Strong Inequivalence of Quantum Nonlocal Resources

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Abstract: In this work [PRL **132**, 250205 (2024), arXiv:2310.16386] we present instances of quantum nonlocal correlations that are incomparable in the strongest sense. Specifically, when starting with an arbitrary many copies of one nonlocal correlation, it becomes impossible to obtain even a single copy of the other correlation, and this incomparability holds in both directions. Such incomparable quantum correlations can be obtained even in the simplest Bell scenario, which involves two parties, each having two dichotomic measurements setups. Notably, there exist an uncountable number of such incomparable correlations. Our result challenges the notion of a 'unique gold coin', often referred to as the 'maximally resourceful state', within the framework of the resource theory of quantum nonlocality. To this end, we provide examples of isotropic quantum correlations that cannot be distilled up-to Tsirelson point,and thus partially answer a long standing open question in nonlocality distillation.

1. Introduction

The nonlocal aspects of quantum correlations, as established by the seminal result of J. S. Bell [1] ; is undoubtedly the most significant departure of quantum theory from the classical description of our world [2,3] . The nonlocal signature of quantum correlations has been verified experimentally with a series of successful experiments. Besides such an extensive study and its immense practical importance in a diverse domain of device-independent technology [4], a resource theoretic formulation of quantum nonlocality is still illusive. In the present work [5] we have addressed the question of maximal nonlocal resource i.e. a 'unique gold coin' and its implications in nonlocality distillation. In a resource theory, two resources R_1 and R_2 will be called equivalent, if R_2 can be obtained from R_1 under the free operations and the vice-verse. Collection of equivalent resources form a equivalent class. When we say R_1 is more resourceful than R_2, means that R_2 can be obtained from R_1 freely but not the other-way around. Finally, R_1 and R_2 are incomparable when neither R_2 can be freely obtained from R_1 nor R_1 from R_2. In such a case resources R_1 and R_2 are treated as inequivalent resources. In a generic resource theory, it is quite possible that a resource R_2 cannot be obtained from one copy of another resource R_1, but can be obtained from its n copies. The symbol $R_1^{\otimes n} \not\rightarrow R_2$ denotes that a single copy of R_2 cannot be obtained from n copy of R_1 under the allowed free operations. This leads to a notion of the strongest form of inequivalence, namely the asymptotically inequivalence between two resources.

In this work, we consider a physically motivated variant of nonlocality theory which we call the resource theory of quantum nonlocality (RTQN). All the correlations allowed in this theory are quantum-realizable, *i.e.*, the resources belong to the set $\mathscr{Q} \setminus \mathscr{L}$. Specifically, we consider the simplest Bell scenario, which involves two parrty, each can perform two dichotomic measurement, namely $2-2-2$ Bell scenario.

2. Main Results

We first consider two specific nonlocal extreme points – the Tsirelson correlation P_T that saturates the maximum quantum value $2\sqrt{2}$ of the Bell-CHSH expression and the Hardy correlation P_H that yields the maximum quantum success $(5\sqrt{5}-11)/2 \approx 0.09$ for the Hardy's argument [6].

Theorem 1 *The quantum correlations P_T and P_H are incomparable in the strongest sense, i.e., $P_T^{\otimes n} \not\leftrightarrow P_H^{\otimes m}$, $\forall\, n,m \in \mathbb{N}$.*

One may wonder whether the inequivalence established in Theorem 1 is specific to the pair of correlations P_T and P_H, and then having a "gold coin" (other than P_T and P_H) can not be ruled out immediately. Nevertheless, our next result shows that there are uncountably many such inequivalent pairs of extreme nonlocal correlations and consequently lead us to conclude about the non-existence of any gold-coin resource.

Theorem 2 *All the pairs of quantum correlations $P_{\phi_2^+}, P_\psi^{st}$ are incomparable in the strongest sense, i.e., $(P_{\phi_2^+})^{\otimes n} \not\leftrightarrow (P_\psi^{st})^{\otimes m}$, $\forall\, n,m \in \mathbb{N}$.*

Here P_{ϕ^+}'s self-test the state ϕ_2^+ and P_ψ^{st}'s self-test the two-qubit non-maximally entangled states ψ.
A comparative discussion with entanglement theory is worthwhile at this point. While entanglement serves as a crucial element for demonstrating nonlocality in quantum theory, here we show that the resource theory of quantum nonlocality (RTQN) significantly diverges from the resource theory of quantum entanglement (RTQE). Particularly, while RTQE allows the notion of a maximally entangled state, a 'unique gold coin', RTQN lacks the concept of such a unique gold coin.
Now, we will see that these two theorems has non trivial implications in non-locality distillation. Consider the class of 222 isotropic correlations defined as,

$$PR_\eta(ab|xy) := \begin{cases} (1+\eta)/4, & \text{if } a \oplus b = xy \\ (1-\eta)/4, & \text{otherwise.} \end{cases}$$

For $0 \le \eta \le 1$ the correlations belong to the set $\mathscr{N}$, for $0 \le \eta \le 1/\sqrt{2}$ they belong to $\mathscr{Q}$, and for $0 \le \eta \le 1/2$ they belong to $\mathscr{L}$. A well-known conjecture, regarding distillability of isotropic correlations is that from arbitrary many copies of PR_{η_1} it is not possible to distill PR_{η_2}, where $1/2 < \eta_1 < \eta_2 < 1$ [7]. While some partial results are known with finite copy manipulation [8, 9], recently the authors in [10] have proved the conjecture for correlations with $1/\sqrt{2} < \eta_1 < \eta_2 < 1$. Our next theorem establishes a nontrivial result to this direction with isotropic quantum nonlocal correlations.

Theorem 3 *There exist isotropic quantum correlations PR_η, with $\eta \in (1/2, 1/\sqrt{2})$, that cannot be distilled up to P_T, even asymptotically.*

3. Discussion

Establishing strong inequivalence among different types of quantum nonlocal correlations carries significant practical implications. These correlations are pivotal for various information-theoretic tasks. Our Theorem 1 elucidates that if a specific quantum correlation is indispensable for the flawless execution of a task, then the same task may not be executed flawlessly even with numerous copies of inequivalent quantum correlations. Instances of such scenarios have been documented in zero-error and reverse-zero-error communication scenarios , as well as in Bayesian game scenarios. Consequently, when deriving nonlocal correlations from entangled quantum states for these tasks, it is imperative to perform the appropriate local measurements on the given state.

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