The Delayed Stabiliser ZX-Calculus

(extended abstract)

Cole Comfort Giovanni de Felice

We introduce the delayed stabiliser ZX-calculus, a formal graphical language for translation-invariant stabiliser quantum processes: enabling compact, finite reasoning about various families of infinite stabiliser-codes, such as quantum convolutional codes and lattice codes. Our calculus extends the odd-prime-dimensional stabiliser ZX-calculus with only a single additional generator, the delay, which feeds data from one time step to the next.

First, we give an interpretation of delayed ZX diagrams as sequences of finite approximations obtained by unrolling the diagram in time. We define a notion of observational equivalence between such sequences and prove that, modulo this equivalence relation, each sequence determines a unique infinite-dimensional stabiliser group.

Second, we interpret the delay as multiplication by a formal polynomial indeterminate, encoding infinite sequences of Pauli operators as fractions of polynomials. This allows us to represent the infinite stabiliser group of a delayed stabiliser circuit in terms of a finite generating tableau. We connect both semantics via geometric-series expansion, showing that composition in the infinite stabiliser group semantics approximates composition in the generating tableau semantics.

Finally, we give an equational theory for delayed stabiliser ZX-calculus, featuring generalised Euler decomposition and colour change rules. We show that this calculus is sound and universal for the generating tableau semantics. Combining delayed ZX calculus with scalable notation [9], we present a novel generalised local complementation rule for translation-invariant graph state. We conjecture that this rule follows from the axioms of our calculus, from which completeness would follow.

Stabiliser quantum mechanics is a fundamental component of quantum information theory and provides the backbone for much of quantum error correction and fault-tolerant quantum computation [32]. The stabiliser formalism offers a compact algebraic description of quantum processes via Pauli operators and their symmetries [18], avoiding the need to manipulate exponentially large matrices. This concise representation makes many encoded operations tractable to describe, analyse, and compare.

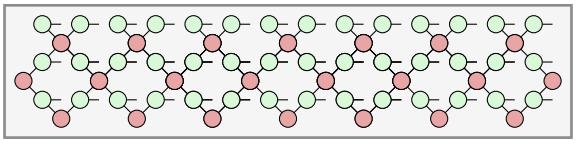
The ZX-calculus, is a graphical language in which quantum processes are represented as graphs such that equalities between processes can be derived by graph-rewriting [14]. Using the ZX-calculus, reasoning about quantum circuits can be carried out graphically rather than through matrix manipulation. Since its introduction by Coecke and Duncan, the ZX-calculus has been an incredibly useful and flexible tool for reasoning about quantum processes: see for example the book of Coecke and Kissinger [15] and the literature review of van de Wetering [38].

Its stabiliser fragment, the stabiliser ZX-calculus, is sound, universal, and complete for stabiliser quantum mechanics in all prime dimensions [1, 6, 35] and this equational theory directly reflects the algebraic structure of stabiliser theory. This has made the calculus particularly useful for reasoning about quantum error correction and measurement-based quantum computation, where diagrammatic manipulations correspond naturally to transformations of graph states and stabiliser codes. See the the book of Kissinger and van de Wetering for an overview of the applications of the ZX-calculus to quantum error correction and fault tolerant quantum computing [27].

However, the stabiliser ZX-calculus becomes inadequate when one attempts to describe quantum processes defined by repeated structure in space or time. The quantum error-correction literature provides several examples, including topological codes [17,28], in which a fixed local stabiliser pattern is repeated across space, as well as convolutional and serial turbo codes [23, 33, 34, 40, 41] where the pattern is repeated accross time. Other examples include quantum channels with memory [29], resource states in

measurement-based quantum computing [31] and quantum cellular automata [22].

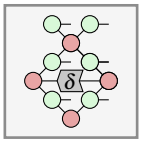
While such processes can be truncated to finite circuits, and therefore represented by ordinary ZX-diagrams, the resulting diagrams quickly become intractable, obscuring the recursive procedure that generates the circuit. For example, consider the ZX-calculus representation of the surface code given in [26]:



Properties of this diagram, representing a single truncation of the lattice, may not extend to larger truncations, forcing unwieldly use of ellipses.

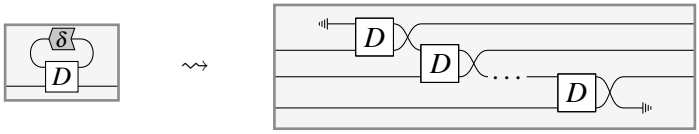
This presents a gap: we want a formal graphical language where these families of codes can be tractably represented and manipulated as finite graphs in a way which captures their translation-invariant symmetry. This would allow us to reduce computational problems in infinite stabiliser theory to problems in rewriting and graph theory.

We provide such a language by extending the odd-prime-dimensional stabiliser ZX-calculus with one additional generator  , called the delay, which passes data from one time step to the next. Diagrams built from this generator describe infinite processes obtained by iterating a finite pattern and connecting successive copies according to the delays. For example, the horizontal translation-invariance of the surface code diagram above is concisely represented by the following delayed ZX-diagram:



Using the delay, a single (finite) delayed ZX-diagram can represent systems such as spin chains, infinite graph states, lattice foliations, and quantum convolutional codes.

While freely adding a delay to the ZX-calculus is easy, in order to provide a useful equational theory, we must first understand what is the correct notion of equivalence between delayed ZX-diagrams. Following Carette et al. [11], we interpret delayed ZX-diagrams as sequences of increasingly better approximations of an infinite process, obtained by iterating the repeating pattern and discarding the boundary:



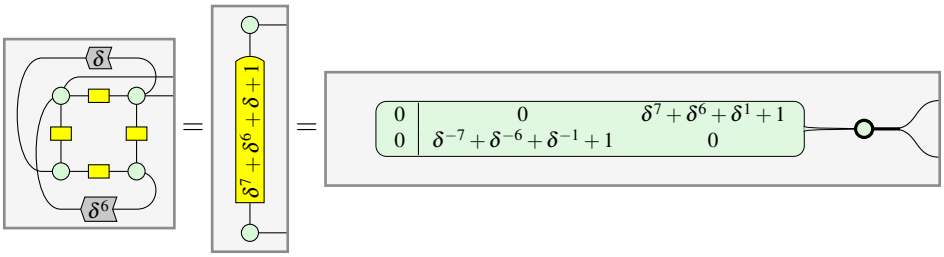
However, the notion of equivalence which they consider is too rigid for our purposes. Every finite approximation gives only *partial* information about an infinite process, and different sequences can determine the same long-term behaviour as more information is revealed. Taking inspiration from domain theory [36], we quotient sequences when they can be seen to approximate each other over time, in which case we say that they have the same *observational behaviour*. We show that two sequences of approximations have the same infinite stabiliser group precisely when they have the same behaviour in this sense.

Due to its infinite nature, this behavioural semantics can be hard to work with and to axiomatise syntactically. For translation-invariant stabiliser quantum processes, however, these problems can be circumvented using techniques from the convolutional codes literature (see the thesis of Wilde [39]). In this

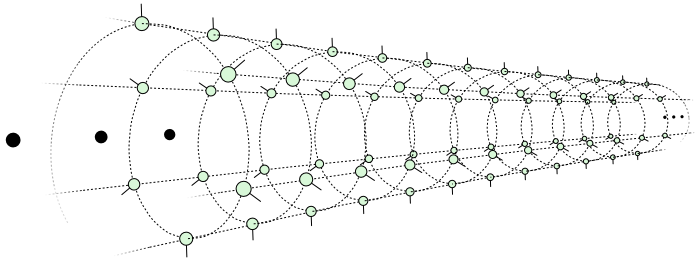
setting, the delay is interpreted as a formal indeterminate δ so that delayed stabiliser circuits are represented as finite *generating tableaux* over the field $\mathbb{F}_p(\delta)$ of fractions of polynomials over the finite field $\mathbb{F}_p$. We provide a compositional semantics for generating tableaux in terms of Lagrangian relations with respect to a shifted symplectic form. We show that the geometric series expansion defines a structure-preserving transformation from generating tableaux to infinite stabiliser groups. Therefore computing the equivalence of these translation-invariant stabiliser processes can be reduced to solving finite systems of polynomial equations. This translation only weakly preserves the structure: composition in the generating tableau semantics is *approximated* by composition in the infinite stabiliser-group semantics.

Thanks to this more tractable semantics, we are able to give an equational theory, δZX, for delayed stabiliser ZX-diagrams, which we call the delayed stabiliser ZX-calculus. In δZX, we can define spiders, multipliers and H-boxes labelled by fractions of polynomials in $\mathbb{F}_p(\delta)$ which encode the translation-invariant structure. The calculus features generalised Euler decomposition and colour change rules, exploiting the conjugation $\widetilde{f(\delta)} := f(\frac{1}{\delta})$ on polynomial fractions.

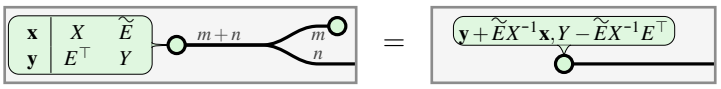
We prove that δZX is universal for the generating tableaux semantics by introducing scalable notation. The scalable notation generalises Haah's polynomial representation of translation-invariant graph states [20–22] as the phases of scalable spiders. For example, the following delayed graph state reduces to a single scalable spider:



which unrolls into the following infinite graph state (where the dotted wires denote Hadamard edges):



Every delayed ZX diagram can be reduced to a (non-unique) graph-like form defined by an adjacency matrix over $\mathbb{F}_p(\delta)$ between the boundary and internal nodes of the diagram. This allows us to express a generalised local complementation rule for translation-invariant graph states:



We conjecture that this rule can be derived from our equational theory, following a similar procedure as in [7]. Since the generating tableaux semantics gives us a normal form for δZX diagrams, it would follow from this conjecture that our calculus is complete.

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1 Introduction

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Contributions. The main contributions of this paper are as follows.

1. We introduce the notion of discard bicategory (definition 12), capturing theories of processes with intrinsic notions of approximation and discarding. Within this framework, we introduce the poset-enriched mixed stabiliser ZX-calculus (definition 16), providing a rewriting theory that captures approximation of stabiliser quantum channels (theorem 6).
2. We introduce a notion of observational behaviour over any discard bicategory (definition 22), capturing processes which evolve over discrete time or space. We show that the behaviour of a time-dependent stabiliser circuit is uniquely determined by its associated infinite stabiliser group (theorem 8 and corollary 2).
3. Recalling that translation-invariant stabiliser processes admit finite presentations in terms of generating tableaux, we show in section 5 that these tableaux compose relationally. We prove that the geometric-series expansion of a generating tableau corresponds to the infinite stabiliser group of the associated translation-invariant stabiliser process, and that composition of infinite stabiliser groups approximates composition of their generating tableaux (theorem 9 and proposition 10).
4. We introduce the delayed stabiliser ZX-calculus, δZX, in section 6, giving translation-invariant analogues of the odd-prime-dimensional ZX-calculus generators (section 6.1) and equations (section 6.2).
5. We introduce a scalable notation for δZX-diagrams in section 6.4, using it to prove universality, so that that every shifted stabiliser tableau can be represented by a δZX-diagram (theorem 10). Using the scalable notation, we formulate translation-invariant analogues of the local complementation and pivoting rules (lemma 14). We conjecture that these rules are derivable in our equational theory, which would imply completeness of δZX.

Contents

2 Pure stabiliser quantum mechanics

Stabiliser quantum mechanics is the fundamental structure which underlies the vast majority of finite-dimensional quantum error correction. See for example the seminal and highly influential work of Gottesman [18]. The stabiliser formalism of quantum error correction exploits the symplectic representation of stabiliser quantum mechanics in order to redundantly store, manipulate and extract quantum information.

