frac{d-\sqrt{d}}{2} \rceil$. In this case the function becomes:

$$\begin{aligned}\sum_j |\lambda_j - \lambda_{\bar{j}}| + \lambda_{\bar{j}}\sqrt{d} &= \sum_{\lambda_j \ge \lambda_{\bar{j}}} (\lambda_j - \lambda_{\bar{j}}) + \sum_{\lambda_j < \lambda_{\bar{j}}} (\lambda_{\bar{j}} - \lambda_j) + \lambda_{\bar{j}}\sqrt{d} &\text{(C5)}\\ &= \sum_{j > \bar{j}} \lambda_j - (d-\bar{j})\lambda_{\bar{j}} - \sum_{j \le \bar{j}} \lambda_j + \bar{j}\lambda_{\bar{j}} + \lambda_{\bar{j}}\sqrt{d} &\text{(C6)}\\ &= (-d + 2\bar{j} + \sqrt{d})\lambda_{\bar{j}} + 1 - 2\sum_{j \le \bar{j}} \lambda_j\,. &\text{(C7)}\end{aligned}$$

1. Further remarks

1. In some situations, it might be better to exclude the measurement in the computational basis. This can happen in particular when λ_{max} is large. This would reduce the number of measurements by one and we arrive at the sufficient condition:

$$\min_{\bar{\lambda}\in\mathbb{R}_+} \left[\sum_j |\lambda_j - \bar{\lambda}| + \bar{\lambda}\sqrt{d}\right]^2 \le \frac{n-3}{2n-4}\,. \tag{C8}$$

Here is an example of a state where this is relevant. Consider $d = 9$. This is a prime power, so $n = d + 1 = 10$. suppose $|\Psi\rangle_{AB} = \sqrt{0.552}\,|00\rangle + \sum_{i=1}^{8}\sqrt{0.056}\,|ii\rangle$, so $\lambda_{min} = 0.056$ and $\lambda_{max} = 0.552$. Choosing $\bar{\lambda} = \lambda_{min}$ we obtain:

$$\min_{\bar{\lambda}\in\mathbb{R}_+}\left[\sum_j |\lambda_j - \bar{\lambda}| + \bar{\lambda}\sqrt{d}\right]^2 \leq (1 - 6\lambda_{min})^2 = (1 - 6\cdot 0.056)^2 \leq \frac{7}{16} = \frac{n-3}{2n-4}\,. \tag{C9}$$

So the state is fully nonlocal by using the criterion without the computational basis measurement. However, the criterion in above theorem cannot detect full nonlocality. However, examples seem to be rare (already for $|\Psi\rangle_{AB} = \sqrt{0.52}\,|00\rangle + \sum_{i=1}^{8}\sqrt{0.06}\,|ii\rangle$ the argument does not work anymore). Note also, that adding an extra measurement setting cannot turn a fully nonlocal state into a state with a local content. So the discrepancy rather comes from the criterion we use for antidistinguishability and is not fundamental.

2. A further remark is that sometimes it might be better to interpret a state as a higher dimensional state. For instance in $d = 6$, there are only three known MUBs. However, $d = 7$ is a prime and there are $n = d + 1 = 8$ MUBs. So, if we set $\lambda_7 = 0$ and apply the MUBs of dimension 7 it might certify full nonlocality of some states in $d = 6$ that cannot be certified directly by using the MUBs in $d = 6$. (We do not give an example here.)

Appendix D: The Peres-Mermin square with antidistinguishability

It might be useful to consider an explicit known example of full nonlocality and show how the results of this paper apply. In the magic square nonlocal game, two noncommunicating players Alice and Bob are asked to fill, with labels either '+' or '−', a row $x \in \{1, 2, 3\}$ and column $y \in \{1, 2, 3\}$, respectively, of a 3×3 grid called the magic square. Alice must always assign an even number of −'s for the row whereas Bob must always assign an odd number of −'s for the column. Additionally, the labels assigned by Alice and Bob must agree in the intersecting element of the grid. In a local deterministic strategy, Alice and Bob agree beforehand the labels for every row and column. However, no local deterministic strategy can win the magic square game perfectly since the total number of −'s in the gird would be even according to Alice and odd according to Bob, leading to a contradiction. Therefore, any no-signaling correlation $p(a, b|x, y)$, where $a \in \{+++, +--, -+-, --+\}$ and $b \in \{---, -++, +-+, ++-\}$, that allows Alice and Bob to win the magic square game perfectly is fully nonlocal (see Appendix A).

There is a quantum correlation p_Q that can perfectly win the magic square game, and hence has full nonlocality [31]. In particular, it is conjectured to be the simplest fully nonlocal strategy in dimensions 4 [28, Conjecture 1]. The quantum strategy uses the maximally entangled state $|\Psi\rangle = (|00\rangle + |11\rangle + |22\rangle + |33\rangle)/2$ and the following projective measurements $\{A_x\}_x$ ($A_x = \{\,|A_{a|x}\rangle\langle A_{a|x}|\}_a$) for Alice and $\{B_y\}_y$ ($B_y = \{\,|B_{b|y}\rangle\langle B_{b|y}|\}_b$) for Bob.

	$\lvert A_{a\lvert x=1}\rangle$	$\lvert A_{a\lvert x=2}\rangle$	$\lvert A_{a\lvert x=3}\rangle$
$a = +++$	$\lvert 0\rangle$	$(\lvert 0\rangle + \lvert 1\rangle + \lvert 2\rangle + \lvert 3\rangle)/2$	$(\lvert 0\rangle - \lvert 1\rangle - \lvert 2\rangle - \lvert 3\rangle)/2$
$a = +--$	$\lvert 2\rangle$	$(\lvert 0\rangle - \lvert 1\rangle + \lvert 2\rangle - \lvert 3\rangle)/2$	$(\lvert 0\rangle + \lvert 1\rangle - \lvert 2\rangle + \lvert 3\rangle)/2$
$a = -+-$	$\lvert 1\rangle$	$(\lvert 0\rangle + \lvert 1\rangle - \lvert 2\rangle - \lvert 3\rangle)/2$	$(\lvert 0\rangle - \lvert 1\rangle + \lvert 2\rangle + \lvert 3\rangle)/2$
$a = --+$	$\lvert 3\rangle$	$(\lvert 0\rangle - \lvert 1\rangle - \lvert 2\rangle + \lvert 3\rangle)/2$	$(\lvert 0\rangle + \lvert 1\rangle + \lvert 2\rangle - \lvert 3\rangle)/2$

	$\lvert B_{b\lvert y=1}\rangle$	$\lvert B_{b\lvert y=2}\rangle$	$\lvert B_{b\lvert y=3}\rangle$
$b = ---$	$(\lvert 1\rangle - \lvert 3\rangle)/\sqrt{2}$	$(\lvert 2\rangle - \lvert 3\rangle)/\sqrt{2}$	$(\lvert 1\rangle - \lvert 2\rangle)/\sqrt{2}$
$b = -++$	$(\lvert 1\rangle + \lvert 3\rangle)/\sqrt{2}$	$(\lvert 2\rangle + \lvert 3\rangle)/\sqrt{2}$	$(\lvert 1\rangle + \lvert 2\rangle)/\sqrt{2}$
$b = +-+$	$(\lvert 0\rangle - \lvert 2\rangle)/\sqrt{2}$	$(\lvert 0\rangle - \lvert 1\rangle)/\sqrt{2}$	$(\lvert 0\rangle - \lvert 3\rangle)/\sqrt{2}$
$b = ++-$	$(\lvert 0\rangle + \lvert 2\rangle)/\sqrt{2}$	$(\lvert 0\rangle + \lvert 1\rangle)/\sqrt{2}$	$(\lvert 0\rangle + \lvert 3\rangle)/\sqrt{2}$