In this section, we first recall the pure stabiliser subtheory of quantum mechanics in terms of linear maps between Hilbert spaces, followed by its symplectic representation in terms of $\mathbb{F}_p$ linear algebra, ending with the corresponding graphical language. To formally relate all three pictures of stabiliser quantum mechanics, we will construct them as instances of the following structure

Definition 1. *A **strict †-compact closed category** (**†-CCC**) is a category $\mathscr{C}$ equipped with a strictly associative unital monoidal product $\otimes$ with unit I, a natural isomorphism $\mathrm{swap}_{X,Y} : X \otimes Y \to Y \otimes X$, an identity-on-objects involutive contravariant functor $(-)^\dagger : \mathscr{C}^{\mathrm{op}} \to \mathscr{C}$, and for each object X a dual X^* satisfying $(X \otimes Y)^* = Y^* \otimes X^*$ and morphisms $\cup_X : I \to X \otimes X^*$ and $\cap_X : X^* \otimes X \to I$, such that:*

$$(g \circ f)^\dagger = f^\dagger \circ g^\dagger, \qquad (f^\dagger)^\dagger = f, \qquad (f \otimes g)^\dagger = f^\dagger \otimes g^\dagger, \qquad \mathrm{swap}_{X,Y}^{-1} = \mathrm{swap}_{Y,X} = \mathrm{swap}_{X,Y}^\dagger,$$
$$(1_X \otimes \cap_X) \circ (\cup_X \otimes 1_X) = 1_X, \qquad (\cap_X \otimes 1_{X^*}) \circ (1_{X^*} \otimes \cup_X) = 1_{X^*}, \qquad \cap_X = (\cup_X)^\dagger \circ \mathrm{swap}_{X^*,X}.$$

Sweeping coherence issues under the rug, we work with strict †-CCCs without loss of generality.

2.1 The stabiliser formalism

We first recall the basic theory of quopit stabiliser quantum mechanics. For reference see Gross [19].

Fix an odd prime p, and write $\mathbb{F}_p$ for the field of integers modulo p. Denote the quopit Hilbert space by $\mathscr{H}_p := \mathrm{span}\{|x\rangle \mid x \in \mathbb{F}_p\}$ and the character function by $\chi(x) := \exp(i2\pi x/p)$.

Definition 2. *The 1 quopit **Pauli operators** are given by $Z|x\rangle := \chi(x)|x\rangle$, and $X|x\rangle := |x+1\rangle$, so that they generate the 1-quopit Pauli group $\mathscr{P}_p := \{\chi(y)X^xZ^z \mid x,y,z \in \mathbb{F}_p\}$. Define the n-quopit **Pauli group** to be the n-fold tensor product $\mathscr{P}_p^{\otimes n}$ of the 1-quopit Pauli group.*

The following bijection underlies the core of the stabiliser formalism:

Lemma 1. *Take a maximal Abelian subgroup $S \subseteq \mathscr{P}_p^{\otimes n}$ such that $\chi(x)1_{\mathscr{H}_p^{\otimes n}} \in S$ iff $x = 0$. Then S determines a unique normalised state $|S\rangle \in \mathscr{H}_p^{\otimes n}$, up to global phase. Its phase class is called the stabiliser state associated to S.*

Stabiliser states and their adjoints assemble themselves into a †-CCC.

Definition 3. *The †-CCC* Stab *of quopit stabiliser maps is the †-CC subcategory of* FHilb *generated by stabiliser states, Clifford operators, and the scalars $1/\sqrt{p}$ and $\sqrt{p}$, under tensor product, composition, and dagger.*

Note that the scalars $1/\sqrt{p}$ and $\sqrt{p}$ are needed for compact closure.

2.2 The symplectic representation

The same structure can be described in phase space. For reference see [19].

Definition 4. *A **symplectic vector space** (V, ω_V) is a finite-dimensional $\mathbb{F}_p$-vector space V equipped with a non-degenerate alternating bilinear form $\omega_V : V \times V \to \mathbb{F}_p$.*

Example 1. *For $n \in \mathbb{N}$, there is a symplectic vector space $(\mathbb{F}_p^{2n}, \omega_n)$ with $\omega_n\big((\mathbf{x},\mathbf{z}),(\mathbf{a},\mathbf{b})\big) \coloneqq \mathbf{x}\cdot\mathbf{b} - \mathbf{z}\cdot\mathbf{a}$.*

Lemma 2. *Every finite-dimensional symplectic vector space is even-dimensional and isomorphic, via a linear map preserving the symplectic form, to $(\mathbb{F}_p^{2n}, \omega_n)$ for $n = \dim(X)/2$.*

To pass from operators to $\mathbb{F}_p$-linear algebra, we use a standard parametrisation of Paulis:

$$\pi : \mathbb{F}_p \oplus \mathbb{F}_p^n \oplus \mathbb{F}_p^n \to \mathscr{P}_p^{\otimes n}, \qquad (a,\mathbf{x},\mathbf{z}) \mapsto \chi(a)\,\chi\big(2^{-1}\mathbf{x}\cdot\mathbf{z}\big)\prod_{k=1}^{n} X_k^{x_k} Z_k^{z_k}. \tag{1}$$

Lemma 3. *Two Paulis $\pi(y,\mathbf{x},\mathbf{z})$ and $\pi(c,\mathbf{x}',\mathbf{z}')$ commute if and only if $\omega_n\big((\mathbf{x},\mathbf{z}),(\mathbf{x}',\mathbf{z}')\big) = 0$.*

Therefore, up to global phase, we can represent an n-quopit Pauli operator by an element of $\mathbb{F}_p^{2n}$. Because the commutation of Pauli operators is captured by the symplectic form, we can use this representation to define stabiliser states by certain subspaces of $\mathbb{F}_p^{2n}$:

Definition 5. *Given a linear subspace $S \subseteq (X,\omega)$, define its symplectic complement*

$$S^{\omega} \coloneqq \{\mathbf{x} \in X \mid \forall \mathbf{s} \in S,\ \omega(\mathbf{x},\mathbf{s}) = 0\}.$$

*The subspace S is **Lagrangian** when $S^{\omega} = S$, and **coisotropic** when $S^{\omega} \subseteq S$. An affine subspace is Lagrangian or coisotropic when it is empty or if its linear part[1] is Lagrangian or coisotropic.*

Proposition 1 ([19])**.** *The map in equation* (1) *induces a bijection between non-empty affine Lagrangian subspaces of $(\mathbb{F}_p^{2n}, \omega_n)$ and stabiliser subgroups of $\mathscr{P}_p^{\otimes n}$, hence stabiliser states up to global phase.*

Affine Lagrangian subspaces are better known in the quantum information literature by their tableaux. We recall the unique tableau associated to an affine Lagrangian subspace:

Proposition 2. *Given an affine Lagrangian subspace $R \subseteq (\mathbb{F}_p^{2n}, \omega_n)$, there exist $m \in \mathbb{N}$, a matrix $L : n-m \to m$, a symmetric matrix $\Sigma^{\top} = \Sigma : n-m \to n-m$, a permutation $\varsigma : n \to n$, and $\mathbf{a} \in \mathbb{F}_p^{2n}$ such that,*

$$R = \ker\left(\left[\begin{array}{cc|cc} I_m & L & \Sigma & 0 \\ 0 & 0 & -L^{\top} & I_{n-m} \end{array}\right]\left[\begin{array}{c|c} \varsigma & 0 \\ \hline 0 & \varsigma \end{array}\right]\right) + \mathbf{a}.$$

*Call this presentation of R the **canonical form** of the tableau associated to R. Given any affine Lagrangian subspace, there is a unique canonical tableau. In case $m = 0$, say that R is the tableau associated to a **graph-state**.*

See proof on page 31. The canonical form of a stabiliser tableau seems to first have been discovered by Cleve and Gottesman [13] for qubits, followed by Kalra in odd prime dimensions [24].

We can compose affine Lagrangian subspaces, following Weinstein [37]:

Definition 6. *The category $\mathsf{ALR}_{\mathsf{fd}}$ of affine Lagrangian relations has:*

- ***objects**: finite-dimensional symplectic vector spaces (V,ω_V);*
- ***morphisms** $(V,\omega_V) \to (W,\omega_W)$: affine Lagrangian subspaces of $(V,-\omega_V)\times(W,\omega_W)$;*
- ***symmetric monoidal structure**: given by the direct sum;*
- ***†-CC structure**: the dagger is given by the relational converse; whereas the cups and caps are induced by the symplectic dual $(V,\omega_V)^* \coloneqq (V,-\omega_V)$.*

[1] The **linear part** of an affine subspace $A \subseteq \mathbb{F}_p^n$ is $\{\mathbf{x}-\mathbf{y} \mid \mathbf{x},\mathbf{y} \in A\}$.

In order to state the equivalence between the Hilbert and symplectic pictures of stabiliser quantum mechanics, we need to quotient stabiliser maps by scalars:

Definition 7. *Given a symmetric monoidal category $\mathscr{C}$, let* $\mathrm{Proj}(\mathscr{C})$*, denote the symmetric monoidal category of* ***projective maps*** *in $\mathscr{C}$. This has the same objects as $\mathscr{C}$, where the morphisms are given by equivalence classes of morphisms $[f]$ in $\mathscr{C}$, so that $[f] = [g]$ if and only if there exists some invertible scalar $\lambda : I \to I$ in $\mathscr{C}$ such that $\lambda f = g$.*

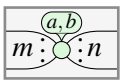
After applying this quotient, the composition of stabiliser maps and their tableaux agree:

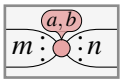
Theorem 1 ([16, 30])**.** *There is a †-CC equivalence* $\mathrm{Proj}(\mathsf{Stab}) \simeq \mathsf{ALR}_{\mathsf{fd}}$.

2.3 The stabiliser ZX-calculus

Affine Lagrangian relations admit a sound, universal, and complete graphical language, called the quopit stabiliser ZX-calculus, which we denote by ZX. The stabiliser ZX-calculus was originally introduced by Booth and Carette [6], after which its presentation was refined by Poór et al. [35]. The interpretation into ZX was later given by Booth et al. [7].

The stabiliser ZX-calculus is the †-CCC generated by green and red **spiders**:



and


, for all $a, b \in \mathbb{F}_p$.

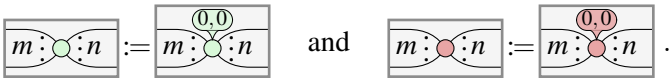
a is called the **affine phase**, and b the **symplectic phase**. These are interpreted in $\mathsf{ALR}_{\mathsf{fd}}$ as:

$$\left[\!\!\left[\quad \right]\!\!\right] := \left\{ \left(\begin{bmatrix} \mathbf{z} \\ \mathbf{x} \end{bmatrix}, \begin{bmatrix} \mathbf{z}' \\ \mathbf{x}' \end{bmatrix} \right) \middle| \forall \mathbf{x}, \mathbf{z} \in \mathbb{F}_p^m, \mathbf{x}', \mathbf{z}' \in \mathbb{F}_p^n : \exists x \in \mathbb{F}_p : x_i = x'_j = x, \sum_{j=0}^{m-1} z_j + bx + a = \sum_{i=0}^{n-1} z'_i \right\}$$

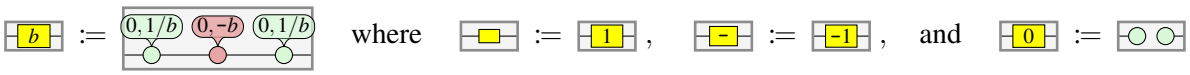
$$\left[\!\!\left[\quad \right]\!\!\right] := \left\{ \left(\begin{bmatrix} \mathbf{z} \\ \mathbf{x} \end{bmatrix}, \begin{bmatrix} \mathbf{z}' \\ \mathbf{x}' \end{bmatrix} \right) \middle| \forall \mathbf{x}, \mathbf{z} \in \mathbb{F}_p^m, \mathbf{x}', \mathbf{z}' \in \mathbb{F}_p^n : \exists z \in \mathbb{F}_p : z_i = z'_j = z, \sum_{j=0}^{m-1} x_j + bz + a = \sum_{i=0}^{n-1} x'_i \right\}$$

To state the equational theory in figure 1 succinctly, we define several notations.

First, for spiders with trivial phases:



We will also make frequent use of **H-boxes**. Given $b \in \mathbb{F}_p$ such that $b \neq 0$:



We will use the following notation to denote relations given by linear transformation:

Definition 8. *Where given a function $f : X \to Y$, its* ***graph*** *(not to be confused with the notation of a graph state) is the relation* $\mathrm{Gr}(f) : \{x, f(x)) \mid x \in X\} \subseteq X \times Y$.