To see that this quantum strategy wins the magic square game, first note that Alice's and Bob's outcomes always satisfy the parity constraints, i.e., the number of −'s in each row is even and in each column is odd. To check that Alice and Bob agree on the intersecting element, we first make the following observation: for the maximally entangled state, the post-measurement states on Bob's side when Alice performs measurement x and gets outcome a are the complex conjugate of the operators $|A_{a|x}\rangle$. Since all the operators $|A_{a|x}\rangle$ are real, we have

$$|\Psi^B_{a|x}\rangle = |A_{a|x}\rangle \ . \tag{D1}$$

It is now easy to see that all the outcomes of Bob that would assign a different label than Alice for the intersecting element of the magic square occur with probability zero. For example, if Alice assigns $a = +++$ for the first row ($x = 1$), then the post-measurement state $|0\rangle$ is orthogonal to $(|1\rangle - |3\rangle)/\sqrt{2}$ and $(|1\rangle + |3\rangle)/\sqrt{2}$. Hence, Bob would never give the outcome $---$ or $-++$ for the first column. The remaining outcomes $+-+$ and $++-$ occur with equal probability and satisfy the constraint that both Alice and Bob assign $+$ in the first column of the first row.

Let us now make the connection with antidistinguishablility, and provide an alternate proof that the quantum correlation defined by the maximally entangled state and the measurements above leads to a fully nonlocal correlation. As we have seen in Appendix A, it is sufficient to show that no deterministic local strategy can have a nonzero weight in the decomposition Eq. (A2) of p_Q to prove that p_Q is fully nonlocal. Indeed, we observe that for every local deterministic strategy $\boldsymbol{\alpha}$ of Alice, the set of post-measurement states corresponding to the outcomes $\boldsymbol{\alpha}$ can be antidistinguished by one of the measurements of Bob. For example, suppose Alice's local deterministic strategy is to output $a = +++$ when $x = 1$, $a = -+-$ when $x = 2$, and $a = +++$ when $x = 3$, i.e., $\boldsymbol{\alpha} = [\alpha_1 = +++, \alpha_2 = -+-, \alpha_3 = +++]$. Now, the measurement $B_{y=2}$ of Bob is an antidistinguishing measurement (see Eq. (7)) for the corresponding set of states

$$\{ |\Psi^B_{a=+++|x=1}\rangle = |0\rangle \ , |\Psi^B_{a=-+-|x=2}\rangle = (|0\rangle + |1\rangle - |2\rangle - |3\rangle)/2 \, , |\Psi^B_{a=+++|x=3}\rangle = (|0\rangle - |1\rangle - |2\rangle - |3\rangle)/2 \} \, . \tag{D2}$$

Therefore, every local strategy of Bob is forbidden (i.e., occurs with zero weight in any decomposition Eq. (A2)) since every outcome b for $y = 2$ leads to a contradiction. Indeed, $b = ---$ is forbidden because, conditioned on Alice giving the outcome $a = +++$ for $x = 1$, the probability of Bob giving the outcome $b = ---$ is 0, i.e., $p_Q(b = ---|a = +++, y = 2, x = 1) = |\langle \Psi^B_{a=+++|x=1} | B_{b=---|y=2}\rangle|^2 = 0$. Similarly, $b = -++$ is forbidden since it implies that Alice's outcome could not have been $a = +++$ for $x = 1$, the outcome $b = +--$ implies that Alice's outcome could not have been $a = -+-$ for $x = 2$, and the outcome $b = ++-$ implies that Alice's outcome could not have been $a = +++$ for $x = 3$. Hence, for $y = 2$, every outcome of Bob is forbidden due to the strategy $\boldsymbol{\alpha}$ of Alice [3]. The proof that p_Q is fully nonlocal then follows from the fact that for every local deterministic strategy $\boldsymbol{\alpha}$ of Alice, the corresponding sets of post-measurement states are antidistinguishable, with one of the measurements of Bob being the antidistinguishing measurement. We make several remarks from this example of full nonlocality.

1. Although the proof of full nonlocality of p_Q based on the magic square construction is more elegant in this example, the connection to antidistinguishability is more general and can always be applied in every scenario due to Theorem III.1.

2. There are $4^3 = 64$ local deterministic strategies for Alice which correspond to an equal number of sets of post-measurement states. However, only 3 measurements on Bob's side are sufficient to antidistiguishing all of them. This is a remarkable reduction in the number of measurements required for Bob.

3. This example also shows explicitly that the antidistiguishing measurement for a set of states is not unique. Indeed, consider the post-measurement states $\Psi^B_{\boldsymbol{\alpha}}$ corresponding to Alice's strategy $\boldsymbol{\alpha} = [\alpha_1 = +++, \alpha_2 = +++, \alpha_3 = +++]$. Since the parity of all the columns are violated, all the 3 measurements of Bob are antidistiguishing for this set of states.

4. In this example, one can see that the inner product between any two post-measurement states is bounded by 1/2 (exactly 1/2 in fact), so we can apply directly the theorems about antidistinguishablity from Ref. [39] or Ref. [48].

Appendix E: Proves for full nonlocality of qutrits

In this appendix, we provide some details concerning the full nonlocality of qutrits, discussed in Section VI, together with some technical proofs. After writing Alice's measurement, we proceed with Lemma 3, which provides the scalar product between two vectors in different bases. We will see how to bound post-measurement states on Bob's side in Lemma 4, and finally, we will provide the proof of Theorem 3, thus demonstrating that partially entangled fully nonlocal states exist in dimension $d = 3$.

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[3] In the magic square, this corresponds to the fact that Alice's assignment $\boldsymbol{\alpha}$ to the magic square violates Bob's parity constraints for the second column.

For each $x \in [5]$, Alice's measurement form a basis in the form (30)-(34), with the components $A_{ab|x}$ that can be expressed as unitary matrices A_x in the form

$$A_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & \omega & \omega^2 \\ 1 & 1 & 1 \\ \omega^2 & \omega & 1 \end{bmatrix}, A_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & \omega & -\omega^2 \\ 1 & 1 & -1 \\ \omega^2 & \omega & -1 \end{bmatrix}, A_4 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -\omega & \omega^2 \\ 1 & -1 & 1 \\ \omega^2 & -\omega & 1 \end{bmatrix}, A_5 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 & \omega & \omega^2 \\ -1 & 1 & 1 \\ -\omega^2 & \omega & 1 \end{bmatrix}, \tag{E1}$$

with $\omega = \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}$, and $A_1 = \mathbb{1}$. Explicitly, the row a of the matrix A_1 contains the components of the vector $|A_{a|x}\rangle$ with respect to the computation basis. In particular, these measurements are used to provide the currently simplest fully nonlocal strategy in dimension $d = 3$, with 5 and 9 orthogonal measurements per party [33].

It is easy to find the scalar product between two vectors of different bases, as expressed in the following lemma.

Lemma 3. *Let us consider two vectors $|A_{a|x}\rangle$, $|A_{a'|x'}\rangle$, elements of two different bases in (30), that is $x \neq x'$. Then, their scalar product has modulus given by:*

$$|\langle A_{a|x}|A_{a'|x'}\rangle| = \begin{cases} \frac{1}{\sqrt{3}} & \text{if } x \neq x' = 1, \\ \delta_{aa'}\frac{1}{3} + (1-\delta_{aa'})\frac{2}{3} & \text{if } x, x' \neq 1. \end{cases} \tag{E2}$$

Proof. The case $x \neq 1$ and $x' = 1$ is trivial. Instead, for the case $x, x' \neq 1$, we can first observe that

$$\langle A_{a|x}|A_{a'|x'}\rangle = \sum_{b=0}^{2} \overline{A_{ab|x}} A_{a'b|x'} = \left(A_{x'} \cdot A_x^\dagger\right)_{a'a}. \tag{E3}$$

Furthermore, any $A_{x'}$ can be obtained from A_x by flipping either one or two columns. Suppose, for example, that $A_{x'}$ is obtained by flipping the third component of A_x. This means that

$$A_{x'} \cdot A_x^\dagger = A_x \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot A_x^\dagger = \mathbb{1} - 2A_x |2\rangle\langle 2| A_x^\dagger. \tag{E4}$$

Here, we used the fact that $A_x \cdot A_x^\dagger = \mathbb{1}$. Since the state $A_x|2\rangle$ is a uniform vector, all the entries of the matrix $A_x|2\rangle\langle 2|A_x^\dagger$ have modulus 1/3, and the diagonal elements are equal to 1/3. This clearly implies a scalar product in the form (E2). A similar proof works also for a flipping of two observables, by writing the flip as $2|a\rangle\langle a| - \mathbb{1}$. □

In a similar way as done in Lemma 2, we can bound the scalar products of the post-measurement states on Bob's side. This is summarized in the following lemma.