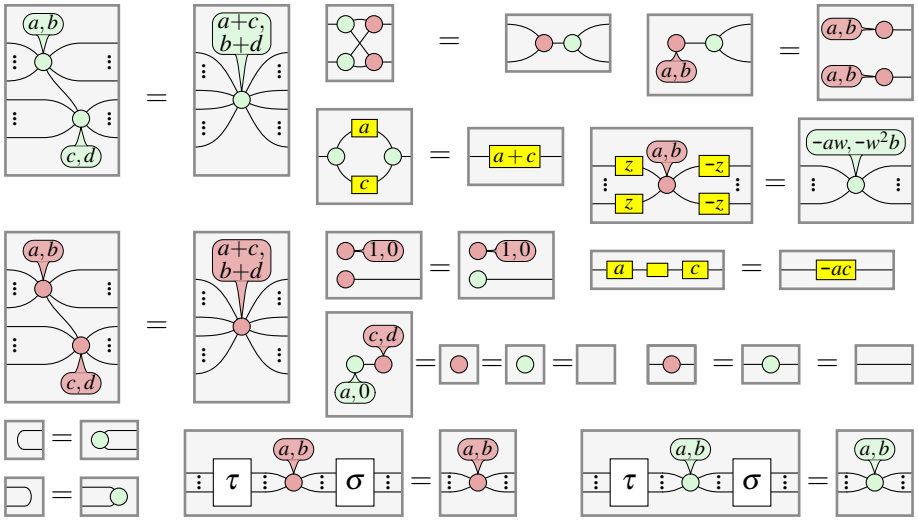


Figure 1: Axioms of ZX, for all $a,b,c,d,z \in \mathbb{F}_p$, with $z \neq 0$ and permutations ς and τ.

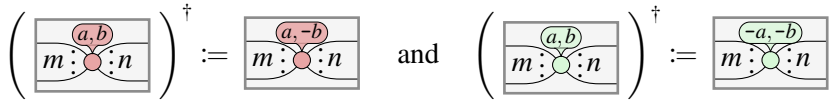
When $b \neq 0$, the H-box is the graph of the *squeezed symplectic Fourier transform*:

$$\left[\!\left[\; \boxed{b} \; \right]\!\right] := \mathrm{Gr}\left[\begin{array}{c|c} 0 & 1/b \\ \hline -b & 0 \end{array}\right]$$

We use the H-boxes to define **multipliers** $\boxed{\rangle b}$ $:=$ $\boxed{b}$ $\boxed{-}$. When $b \neq 0$, the multiplier is interpreted as the graph of the **squeezing map**:

$$\left[\!\left[\; \boxed{\rangle b} \; \right]\!\right] := \mathrm{Gr}\left[\begin{array}{c|c} b & 0 \\ \hline 0 & 1/b \end{array}\right]$$

The **zero map** $\boxed{1,0}$ is interpreted as the empty subspace. Finally, the dagger is given by:



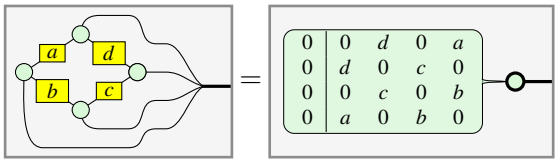
2.4 Scalable notation and normal form for ZX

In this subsection we recall Booth et al.'s scalable notation for ZX [7, § 3.4], which allows diagrams to be concisely rewritten to a unique normal form.

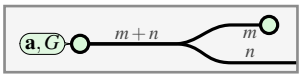
Definition 9. *Given a symmetric matrix $G^\top = G : n \to n$ over $\mathbb{F}_p$ and a vector $\mathbf{a} \in \mathbb{F}_p^n$, the **graph state** associated to G is constructed by first constructing the δZX diagram $\bigotimes_{j=0}^{n-1}$* $\boxed{a_j, G_{j,j}}$ *. Then, for each $0 \leq j < k < n$ connect the jth and kth wires with the following δZX diagrams:*

In the scalable notation the shifted graph state is denoted  .

Example 2.

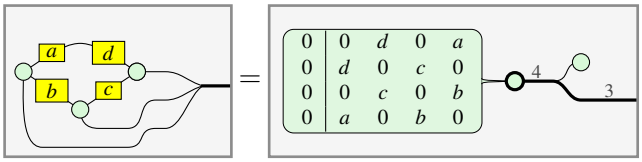


Definition 10. *A* ZX*-diagram is in* ***graph-like*** *form, when it is composed of a graph state where some of the outputs have been capped off by*  *'s. In the scalable notation, graph-like* ZX*-diagrams take the following form, where we split off the first m wires of the graph-state and cap them off:*



Returning to our example, we can remove one of the vertices to produce a graph-like diagram:

Example 3.

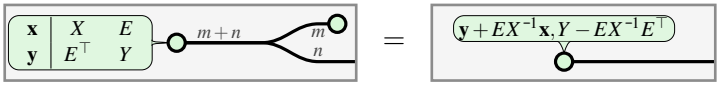


Every diagram can be put into this non-unique form:

Lemma 4 ([7, Prop. 34])**.** *All* ZX*-diagrams can be rewritten to graph-like form.*

To reduce graph-like states to a unique normal form, there are two ingredients which are needed.

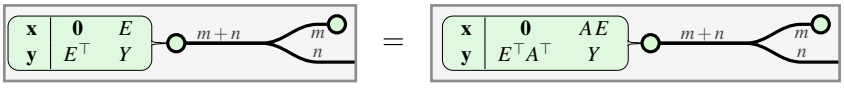
Proposition 3 ([7, Prop. 35])**.** *Given matrices $X : m \to m$, $Y : n \to n$, $E : n \to m$ over $\mathbb{F}_p$ and vectors $\mathbf{x} \in \mathbb{F}_p^m$, $\mathbf{y} \in \mathbb{F}_p^n$, such that X and Y are symmetric and X is invertible, we rewrite the graph-like diagram on the left into the graph-state on the right:*



This transformation is called ***Schur complementation***

Schur-complementation simultaneously generalises the more well-known *local complementation* ([7, Prop. 36]) and *pivoting* ([7, Prop. 37]) rules, which can be seen to apply symplectic row-operations on the diagonal. We can also apply symplectic row-operations on the anti-diagonal:

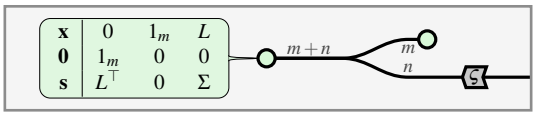
Proposition 4 ([7, Lem. 38])**.** *Take a matrix $E : n \to m$, a symmetric matrix $Y : n \to n$ and vectors $\mathbf{x} \in \mathbb{F}_p^m$, $\mathbf{y} \in \mathbb{F}_p^n$. Given any invertible matrix A:*



Theorem 2 ([7, Thm. 45])**.** *By repeatedly applying Schur-complementation and by row reduction, any nonzero state in* ZX *corresponding to an affine Lagrangian subspace with canonical form*

$$L = \ker\left(\left[\begin{array}{cc|cc} I_m & L & \Sigma & 0 \\ 0 & 0 & -L^\top & I_{n-m} \end{array}\right]\left[\begin{array}{c|c} \varsigma & 0 \\ \hline 0 & \varsigma \end{array}\right]\right) + \mathbf{a}$$

can be rewritten in to the following unique normal form, where $(\mathbf{x},\mathbf{s}) = \varsigma \circ \mathbf{a}$*:*



Because the canonical form is unique, it follows immediately that:

Theorem 3. *There is a †-CC equivalence* $\mathsf{ZX} \simeq \mathsf{ALR}_{\mathsf{fd}} \simeq \mathsf{Proj}(\mathsf{Stab})$.

3 Noise and approximation in stabiliser quantum mechanics

Pure quantum mechanics fails to capture noise, which a fundamental feature of quantum mechanics. In this section, we review mixed stabiliser quantum mechanics which captures such notions as noise and measurement. We frame mixed state quantum mechanics within categories with a notion of approximation and discarding:

Definition 11. *A symmetric monoidal category is* ***poset-enriched****, in case for all objects* X *and* Y*, there is a poset* $\subseteq$ *structure on the hom-set* $\mathscr{C}(X,Y)$ *so that:*

- *given* $f \subseteq g$ *and* $h \subseteq k$ *of appropriate type, then* $f \otimes h \subseteq g \otimes h$*;*
- *and given* $f \subseteq g$ *and* $h \subseteq k$ *of appropriate type, then* $h \circ f \subseteq k \circ g$*.*

Given two processes f and g of the same type, we interpret $f \subseteq g$ to denote that f is a less noisy version of g, so that g *approximates* f. To model discarding of information, we use the following structure:

Definition 12. *A* ***discard bicategory*** *is a poset-enriched symmetric monoidal category, such that for all objects* X*, there exists a* ***discard map*** $\mathtt{disc}_X : X \to I$*, denoted*  *, where for all* $f : X \to Y$*,*

$$\mathtt{disc}_Y \circ f \subseteq \mathtt{disc}_X, \quad \mathtt{disc}_I = 1_I \quad \textit{and} \quad \mathtt{disc}_{X\otimes Y} = \mathtt{disc}_X \otimes \mathtt{disc}_Y.$$

In a discard †-CCC, denote the dagger of the discard $\mathtt{codisc} \coloneqq \mathtt{disc}^\dagger$ *by*  .

Interpreting the $f \subseteq g$ to denote that g approximates f, the discard map approximates all effects, so that it is maximally uninformative. In the following subsections we upgrade the three pictures of stabiliser theory reviewed in the previous section from pure to mixed processes. We show that this naturally gives rise to discard bicategory structures.

3.1 Completely positive maps

In standard quantum theory, noisy processes are modeled by completely positive maps [32]. The category of completely positive maps has a rich structure:

Definition 13. *The poset enriched †-CCC* CPM *has finite-dimensional Hilbert space as objects, where morphisms* $\Phi : \mathscr{H} \to \mathscr{K}$ *are* ***completely positive maps*** $\Phi : \mathscr{H} \otimes \mathscr{H}^* \to \mathscr{K} \otimes \mathscr{K}^*$*. Here an operator* $\rho \in \mathscr{H} \otimes \mathscr{H}^*$ *is positive when* $\langle \psi, \rho\,\psi \rangle \geq 0$ *for all* $\psi \in \mathscr{H}$*, and* Φ *is* ***completely positive*** *when* $1_{\mathscr{E}\otimes\mathscr{E}^*} \otimes \Phi$ *sends positive operators to positive operators for all f.d. Hilbert spaces* $\mathscr{E}$*.*

The †-CC structure of CPM *is inherited from* FHilb*. The poset enrichment is given by the* ***Löwner order*** *so that* $\Phi \preceq \Psi$ *in case* $\Psi - \Phi$ *is completely positive.*

Given a linear map $T : \mathscr{H} \to \mathscr{K}$ *between finite-dimensional Hilbert spaces, the linear map* $T \otimes (\overline{T})^* : \mathscr{H} \otimes \mathscr{H}^* \to \mathscr{K} \otimes \mathscr{K}^*$ *is completely positive. Such a completely positive map is called* ***pure****.*

The Stinespring dilation theorem is an important result characterising completely positive maps as the partial trace of a *pure* quantum process. Intuitively, any noisy process $X \to Y$ is a pure process on a larger space $X \to Y \otimes E$ where the environment E is discarded or unobserved.