Lemma 4. *Let Alice be performing measurements from the basis (30). If she applies these measurements on a given state $|\Psi\rangle_{AB} = \sum_{i=0}^{2}\sqrt{\lambda_i}|ii\rangle$, the post measurement states on Bob's side fulfill:*

$$|\langle \Psi^B_{a|x=1}|\Psi^B_{a'|x\neq 1}\rangle| \leq \sqrt{\lambda_{max}} \tag{E5}$$

$$|\langle \Psi^B_{a|x=i}|\Psi^B_{a'|x'=k}\rangle| \leq \min_{\overline{\lambda}\in\mathbb{R}_+}\left[\sum_{j=0}^{2}|\lambda_j - \overline{\lambda}| + 3\overline{\lambda}|\langle A_{a|x=i}|A_{a'|x'=k}\rangle|\right], \tag{E6}$$

with $i \neq k$ and $i, k \neq 1$. The minimum of the second inequality (E6) is reached when $\overline{\lambda} = \lambda_{min}$.

Proof. The bounds (E5) and (E6) can be obtained by mimicking the proof of Lemma 2. We only need to prove the last statement, for which the explicit form of (E2) plays a key role. Assume $\lambda_0 \leq \lambda_1 \leq \lambda_2$, and suppose that $a = a'$, so that $\left|\langle A_{a|x}|A_{a|x'}\rangle\right| = \frac{1}{3}$. As in the argument presented after Lemma 2, which in this case can be made more explicit, we need to minimize the convex and piecewise linear function

$$\sum_{j=0}^{2}|\lambda_j - \overline{\lambda}| + \overline{\lambda} = \begin{cases} 1 - 2\overline{\lambda} & \text{for } 0 \leq \overline{\lambda} \leq \lambda_0 \\ 1 - 2\lambda_0 & \text{for } \lambda_0 \leq \overline{\lambda} \leq \lambda_1 \\ 2\lambda_2 - 1 + 2\overline{\lambda} & \text{for } \lambda_1 \leq \overline{\lambda} \leq \lambda_2 \\ 4\overline{\lambda} - 1 & \text{for } \overline{\lambda} \geq \lambda_2 \end{cases}. \tag{E7}$$

In particular, this function is decreasing for $\overline{\lambda} \leqslant \lambda_0$, it is constant in the interval $\lambda_0 \leq \overline{\lambda} \leq \lambda_1$ and it increases for $\overline{\lambda} \geq \lambda_1$. Its minimum is therefore $1 - 2\lambda_0$, which is also the value it takes in $\overline{\lambda} = \lambda_0 = \lambda_{min}$. A similar computation can be repeated for $a \neq a'$, for which $\left|\langle A_{a|x}|A_{a'|x'}\rangle\right| = \frac{2}{3}$. Again, the minimal value is attained for $\overline{\lambda} = \lambda_0 = \lambda_{min}$, here the slope changes sign and becomes positive. □

The previous bounds can be used to attain a sufficient condition for the antidistinguishability of the post-measurement states. Specifically, using Theorem III.2, we can now provide a proof for Result 3.

Result 3. *A bipartite qutrit state* $|\Psi\rangle_{AB} = \sum_{i=0}^{2} \sqrt{\lambda_i}\, |ii\rangle$ *is fully nonlocal if*

$$0.328 \approx \frac{22-\sqrt{259}}{18} \leq \lambda_{min} \leq \frac{1}{3}\,. \tag{E8}$$

Proof. Let us consider, for each measurement $i \in [5]$, an output a_i. We are interested in bounding the Frobenius norm of the Gram matrix G_{ik} constructed from the post measurement states:

$$(\|G\|_F)^2 = \sum_{i=1}^{5}\sum_{k=1}^{5} |\langle \Psi^B_{a_i|x=i} | \Psi^B_{a_k|x=k}\rangle|^2. \tag{E9}$$

The bound on $|\langle \Psi^B_{a_i|x=i} | \Psi^B_{a_j|x=j}\rangle|$ can be obtained from Lemma 4. In particular, the main complication is that there are some i and k for which $a_i = a_k$ and others for which $a_i \neq a_k$, which makes the second inequality (E6) more complicated to manage. Nevertheless, for $i,k = 2,3,4,5$, there exists at least a couple i,k, with $i \neq k$, such that $a_i = a_k$. Therefore, there are at least two terms in the sum (E9) (i,k and k,i) where the condition $\langle A_{a_i|x=i} | A_{a_k|x=k}\rangle = \frac{1}{3}$ holds. For the other terms, we use the worse case scenario $|\langle A_{a_i|x=i} | A_{a_k|x=k}\rangle| = \frac{2}{3}$. Using Lemma 4, we can therefore bound the Frobenius norm (E9) as

$$(\|G\|_F)^2 \leq 5 + 8\lambda_{max} + 2\cdot(1-\lambda_{min})^2 + 10\cdot(1-2\lambda_{min})^2 = 18\lambda_{min}^2 - 28\lambda_{min} + 17 + 8\lambda_{max}. \tag{E10}$$

Imposing $(\|G\|_F)^2 \leq 25/2$ from III.2, we obtain

$$18\lambda_{min}^2 - 28\lambda_{min} + 8\lambda_{max} + \frac{9}{2} \leq 0. \tag{E11}$$

Using $\lambda_{max} \leq 1 - 2\lambda_{min}$, we get the sufficient condition for antidistinguishability

$$18\lambda_{min}^2 - 44\lambda_{min} + \frac{25}{2} \leq 0, \tag{E12}$$

with solution given by Equation (E8). □

Appendix F: Proof of the Local Content Bound

In this appendix, we are going to prove Lemma 1, which provides the fundamental tool to obtain a lower bound on the local content of entangled states.

Lemma 1. *Consider a d-dimensional entangled state $|\Psi\rangle = \sum_{i=0}^{d-1} \sqrt{\lambda_i}\, |ii\rangle$, where $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{d-1}$. If Alice and Bob perform rank-1 measurements with at most k_A and k_B outcomes respectively on this state, then, for every measurement choice of Alice and Bob, there exists a POVM element $A = |A\rangle\langle A|$ for Alice and $B = |B\rangle\langle B|$ for Bob such that*

$$|\langle \Psi | A \otimes B\rangle| \geq \frac{\sqrt{\lambda_0} - \sqrt{\lambda_1}\sqrt{(k_A-1)(k_B-1)}}{\sqrt{k_A k_B}}\,. \tag{F1}$$

The inequality (F1) is nontrivial when $\lambda_0 > \lambda_1(k_A-1)(k_B-1)$, which implies $|\langle \Psi | A\otimes B\rangle|^2 = \mathrm{Tr}[|\Psi\rangle\langle\Psi|\, A\otimes B] > 0$ and that there is a nonzero local content for the correlations generated.

Proof. Let us begin by fixing explicitly the POVM elements $A = |A\rangle\langle A|$ and $B = |B\rangle\langle B|$ used in the statement of the theorem. For any k_A-outcome, rank-1 POVM $\{A_1, \ldots, A_{k_A}\}$ of Alice, we must have

$$\langle 0| \left(\sum_{l=1}^{k_A} |A_l\rangle\langle A_l| \right) |0\rangle = \langle 0|\, \mathbb{1}\, |0\rangle \implies \sum_{l=1}^{k_A} |\langle 0|A_l\rangle|^2 = 1\,. \tag{F2}$$