Lemma 5. *Given any completely positive map $\Phi : \mathscr{H} \otimes \mathscr{H}^* \to \mathscr{K} \otimes \mathscr{K}^*$, there exists a Hilbert space $\mathscr{E}$ and a linear map $T : \mathscr{H} \to \mathscr{K} \otimes \mathscr{E}$, called a **minimal Stinespring dilation** of Φ, such that $\Phi = \mathrm{Tr}_{\mathscr{E}}(T \otimes (\overline{T})^*)$, where $\mathrm{Tr}_{\mathscr{E}} = 1_{\mathscr{K}} \otimes \mathrm{ev}_{\mathscr{E}} \otimes 1_{\mathscr{K}^*}$ is the **partial trace** and $\mathrm{ev}_{\mathscr{E}} : \mathscr{E} \otimes \mathscr{E}^* \to \mathbb{C}$ is evaluation $v \otimes \varphi \mapsto \varphi(v)$.*

Moreover, for any $S : \mathscr{H} \to \mathscr{K} \otimes \mathscr{E}'$ satisfying $\Phi = \mathrm{Tr}_{\mathscr{E}'}(S \otimes (\overline{S})^*)$ there exists an isometry $V : \mathscr{E} \to \mathscr{E}'$ such that $(1 \otimes V) \circ T = S$.

Definition 14. *The †-CCC $\mathsf{Stab}^{+!!}$ of **completely positive stabiliser maps** is the subcategory of* CPM *whose objects are given by quopit Hilbert spaces $\mathscr{H}_p^{\otimes n}$ and whose morphisms are given by completely positive maps which admit a pure stabiliser Stinespring dilations.*

Carette et al. [10, Def. 5] introduced a purification *preorder* between completely positive maps.After quotienting by nonzero scalars, this becomes a *partial order* and gives $\mathrm{Proj}(\mathrm{CPM})$ and $\mathrm{Proj}(\mathsf{Stab}^{+!!})$ the structure of discard bicategories:

Proposition 5. $\mathrm{Proj}(\mathrm{CPM})$ *is a discard bicategory with respect to the trace and the **purification order**. Given projective completely positive maps $\Phi, \Psi : \mathscr{H} \to \mathscr{K}$ in* $\mathrm{Proj}(\mathrm{CPM})$*, $\Phi \preceq_{\mathsf{P}} \Psi$ in case there exists some Hilbert space $\mathscr{E}$ and maps $\Psi_0 : \mathscr{H} \to \mathscr{K} \otimes \mathscr{E}$ and $\Phi_0 : \mathscr{E} \to \mathbb{C}$ in* $\mathrm{Proj}(\mathrm{CPM})$ *such that $\Psi = \mathrm{Tr}_{\mathscr{E}}(\Psi_0)$ and $\Phi = (1_{\mathscr{K}} \otimes \Phi_0) \circ \Psi_0$.*

Moreover, $\mathrm{Proj}(\mathrm{CPM})$ *is a discard bicategory with respect to the trace.*

See proof on page 32.

Corollary 1. $\mathrm{Proj}(\mathsf{Stab}^{+!!})$ *is a discard bicategory with respect to the purification order and the trace, and the canonical functor* $\mathrm{Proj}(\mathsf{Stab}^{+!!}) \to \mathrm{Proj}(\mathrm{CPM})$ *preserves the order.*

See proof on page 32.

Carrette et al. [10] claim that that the purification order and the Löwner order are "tightly linked". We prove this claim formally:

Theorem 4. $[\Phi] \preceq_{\mathsf{P}} [\Psi]$ *in* $\mathrm{Proj}(\mathrm{CPM})$ *if and only if there is some $\lambda \in \mathbb{R}^+$ such that $\Phi \preceq_{\mathsf{L}} \lambda\Psi$ in* CPM.

See proof on page 32.

3.2 The symplectic picture

What are the stabiliser groups of completely positive stabiliser maps? The quantum trace is stabilised by all Pauli operators, so that its stabiliser group is the entire Pauli group. Therefore, completely positive stabiliser maps should have larger stabiliser groups than pure stabiliser maps:

Definition 15. *Let $\mathsf{ALR}_{\mathsf{fd}}^{+!!}$ denote the †-CCC with the same structure as $\mathsf{ALR}_{\mathsf{fd}}$, except now, whose morphisms are **affine coisotropic relations**. That is to say, these are affine relations which are either empty, or an affine relation whose linear component S satisfies $S^{\omega} \subseteq S$.*

Booth and Comfort proved that $\mathsf{ALR}_{\mathsf{fd}}^{+!!}$ admits a notion of Stinespring dilation [8, Prop. 38]:

Lemma 6. *Given an affine coisotropic relation $R : (V, \omega_V) \to \{\bullet\}$ there is exists a symplectic vector space (E, ω_E) and an affine Lagrangian coisometry $S : (V, \omega_V) \to (E, \omega_E)$ such that:*

$$R = \mathtt{disc}_{(E,\omega_E)} \circ S \quad \textit{where} \quad \mathtt{disc}_{(E,\omega_E)} \coloneqq \{(\mathbf{v}, \bullet) \mid \mathbf{v} \in E\} : (E, \omega_E) \to \{\bullet\}.$$

It follows immediately that:

Proposition 6. $\mathsf{ALR}^{\dagger}_{\mathsf{fd}}$ *is discard bicategory with respect to subspace inclusion and* `disc`.

The relational picture is equivalent to the projective Hilbert space picture:

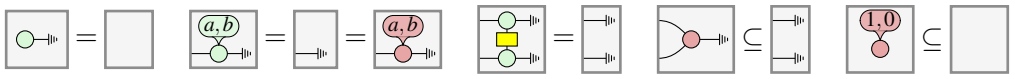
Theorem 5. *There is an equivalence of †-CC discard bicategories* $\mathsf{Proj}(\mathsf{Stab}^{\dagger}) \cong \mathsf{ALR}^{\dagger}_{\mathsf{fd}}$.

See proof on page 33.

3.3 The mixed stabiliser ZX-calculus

Finally, building on the work of Carette et al. [10] and Booth et al. [7, Cor. 54], we can add discarding to the stabiliser ZX-calculus:

Definition 16. *The* ***mixed stabiliser ZX-calculus****,* $\mathsf{ZX}^{\dagger}$*, is the †-CC discard category presented by adding a generator* $\boxed{\dagger}$ $: 1 \to 0$ *to* ZX*. We impose that isometries are discarded, all maps are laxly discarded, and that the zero map is approximated by all morphisms, so that for all* $a, b \in \mathbb{F}_p$*,*



Putting everything together, we have:

Theorem 6. *There is an equivalence of †-CC discard bicategories* $\mathsf{ZX}^{\dagger} \simeq \mathsf{ALR}^{\dagger}_{\mathsf{fd}} \simeq \mathsf{Proj}(\mathsf{Stab}^{\dagger})$.

See proof on page 33.

4 Behaviours of time-delayed quantum circuits

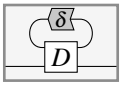
In this section, we add a notion of time to ZX via the *state construction*, originally introduced by Katis et al. [25]:

Definition 17. *The †-CCC* $\mathrm{St}(\mathsf{ZX})$*, called the* ***stateful stabiliser ZX-calculus*** *(or* ***stateful ZX-calculus*** *for short), is given by adding a* ***delay*** *generator* $\boxed{\delta}$ *to* ZX*, modulo*

(2)

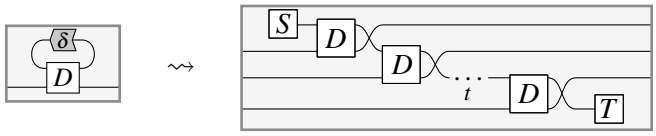
The Pauli operators are interpreted as being applied at a single time step, so that they can not be slid through the delay: applying a Pauli after the discard is interpreted as applying the Pauli at the next time step, which is not the same as applying a Pauli as the current time step.

By isolating the stabiliser ZX-diagrams from the delay, every morphism of $\mathrm{St}(\mathsf{ZX})$ can be represented (non-uniquely) in **delayed trace** form for some ZX-diagram $D : s+n \to s+m$:



The "feedback wire" on top is interpreted as taking the the output at time t and feeding it back to the input at time $t+1$. This allows us to interpret stateful ZX-diagrams as discrete-time quantum automata.

we interpret a a ZX diagram D with no Pauli operators as a the transition function and the delayed wire denotes the internal state which evolves over time. By providing an initial state S, a final state T and number of time steps n, we can unroll the automata as follows:

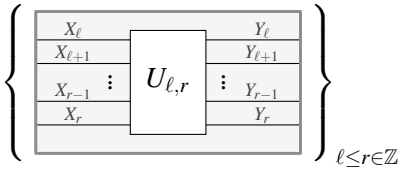


Notice that two different such automata can have distinct denotations in $\mathrm{St}(\mathsf{ZX})$, even if they produce the same behaviour over time. In the following subsections, we provide semantics which equate stateful ZX-diagrams when they have the same behaviour over arbitrary time, for all possible initial and final states.

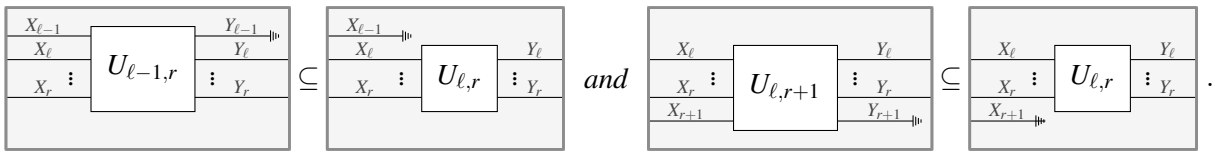
4.1 Monotone sequences

To give a time-dependent semantics to $\mathrm{St}(\mathsf{ZX})$, following Carette et al. [11, P. 4], we use the discard bicategory structure of $\mathsf{ZX}^{\Vdash}$:

Definition 18. *Let $\mathscr{C}$ be a discard bicategory, and take two bi-infinite sequence of objects in $\mathscr{C}$, $\mathbf{X},\mathbf{Y} \in (\mathrm{ob}(\mathscr{C}))^{\mathbb{Z}}$. A **monotone sequence** $\mathbf{U} : \mathbf{X} \to \mathbf{Y}$ is a morphisms in $\mathscr{C}$ indexed by finite intervals of $\mathbb{Z}$*

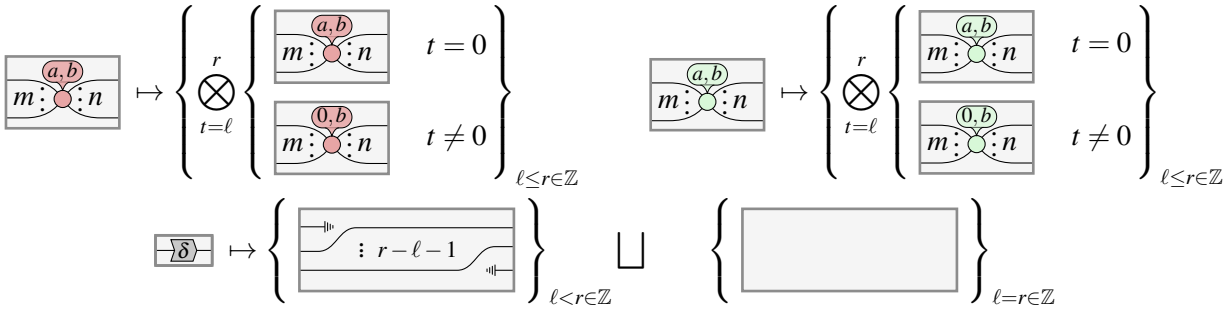


such that for all $\ell \leq r \in \mathbb{Z}$:



*A monotone sequence is **causal** when all such inclusions are equalities.*

Definition 19. *Given a stateful stabiliser ZX-diagram $D : n \to m$, define the unrolling $\mathtt{Unroll}(D) : n^{\mathbb{Z}} \to m^{\mathbb{Z}}$ to be the monotone sequence given on generators by:*

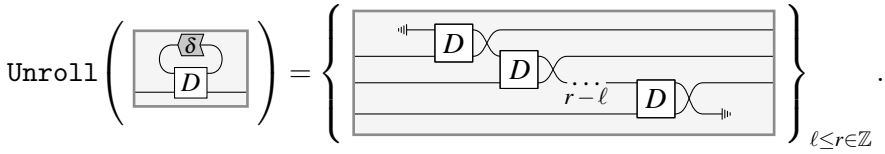


where the tensor product and composition of unrollings are defined pointwise.