Hence, for some POVM element $A \in \{A_1, \ldots, A_{k_A}\}$, $|\langle 0|A\rangle|^2 \geq 1/k_A$. Similarly, for some POVM element $B \in \{B_1, \ldots, B_{k_B}\}$ of Bob, $|\langle 0|B\rangle|^2 \geq 1/k_B$. Let us denote by a_j the j^{th} component of $|A\rangle$ in the Schmidt basis of $|\Psi\rangle$, i.e., $a_j = \langle j|A\rangle$, and by b_j the j^{th} component of $|B\rangle$, i.e., $b_j = \langle j|B\rangle$. Hence,

$$|a_0|^2 \geq \frac{1}{k_A} \quad \text{and} \quad |b_0|^2 \geq \frac{1}{k_B}\,. \tag{F3}$$

We now consider the scalar product

$$\langle\Psi|A\otimes B\rangle = \sum_{j=0}^{d-1}\sqrt{\lambda_j}\,\langle j|A\rangle\,\langle j|B\rangle = \sqrt{\lambda_0}\,a_0b_0 + \sum_{j=1}^{d-1}\sqrt{\lambda_j}\,a_jb_j\,, \tag{F4}$$

and take its absolute value, which satisfies

$$|\langle\Psi|A\otimes B\rangle| \geq \sqrt{\lambda_0}\,|a_0b_0| - \left|\sum_{j=1}^{d-1}\sqrt{\lambda_j}\,a_jb_j\right|. \tag{F5}$$

Furthermore,

$$\left|\sum_{j=1}^{d-1}\sqrt{\lambda_j}\,a_jb_j\right| \leq \sum_{j=1}^{d-1}\sqrt{\lambda_j}\,|a_jb_j| \leq \sqrt{\lambda_1}\sum_{j=1}^{d-1}|a_jb_j|\,, \tag{F6}$$

where in the final step we have used the fact that $\lambda_1 = \max\{\lambda_1,\ldots,\lambda_{d-1}\}$. Defining $|u\rangle := (|a_1|,\ldots,|a_{d-1}|)$, $|v\rangle := (|b_1|,\ldots,|b_{d-1}|)$ and applying the Cauchy–Schwarz inequality $|\langle u|v\rangle| \leq \sqrt{\langle u|u\rangle\,\langle v|v\rangle}$ we get,

$$\sqrt{\lambda_1}\sum_{j=1}^{d-1}|a_jb_j| \leq \sqrt{\lambda_1}\left(\sum_{i=1}^{d-1}|a_i|^2\right)^{1/2}\left(\sum_{j=1}^{d-1}|b_j|^2\right)^{1/2}. \tag{F7}$$

Inequalities (F6),(F7) imply that we can relax the inequality (F5) as

$$|\langle\Psi|A\otimes B\rangle| \geq \sqrt{\lambda_0}\,|a_0b_0| - \sqrt{\lambda_1}\left(\sum_{i=1}^{d-1}|a_i|^2\right)^{1/2}\left(\sum_{j=1}^{d-1}|b_j|^2\right)^{1/2}. \tag{F8}$$

Moreover, using Eq. (F3) we obtain

$$\sum_{j=1}^{d-1}|a_j|^2 = \left(\sum_{j=0}^{d-1}|a_j|^2\right) - |a_0|^2 = \langle A|A\rangle - |a_0|^2 \leq \langle A|A\rangle - \frac{1}{k_A} \leq \frac{k_A-1}{k_A}\,, \tag{F9}$$

where in the final step we have used $\langle A|A\rangle \leq 1$ since

$$|A\rangle\!\langle A| \preceq \mathbb{1} \implies \langle A|A\rangle\,\langle A|A\rangle \leq \langle A|\,\mathbb{1}\,|A\rangle \implies \langle A|A\rangle^2 \leq \langle A|A\rangle\,. \tag{F10}$$

Similarly, on Bob's side we obtain

$$\sum_{j=1}^{d-1}|b_j|^2 \leq \frac{k_B-1}{k_B}\,. \tag{F11}$$

Substituting the RHS of inequalities (F9),(F11) in (F8), and using $|a_0b_0| \geq 1/\sqrt{k_Ak_B}$, we obtain the desired result (F1). □