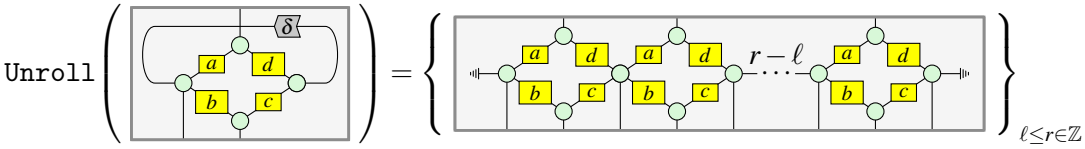
The linear generators are repeated at each time step. The affine shifts, which represent the Paulis, act only at time $t = 0$. The delay feeds the input at time t forward to the output at time $t+1$, discarding the first input and returning the state of complete uncertainty, the maximally mixed state, on the last output.

The delayed trace of a linear ZX-diagram has a particularly simple unrolling: the linear part of the diagram is repeated at each time step, with the state being fed back to the next time step:

Example 4. *Given a linear ZX diagram $D : s+n \to s+m$, the unrolling of the delayed trace of D is:*



Example 5. *We can glue graph states to each other as plaquettes of an increasingly larger graph state:*

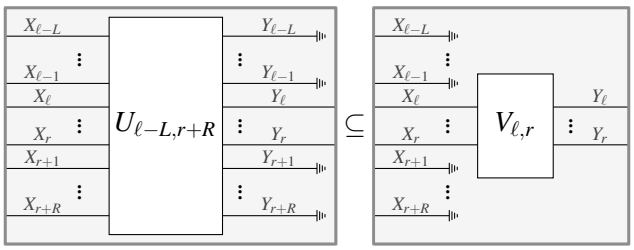


4.2 Observational equivalence and behaviours

The natural number indexed variation of this construction was used by Carette et al. to model quantum channels with finite memory, where they add a similar notion of delay to the ZX-calculus [11]. However, monotone sequences are too rigid to fully reflect the behaviour of a stateful ZX-diagram evolving over time. For example, if at some point in the future, an element of the sequence is zero, then the whole process should be zero. However, such a sequence containing zero, in general, will not be equal to the everywhere-zero sequence.

On the other hand, taking inspiration from the field of domain theory (see Scott [36]), we interpret monotone sequences as representing partial information about some infinite process, so that the value of a sequence at some interval denotes what is known at that interval of time. However, what is known at some interval of time may be an incomplete state of knowledge about the infinite process. Therefore, two compare two monotone sequences, we must take into account what additional information can be revealed at some larger time-interval:

Definition 20. *Given monotone sequences $\mathbf{U}, \mathbf{V} : \mathbf{X} \to \mathbf{Y}$, say that $\mathbf{V}$ **approximates** $\mathbf{U}$, written $\mathbf{U} \subseteq \mathbf{V}$, if for every $\ell \leq r \in \mathbb{Z}$, there exists some **lookahead** $L, R \in \mathbb{N}$ such that*



Therefore, the correct notion of equivalence of monotone sequences is not their equality. Rather, two infinite processes represented by monotone sequences have the same behaviour over time precisely when they approximate each other:

Definition 21. *Two monotone sequences* $\mathbf{U}, \mathbf{V} : \mathbf{X} \to \mathbf{Y}$ *are* ***observationally equivalent*** *in case they approximate each other so that* $\mathbf{U} \subseteq \mathbf{V}$ *and* $\mathbf{V} \subseteq \mathbf{U}$.

Monotone sequences modulo observational equivalence can be composed:

Definition 22. *Given a discard bicategory* $\mathscr{C}$*, the discard bicategory* ***observational behaviours****,* $\mathrm{Obs}(\mathscr{C})$*, has:*

- *objects:* $\mathbb{Z}$*-indexed sets of objects in* $\mathscr{C}$*:* $\mathbf{X} = \{X_k\}_{k\in\mathbb{Z}}$*;*
- *morphisms: observational equivalence classes of monotone sequences* $[\mathbf{U}] : \mathbf{X} \to \mathbf{Y}$*;*
- *poset enrichment: given by approximation so that* $[\mathbf{V}] \subseteq [\mathbf{U}]$ *if and only if* $\mathbf{V} \subseteq \mathbf{U}$*;*
- *discard bicategory structure: defined pointwise.*

Theorem 7. *The data in definition 22 makes* $\mathrm{Obs}(\mathscr{C})$ *a discard bicategory.*

See proof on page 33.

Proposition 7. *The unrolling of a stateful ZX-diagram is a* †*-CC functor* $\mathtt{Unroll} : \mathrm{St}(\mathsf{ZX}) \to \mathrm{Obs}(\mathsf{ZX}^{\dashv\!\vdash_\omega})$.

See proof on page 34. The quotient by observational equivalence is essential for the preservation of equation (2) which is needed to prove the functoriality of `Unroll`.

Moreover, in contrast to the work of Carette et al., this quotient makes the delay natural so that

$$\mathtt{Unroll}(\ \boxed{\delta}\ ) \circ \{U_{\ell,r}\}_{\ell\le r\in\mathbb{Z}} = \{U_{\ell+1,r+1}\}_{\ell\le r\in\mathbb{Z}} \circ \mathtt{Unroll}(\ \boxed{\delta}\ ).$$

This means that the delay can be interpreted as a formal reindexing of time without any caveats.

4.3 Infinite stabiliser groups

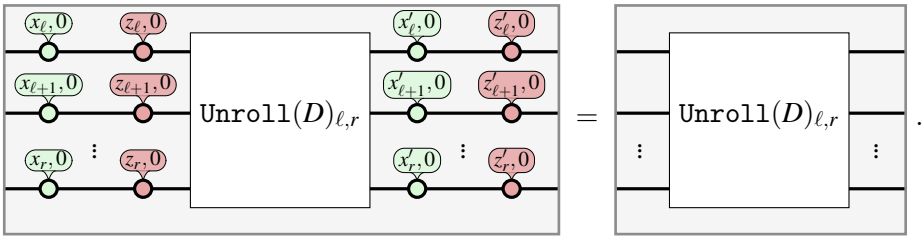
Given a stateful ZX-diagram, each stage of the unrolling can be interpreted as the affine coisotropic relation corresponding to its group of stabilisers. Remarkably, and nontrivially, observational equivalence classes of sequences of stabiliser groups uniquely correspond to an infinite set of stabilisers.

Definition 23. *Let* AR_∞ *denote the* †*-CCC whose objects are (possibly infinite-dimensional) vector spaces over* $\mathbb{F}_p$*, whose morphisms* $R : X \to Y$ *are affine subspaces* $R \subseteq X \times Y$.

Theorem 8. *There is a faithful* †*-CC functor* $\mathtt{Lim} : \mathrm{Obs}(\mathsf{ALR}^{\dashv\!\vdash_\omega}_{\mathsf{fd}}) \rightarrowtail \mathsf{AR}_\infty$ *sending morphisms* $[\mathbf{R}] : \mathbf{X} \to \mathbf{Y}$ *to the relation* $\left\{(\mathbf{x},\mathbf{y}) \;\middle|\; \forall \ell \le r \in \mathbb{Z} :\ \big((x_\ell,\dots,x_r),(y_\ell,\dots,y_r)\big) \in R_{\ell,r}\right\} \subseteq \prod_{j\in\mathbb{Z}} X_j \times \prod_{j\in\mathbb{Z}} Y_j$.

See proof on page 38. Because we built these infinite relations from observational equivalence classes of finite relations, they are are closed subspaces in the product topology of discrete topologies.

Corollary 2. *This induces a* †*-CC functor* $\mathtt{StabGrp} : \mathrm{St}(\mathsf{ZX}) \to \mathrm{Obs}(\mathsf{ZX}^{\dashv\!\vdash_\omega}) \simeq \mathrm{Obs}(\mathsf{ALR}^{\dashv\!\vdash_\omega}_{\mathsf{fd}}) \to \mathsf{AR}_\infty$*, such that given any stateful ZX-diagram* $D : n \to m$*, a sequence* $\left(\left[\begin{smallmatrix}\mathbf{x}\\ \mathbf{z}\end{smallmatrix}\right], \left[\begin{smallmatrix}\mathbf{x}'\\ \mathbf{z}'\end{smallmatrix}\right]\right) \in ((\mathbb{F}_p^n)^{\mathbb{Z}})^2 \times ((\mathbb{F}_p^m)^{\mathbb{Z}})^2$ *of Pauli operators is an element of* $\mathtt{StabGrp}(D)$ *if and only if for all* $\ell \le r \in \mathbb{Z}$*:*



Where the thick spiders denote multi-quopit Pauli operators in the scalable notation for ZX *(see section 2.4 for reference). This means that* $\mathtt{StabGrp}(D)$ *is the space of Pauli operators which stabilise each stage of the unrolling of the stateful ZX-diagram D.*

See proof on page 38.

Example 6. *The delay is sent to the relation which reindexes the Paulis:*

$$\texttt{StabGrp}\left(\ \boxed{\delta}\ \right) = \left\{ \left(\begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix}, \begin{bmatrix} \mathbf{x}' \\ \mathbf{z}' \end{bmatrix} \right) \;\middle|\; \forall t \in \mathbb{Z},\, x'_t = x_{t-1},\, z'_t = z_{t-1} \right\} \subseteq (\mathbb{F}_p^{\mathbb{Z}})^2 \times (\mathbb{F}_p^{\mathbb{Z}})^2.$$

5 Generating tableaux for time-delayed circuits

The functor of corollary 2 takes a stateful ZX-diagram and produces an infinite set of stabilisers. However, from this description alone, it is not clear how to represent such infinite sets of stabilisers in a finite way, for example via some infinite analogue of a tableau. In this section, we show that stateful ZX-diagrams do admit a notion of tableau, which we call a **generating tableau**.

To motivate the idea of a generating tableau, we recall the basic idea behind generating functions for geometric sequences. Fix a field $\mathbb{K}$. The field of formal Laurent series has elements given by formal sums of the form $\mathbb{K}((\delta)) \coloneqq \{\sum_{t \geq N} a_t \delta^t \,|\, N \in \mathbb{Z},\ a_t \in \mathbb{K}\}$. Any infinite sequence $(x_t)_{t \in \mathbb{N}}$ in $\mathbb{K}$ can be represented by a formal Laurent series $\sum_{t \in \mathbb{Z}} x_t \delta^t \in \mathbb{K}((\delta))$, called its **generating function**.

For example, the sequence $x_t \coloneqq t$ is represented by the formal Laurent series $X(\delta) = \sum_{t \geq 0} t\, \delta^t$. Moreover, because $(x_t)_{t \in \mathbb{N}}$ is also a geometric series, the formal Laurent series $X(\delta)$ can be expressed as a fraction of polynomials in δ: $X(\delta) = \frac{\delta}{(1-\delta)^2}$. Therefore, the infinite sequences whose generating functions happen to be geometric sequences do admit a finite description.

In this section, we extend the interpretation $[\![-]\!] : \mathsf{ZX} \to \mathsf{ALR}_{\mathsf{fd}}$ to the delayed setting, so that rather than interpreting the delay as the $\mathbb{F}_p$-linear relation which *reindexes* the vector space $\mathbb{F}_p^{\mathbb{Z}}$ as in example 6, it is interpreted as the $\mathbb{F}_p(\delta)$-linear relation that *multiplies* by δ:

$$\left[\!\!\left[\ \boxed{\delta}\ \right]\!\!\right] \coloneqq \left\{ \left(\begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix}, \delta \cdot \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix} \right) \;\middle|\; \mathbf{x}, \mathbf{z} \in \mathbb{F}_p(\delta) \right\} \subseteq \mathbb{F}_p(\delta)^2 \times \mathbb{F}_p(\delta)^2 \tag{3}$$

5.1 Generating tableaux and the shifted symplectic form

Interpreting stateful ZX-diagrams as $\mathbb{F}_p(\delta)$ affine relation according to equation (3) produces a special class of affine relations, which we call *shifted affine Lagrangian relations*. These are analogous to affine Lagrangian relations over $\mathbb{F}_p$, except where the symplectic form is twisted by the involution $\widetilde{(-)} : \mathbb{F}_p(\delta) \to \mathbb{F}_p(\delta)$ given by $\widetilde{f}(\delta) \coloneqq f(1/\delta)$.

Definition 24. *Given some $n \in \mathbb{N}$, the standard **shifted symplectic form** on $\mathbb{F}_p(\delta)^{2n}$ is the sequilinear form given by twisting the standard symplectic form on $\mathbb{F}_p(\delta)^{2n}$ with the conjugation $\widetilde{(-)}$:*

$$\widetilde{\omega}_n(f(\delta), g(\delta)) \coloneqq \omega_n(f(\delta), g(1/\delta)) \tag{4}$$

More generally, given any $\mathbb{F}_p$-symplectic vector space, one can obtain a shifted symplectic vector space by extending scalars along $\mathbb{F}_p(\delta)/\mathbb{F}_p$ and twisting the symplectic form.

The standard shifted symplectic form, was originally formulated by Wilde et al. [40, 41] to express the commutation of infinite streams of Pauli operators.

Definition 25. *Given a $\mathbb{F}_p(\delta)$-linear subspace $S + \mathbf{a}(\delta) \subseteq (V, \widetilde{\omega})$ of a shifted symplectic vector space, the **shifted symplectic complement** is the linear subspace:*

$$S^{\widetilde{\omega}} = \{\mathbf{f}(\delta) \in V \mid \forall \mathbf{g}(\delta) \in S,\, \widetilde{\omega}(\mathbf{f}(\delta), \mathbf{g}(\delta)) = 0\} \subseteq (V, \widetilde{\omega}).$$

*A **shifted affine Lagrangian subspace** $S + \mathbf{a}(\delta) \subseteq (V, \widetilde{\omega})$ is an affine subspace with $S^{\widetilde{\omega}} = S$.*

This notion generalises the unshifted notion of a Lagrangian subspace

Example 7. *Given an $\mathbb{F}_p$-affine Lagrangian subspace of $(\mathbb{F}_p^{2n}, \omega_n)$, its field extension along $\mathbb{F}_p(\delta)/\mathbb{F}_p$ is a shifted affine Lagrangian subspace of $(\mathbb{F}_p(\delta)^{2n}, \widetilde{\omega}_n)$.*

Similarly, for the interpretation of the delay in equation (3).

Example 8. *The interpretation of the delay is a shifted affine Lagrangian subspace:*

$$\left\{ \left(\begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix}, \delta \cdot \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix} \right) \;\middle|\; \mathbf{x}, \mathbf{z} \in \mathbb{F}_p(\delta) \right\} \subseteq (\mathbb{F}_p(\delta)^{2n}, -\widetilde{\omega}_n) \times (\mathbb{F}_p(\delta)^{2m}, \widetilde{\omega}_m).$$

As in the unshifted case given in proposition 2:

Proposition 8. *Given a shifted affine Lagrangian subspace $R \subseteq (\mathbb{F}_p(\delta)^{2n}, \widetilde{\omega}_n)$, there exist $m \in \mathbb{N}$, a matrix $L : n-m \to m$, a sesqui-symmetric matrix $\Sigma^\top = \widetilde{\Sigma} : n-m \to n-m$, a permutation $\varsigma : n \to n$, and $\mathbf{a} \in \mathbb{F}_p(\delta)^{2n}$ such that,*

$$R = \ker \left(\left[\begin{array}{cc|cc} I_m & \widetilde{L} & \Sigma & 0 \\ 0 & 0 & -L^\top & I_{n-m} \end{array} \right] \left[\begin{array}{c|c} \varsigma & 0 \\ \hline 0 & \varsigma \end{array} \right] \right) + \mathbf{a}.$$

*Call this presentation of R the **canonical form** of the **tableau** associated to R. Given any shifted affine Lagrangian subspace, there is a unique canonical tableau. In case $m = 0$, say that R is the tableau associated to a **shifted graph-state**.*

See proof on page 39.

Such a shifted Lagrangian subspace is known as the tableau associated to a quantum convolutional code, which is the infinite-time, translation-invariant analogue to a stabiliser state (see Wilde's thesis for reference [39]). There is a breadth of literature on quantum convolutional codes in the qubit setting [23, 33, 40, 41]; however, due to their infinite nature, the compositional structure of quantum convolutional coding theory remains unexplored. By working in odd-prime dimensions, we show that quantum convolutional processes can be composed as relations, in the same way as their finite-dimensional analogue:

Definition 26. *The †-CC feedback category $\mathsf{sALR}_{\mathsf{fd}}$ of shifted $\mathbb{F}_p(\delta)$-affine Lagrangian relations between finite-dimensional $\mathbb{F}_p(\delta)$-shifted symplectic vector spaces has objects given by shifted symplectic vector spaces, whose morphisms $(V, \widetilde{\omega}_V) \to (W, \widetilde{\omega}_W)$ are either shifted affine Lagrangian subspaces of $(V, -\widetilde{\omega}_V) \times (W, \widetilde{\omega}_W)$ or the empty set. The rest of the †-CC structure is the same as for $\mathsf{ALR}_{\mathsf{fd}}$.*

Using equation (3), we can interpret the stateful ZX-calculus within this category:

Proposition 9. *There is a †-CC functor $\mathtt{Ext} : \mathrm{St}(\mathsf{ZX}) \to \mathsf{sALR}_{\mathsf{fd}}$.*

Functoriality is immediate, as given any $\mathbb{F}_p(\delta)$-linear vector space S, by linearity, we have $\delta \cdot S = S$.

Example 9. *Returning to example 5, we obtain a* shifted *graph-state:*

$$\mathtt{Ext}\left(\begin{array}{c} \\ \\ \\ \end{array} \right) = \ker \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \delta a + d & 0 \\ 0 & 1 & 0 & \delta^{-1} a + d & 0 & \delta^{-1} b + c \\ 0 & 0 & 1 & 0 & \delta b + c & 0 \end{array} \right],$$

Example 10. *The delayed controlled X gate has the following interpretation:*

$$\mathtt{Ext}\left(\begin{array}{c} \\ \\ \end{array} \right) = \mathrm{Gr} \begin{bmatrix} (1+\delta)^{-1} & 0 \\ 0 & 1+\delta^{-1} \end{bmatrix}$$

These shifted graph states have been extensively studied by Haah in the context of lattice codes, topological quantum computing, and quantum cellular automata [20–22]. In his work, the entries of the graph are restricted to the PID $\mathbb{F}_p[\delta,\delta^{-1}] \subset \mathbb{F}_p(\delta)$ of Laurent polynomials as well as related group algebras and modules.

5.2 From generating tableaux to behaviours

In this section, we show that the stateful ZX-calculus can be interpreted functorially in shifted affine Lagrangian relations; after which, we show that shifted affine Lagrangian relations can be interpreted oplaxly in AR_∞ via the geometric series expansion.

Theorem 9. *There is a †-CC oplax normal functor* $\Gamma : \mathsf{sALR}_{\mathsf{fd}} \to \mathsf{AR}_\infty$*, which factors through* $\mathsf{Obs}(\mathsf{ACR}_{\mathsf{fd}})$*, sending objects* $(\mathbb{F}_p^{2n}(\delta), \widetilde{\omega}_n) \mapsto (\mathbb{F}_p^{\mathbb{Z}})^{2n}$ *and sending morphisms* $R \subseteq (\mathbb{F}_p^{2n}(\delta), -\widetilde{\omega}_n) \times (\mathbb{F}_p^{2m}(\delta), \widetilde{\omega}_m)$ *to the* $\mathbb{F}_p$*-affine subspace of* $(\mathbb{F}_p^{\mathbb{Z}})^{2n} \times (\mathbb{F}_p^{\mathbb{Z}})^{2m}$ *given on elements by computing the geometric series expansion pointwise and then regarding formal Laurent series as* $\mathbb{Z}$ *indexed sequences over* $\mathbb{F}_p$*.*

See proof on page 40. The condition that Γ is oplax normal means that the identity is preserved and for all relations R and S of the same type $\Gamma(S \circ R) \subseteq \Gamma(S) \circ \Gamma(R)$. This is interpreted as the composition in $\mathsf{sALR}_{\mathsf{fd}}$ is only *approximated* by composition in AR_∞ along Γ. For example, the interpretations of rational functions in general are only partially invertible in AR_∞:

Example 11. *Consider the graph of squeezing matrix by* $\delta+1$*:* $\mathrm{Gr}(\mathrm{diag}(\delta+1), \widetilde{(\delta+1)^{-1}})$.

This is invertible in $\mathsf{sALR}_{\mathsf{fd}}$*:*

$$\texttt{Ext}\Big(\ \Big)^{-1} = \texttt{Ext}\Big(\ \Big).$$

However it is only partly invertible in AR_∞*:*

$$\Gamma\big(\texttt{Ext}\big(\ \big)\big) \circ \Gamma\big(\texttt{Ext}\big(\ \big)\big) = 1,$$

where we only have the strict inequality:

$$\Gamma\big(\texttt{Ext}\big(\ \big)\big) \circ \Gamma\big(\texttt{Ext}\big(\ \big)\big) \supsetneq 1.$$

Because observational equivalence is defined in terms of finite approximations, in order to prove the left inverse, we would need infinite lookahead to expand the geometric series. On the other hand, the right inverse can be computed lazily with finite lookahead without needed to fully expand the geometric series at each stage of the unrolling. Therefore, the oplaxness of Γ means that the finitary nature of composition in AR_∞ approximates the formally infinite nature of composition in $\mathsf{sALR}_{\mathsf{fd}}$.

We can also relate the unrolling semantics to the generating-tableau semantics:

Proposition 10. *For all* $D \in \mathrm{St}(\mathsf{ZX})$*,* $\Gamma(\texttt{Ext}(D)) \subseteq \texttt{StabGrp}(D)$.

See proof on page 41.

Given a shifted affine Lagrangian relation associated to a stateful ZX-diagram, the geometric series expansion of this subspace produces the stream of Pauli operators which stabilise its unrollings. This justifies our claim that the generating-tableau is a finite description of the infinite space of stabilisers.

Example 12. *Continuing example 9, let*

$$\texttt{Ext}\left(\ \right) = \ker \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \delta a + d & 0 \\ 0 & 1 & 0 & \delta^{-1}a + d & 0 & \delta^{-1}b + c \\ 0 & 0 & 1 & 0 & \delta b + c & 0 \end{array}\right] =: R.$$

Applying the geometric series expansion, a stream $(\mathbf{x},\mathbf{z}) \in ((\mathbb{F}_p^3)^{\mathbb{Z}})^2$ *is in* $\Gamma(R)$ *if and only if, for all* $t \in \mathbb{Z}$,

$$x_{0,t} = -a\,z_{1,t-1} - d\,z_{1,t}, \quad x_{1,t} = -a\,z_{0,t+1} - d\,z_{0,t} - b\,z_{2,t+1} - c\,z_{2,t}, \quad \textit{and} \quad x_{2,t} = -b\,z_{1,t-1} - c\,z_{1,t}.$$

Therefore, the associated finitely-supported stabiliser group associated to this graph state is generated by finite products of the following Pauli operators, for all $t \in \mathbb{Z}$

$$K_{0,t} := X_{1,t-1}^{-a} X_{1,t}^{-d} Z_{0,t}, \quad K_{1,t} := X_{0,t+1}^{-a} X_{0,t}^{-d} X_{2,t+1}^{-b} X_{2,t}^{-c} Z_{1,t}, \quad \textit{and} \quad K_{2,t} := X_{1,t-1}^{-b} X_{1,t}^{-c} Z_{2,t},$$

where $X_{n,t}$ *(respectively* $Z_{n,t}$*) denotes the Pauli X (respectively Pauli Z) on wire* $n \in \{0,1,2\}$ *at index* t.

Example 13. *Continuing example 10,*

$$\texttt{Ext}\left(\quad\right) = \mathrm{Gr}\begin{bmatrix}(1+\delta)^{-1} & 0\\ 0 & 1+\delta^{-1}\end{bmatrix}$$

Applying the geometric series expansion, a pair $\big((\mathbf{x},\mathbf{z}),(\mathbf{x}',\mathbf{z}')\big) \in ((\mathbb{F}_p^{\mathbb{Z}})^2)^2$ *is in this relation if and only if, for all* $t \in \mathbb{Z}$,

$$x_t = x'_t + x'_{t-1}, \qquad z'_t = z_t + z_{t+1}.$$

Therefore, the associated finitely-supported stabiliser subgroup on the input-output chain is generated by finite products of the following Pauli operators, for all $t \in \mathbb{Z}$

$$L_t^X := X_t^{\mathrm{in}} X_{t+1}^{\mathrm{in}} X_t^{\mathrm{out}}, \qquad L_t^Z := Z_t^{\mathrm{in}} Z_t^{\mathrm{out}} Z_{t-1}^{\mathrm{out}}.$$

As a technical detail, the Pauli operators associated should be seen to act on a quasi-local algebra rather than on an infinite-dimensional Hilbert space (see [12, 29]). However, because we are ignoring normalisation, the corresponding morphisms need not be bounded.

6 The delayed ZX-calculus: δZX

In this section, we add enough equations to $\mathrm{St}(\mathsf{ZX})$ to obtain a complete axiomatisation of $\mathsf{sALR}_{\mathsf{fd}}$. We call the resulting language the **delayed stabiliser ZX-calculus** (or **delayed ZX-calculus** for short), denoted δZX. Our delayed ZX-calculus can be regarded as a quantum analogue of the graphical calculus for signal flow graphs [2–5]. Whereas signal flow graphs are interpreted in affine relations between finite dimensional vector spaces over the field of rational functions; δZX is interpreted in shifted affine Lagrangian relations between finite-dimensional shifted symplectic vector spaces.

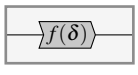
6.1 Derived notation

In this subsection, we define generalised spiders, H-boxes and multipliers in $\mathrm{St}(\mathsf{ZX})$, giving their interpretations in $\mathsf{sALR}_{\mathsf{fd}}$. This will allow us to state the equational theory of δZX in the following subsection. We prove that these notations are well-defined in section 6.3.

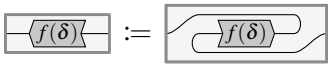
Definition 27. *Multipliers by polynomials in* $\mathbb{F}_p[\delta]$ *are defined by induction on the degree. The base case is given by the polynomial* 0*. For the inductive hypothesis, suppose that we have multipliers for degree* $n \in \mathbb{N}$ *polynomials* $f(\delta) \in \mathbb{F}_p[\delta]$*. Given an* $n+1$*-degree polynomial* $f(\delta) + b\delta^{n+1} \in \mathbb{F}_p[\delta]$*, define:*

$$\quad := \quad$$

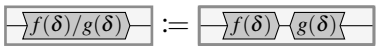
A multiplier by $f(\delta) \in \mathbb{F}_p[\delta]$ is interpreted as the graph of the following matrix over $\mathbb{F}_p(\delta)$:

$$\texttt{Ext}\left(\text{  } \right) = \text{Gr}\left[\begin{array}{c|c} f(\delta) & 0 \\ \hline 0 & 1/f(1/\delta) \end{array}\right]$$

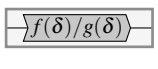
Multipliers by polynomials $f(\delta) \in \mathbb{F}_p[\delta]$ simultaneously generalize the ZX multipliers when $f(\delta) = a \in \mathbb{F}_p$, and the delay when $f(\delta) = \delta$. Note that because $\mathbb{F}_p$ multipliers and the delay all commute with each other, and because the red and green spiders are commutative Frobenius algebras, this notation is well defined. Next, we define the converse of a polynomial multiplier:

Definition 28. *Given a polynomial $f(\delta) \in \mathbb{F}_p[\delta]$ define*  .

We introduce notation for multipliers over rational functions:

Definition 29. *Given $f(\delta), g(\delta) \in \mathbb{F}_p[\delta]$, define:*  .

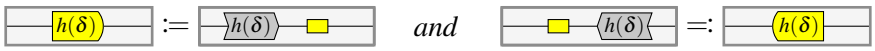
Where

$$\texttt{Ext}\left(\text{  } \right) = \text{Gr}\left[\begin{array}{c|c} f(\delta)/g(\delta) & 0 \\ \hline 0 & g(1/\delta)/f(1/\delta) \end{array}\right].$$

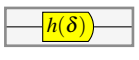
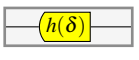
Because we have not yet imposed any equations governing the division of polynomials, these multipliers are not well-defined with respect to rational function labels. Throughout this subsection, whenever we label notation by a rational function, we take this to denote a formal pair of polynomials. In the following subsection, we impose axioms which make these derived notations labelled by rational functions well-defined.

We define generalized, *directed* H-boxes labelled by rational functions:

Definition 30. *Given a rational function $h(\delta) \in \mathbb{F}_p(\delta)$, define:*



These are interpreted as the graphs of the matrices, which are converses to each other as relations:

$$\texttt{Ext}\left(\text{  } \right) = \text{Gr}\left[\begin{array}{c|c} 0 & 1/h(1/\delta) \\ \hline -h(\delta) & 0 \end{array}\right] \quad \text{and} \quad \texttt{Ext}\left(\text{  } \right) = \text{Gr}\left[\begin{array}{c|c} 0 & 1/h(\delta) \\ \hline -h(1/\delta) & 0 \end{array}\right]$$

To define undirected multipliers we make the following observations:

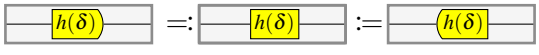
Lemma 7. *The ring of self-conjugate Laurent polynomials $h(\delta) = h(1/\delta) \in \mathbb{F}_p[\delta, 1/\delta]$ is isomorphic to the ring $\mathbb{F}_p[\delta + 1/\delta] \subset \mathbb{F}_p[\delta, 1/\delta]$ of polynomials in $\delta + 1/\delta$.*

See proof on page 41.

Lemma 8. *The field of self-conjugate rational functions is isomorphic to the field of fractions $\mathbb{F}_p(\delta + 1/\delta) \subset \mathbb{F}_p(\delta)$ of the ring of self-conjugate Laurent polynomials $\mathbb{F}_p[\delta + 1/\delta] \subset \mathbb{F}_p[\delta, 1/\delta]$ in $\delta + 1/\delta$.*

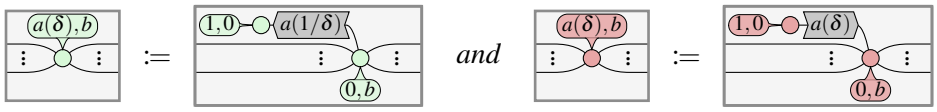
See proof on page 42. When the H-boxes are labelled by a self-conjugate rational function the left and right-facing H-boxes coincide, making undirected H-boxes unambiguous:

Definition 31. *Denote the undirected $h(\delta) \in \mathbb{F}_p(\delta + 1/\delta)$ labelled H-box by:*



We define affine phases labelled by rational functions:

Definition 32. *Given $a(\delta) \in \mathbb{F}_p(\delta)$ and $b \in \mathbb{F}_p$, define:*



Defining generalised symplectic phases is more intricate. To this end, observe:

Remark 1. *For any rational function $f(\delta) \in \mathbb{F}_p(\delta)$:*

$$\texttt{Ext}\left(\boxed{f(\delta)}\right) = \mathrm{Gr}\left[\begin{array}{c|c} \frac{1}{f(\delta)+f(1/\delta)} & 0 \\ \hline 0 & 1 \end{array}\right] \quad \textit{and} \quad \texttt{Ext}\left(\boxed{f(\delta)}\right) = \mathrm{Gr}\left[\begin{array}{c|c} 1 & f(\delta)+f(1/\delta) \\ \hline 0 & 1 \end{array}\right],$$

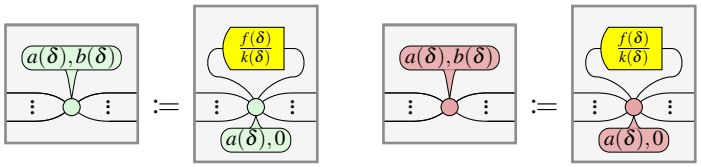
Which are analogous to the the following spiders in ZX*, for all scalars $a \in \mathbb{F}_p$:*

$$\texttt{Ext}\left(\boxed{0,b}\right) = \mathrm{Gr}\left[\begin{array}{c|c} 1 & 0 \\ \hline b & 1 \end{array}\right] \quad \textit{and} \quad \texttt{Ext}\left(\boxed{0,b}\right) = \mathrm{Gr}\left[\begin{array}{c|c} 1 & b \\ \hline 0 & 1 \end{array}\right]$$

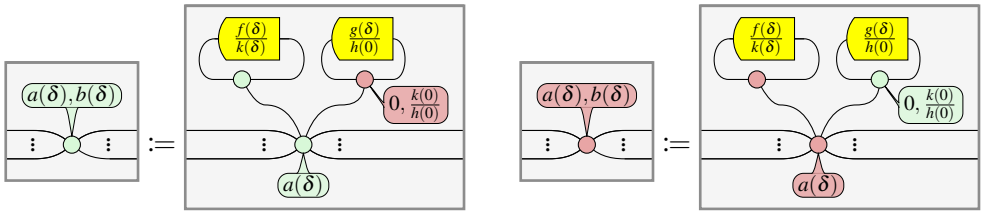
Therefore, we can define spiders with symplectic phase labelled by $f(\delta)+f(1/\delta) \in \mathbb{F}_p(\delta+1/\delta)$ for all $f \in \mathbb{F}_p(\delta)$. More generally, we can do so for *arbitrary* self-conjugate rational functions:

Definition 33. *Take some $a(\delta) \in \mathbb{F}_p(\delta)$ and $b(\delta) \in \mathbb{F}_p(\delta+1/\delta)$.*
Choose $h(\delta),k(\delta) \in \mathbb{F}_p[\delta+1/\delta]$ such that $b(\delta) = \frac{h(\delta)}{k(\delta)}$.
Choose $f(\delta),g(\delta) \in \mathbb{F}_p[\delta]$ such that $h(\delta) = f(\delta)+f(1/\delta)+h(0)$ and $k(\delta) = g(\delta)+g(1/\delta)+k(0)$.
We define $(a(\delta),b(\delta))$-labelled spiders by case distinction:

- *If $h(0) = 0$, we can use our insight from remark 1:*



- *Otherwise, if $h(0) \neq 0$, we need to be more clever:*



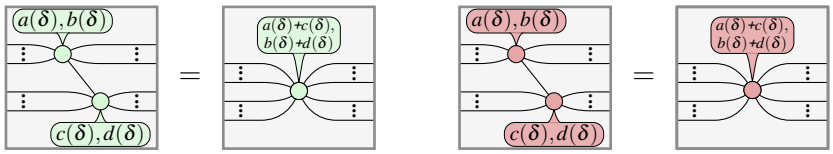
These are the correct definition of generalised symplectic phases as they correspond to shearing operators; given a self-conjugate rational function $h(\delta) \in \mathbb{F}_p(\delta+1/\delta)$:

$$\texttt{Ext}\left(\boxed{0,h(\delta)}\right) = \mathrm{Gr}\left[\begin{array}{c|c} 1 & 0 \\ \hline h(\delta) & 1 \end{array}\right] \quad \text{and} \quad \texttt{Ext}\left(\boxed{0,h(\delta)}\right) = \mathrm{Gr}\left[\begin{array}{c|c} 1 & h(\delta) \\ \hline 0 & 1 \end{array}\right]$$

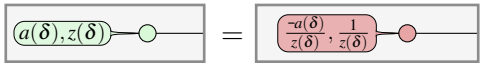
6.2 Axioms

Using the notation from the previous subsection, we impose the additional equations:

Axiom 1 (Spider-fusion). *Given* $(a(\delta),b(\delta)),(c(\delta),d(\delta)) \in \mathbb{F}_p(\delta) \times \mathbb{F}_p(\delta+1/\delta)$*:*



Axiom 2 (Phase-inversion). *Given any* $a \in \mathbb{F}_p(\delta)$ *and nonzero* $b(\delta) \in \mathbb{F}_p(\delta+1/\delta)$*:*



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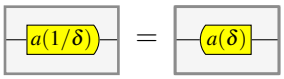
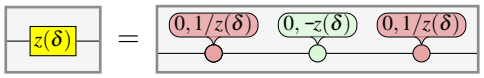
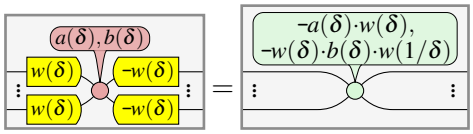
Axiom 3 (H-flip). *Given any* $a(\delta) \in \mathbb{F}_p(\delta)$*:*



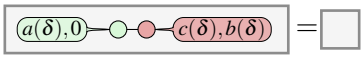
Axiom 4 (Euler-decomposition). *Given any* $h(\delta) \in \mathbb{F}_p(\delta+1/\delta)$*:*



Axiom 5 (Colour-change). *Given any* $a(\delta) \in \mathbb{F}_p(\delta)$*,* $b(\delta) \in \mathbb{F}_p(\delta+1/\delta)$ *and nonzero* $w(\delta) \in \mathbb{F}_p(\delta)$*:*



Axiom 6 (Scalar-elimination). *For all* $a(\delta),c(\delta) \in \mathbb{F}_p(\delta)$ *and* $b(\delta) \in \mathbb{F}_p(\delta+1/\delta)$*:*



All of the axioms can be easily verified to be sound.

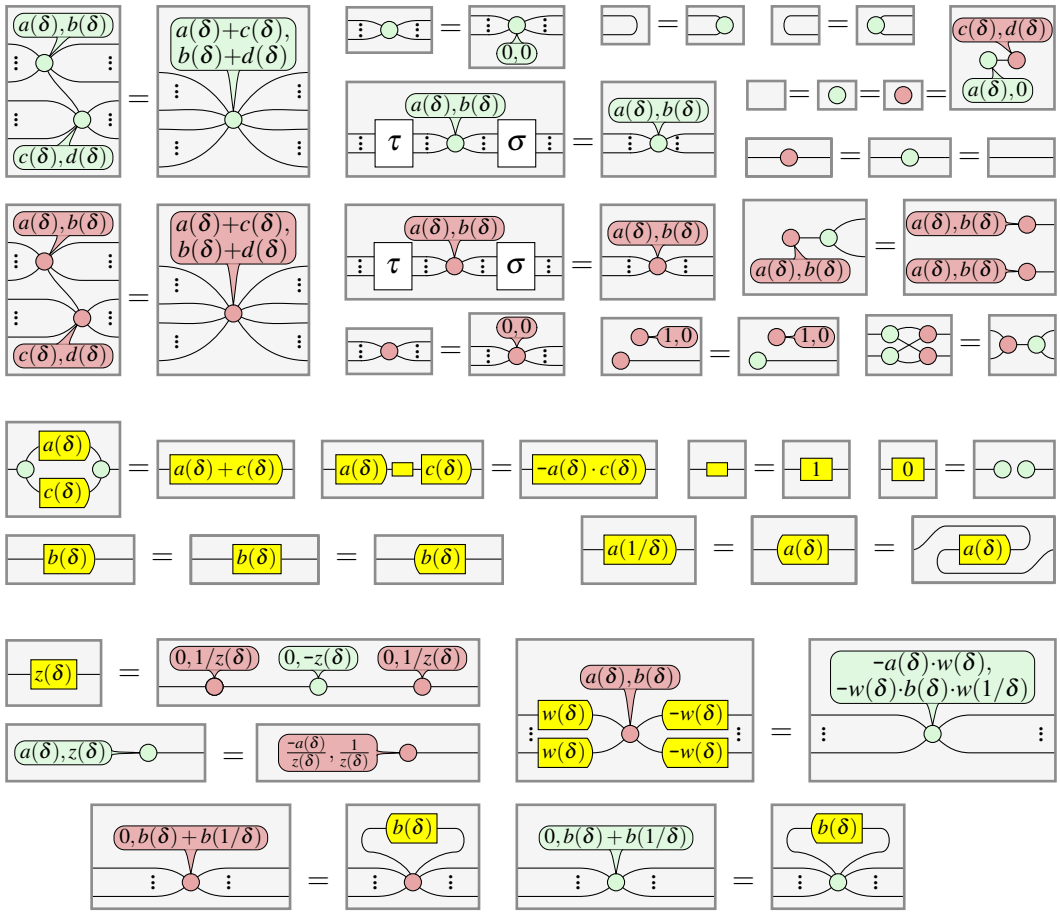
We reframe the presentation of δZX by taking the rational spiders and H-boxes as primitives, providing a self-contained axiomatisation in figure 2.

6.3 Well-definedness of notation, and equivalence of presentations

In this section, we prove some useful lemmas for δZX, showing that our derived notation in the previous subsection is well-defined, and that the additional axioms on top of $\mathrm{St}(\mathsf{ZX})$ given in the previous section, are equivalent to the equational theory in figure 2.

First, we show that the rational function multipliers are well-defined.

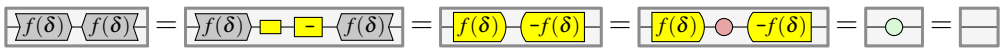
By axiom 5, and by definition of the H-box it follows that:



Parameter class	**Parameters**	**Conditions**
Permutations	σ, τ	—
Rational functions	$a(\delta), c(\delta), w(\delta) \in \mathbb{F}_p(\delta)$	$w(\delta) \neq 0$
Self-conjugate rational functions	$b(\delta), d(\delta), z(\delta) \in \mathbb{F}_p(\delta + 1/\delta)$	$z(\delta) \neq 0$

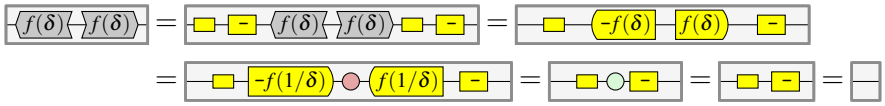
Figure 2: Axioms of the delayed stabiliser ZX-calculus.

Lemma 9. *For any nonzero $f(\delta) \in \mathbb{F}_p[\delta]$:*



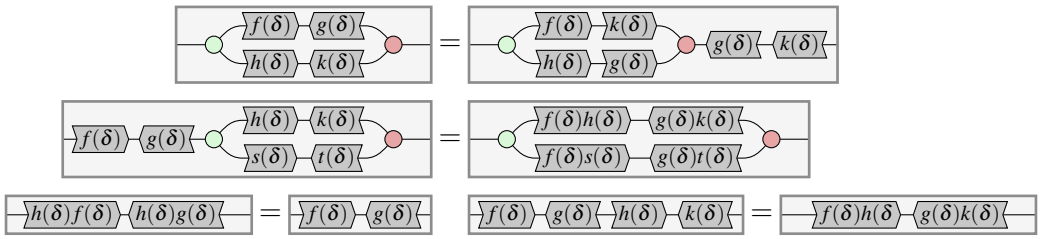
Moreover, by axiom 3:

Lemma 10. *For any nonzero $f(\delta) \in \mathbb{F}_p[\delta]$:*



Moreover, it follows from several simple induction arguments that rational arithmetic can be derived by polynomial multipliers and their duals:

Lemma 11. *For all $f(\delta), g(\delta), h(\delta), k(\delta), s(\delta), t(\delta) \in \mathbb{F}_p[\delta]$ with $g(\delta), k(\delta), t(\delta)$ nonzero:*



Therefore rational function multipliers are well-defined, and thus, so are directed H-boxes. By axiom 3 it follows that undirected H-boxes are also well-defined.

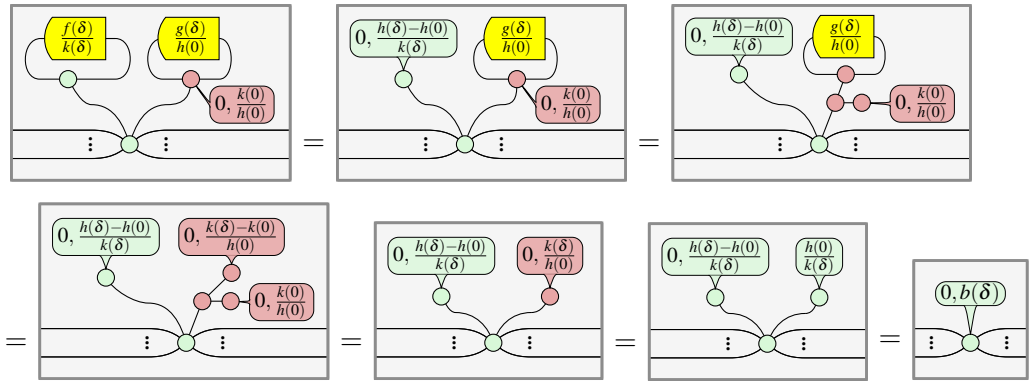
The well-definedness of rational multipliers also makes the rational affine phases well-defined.

Now we show that the self-conjugate rational symplectic phases are well-defined, this follows from the following equation, using axioms 1 and 2:

Lemma 12. *Take some $b(\delta) \in \mathbb{F}_p(\delta + 1/\delta)$.*

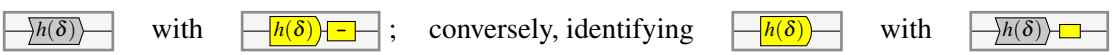
Choose $h(\delta), k(\delta) \in \mathbb{F}_p[\delta + 1/\delta]$ such that $b(\delta) = \frac{h(\delta)}{k(\delta)}$.

Choose $f(\delta), g(\delta) \in \mathbb{F}_p[\delta]$ such that $h(\delta) = f(\delta) + f(1/\delta) + h(0)$ and $k(\delta) = g(\delta) + g(1/\delta) + k(0)$.



Similarly, for the red spider.

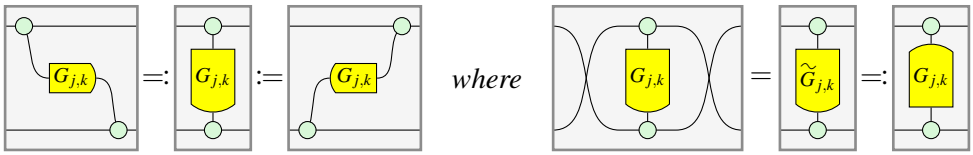
Using these equations it follows easily that both presentations are equivalent, where we identify



6.4 Scalable notation and universality

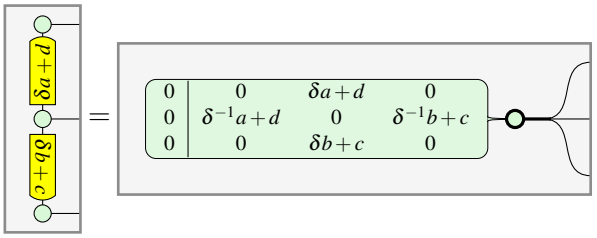
In this subsection, we introduce scalable notation for δZX, generalising the scalable notation for ZX given in section 2.4. We use the scalable notation to prove that δZX is universal for $\mathsf{sALR}_{\mathsf{fd}}$, conjecturing completeness.

Definition 34. *Given a shifted-symmetric matrix $G^\top = \widetilde{G} : n \to n$ over $\mathbb{F}_p(\delta)$ and a vector $\mathbf{a} \in \mathbb{F}_p(\delta)^n$, the **shifted graph state** associated to G is constructed by first constructing the δZX diagram $\bigotimes_{j=0}^{n-1}$* *. Then, for each $0 \le j < k < n$ connect the jth and kth wires with the following δZX diagrams:*

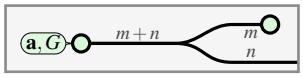


In the scalable notation the shifted graph state is denoted  *.*

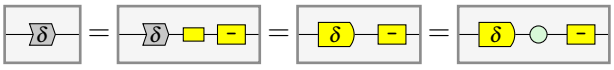
Example 14.



A δZX-diagram in in **graph-like** form when it can be decomposed into a graph-state with affine shift where some of the output wires are capped off by  s:



First, we already know that all of the non-delayed components can be put into graph-like form. Moreover, the delay can be rewritten to a directed Hadamard edge by adding an internal node:



Therefore:

Lemma 13. *All δZX-diagrams can be rewritten to graph-like form.*

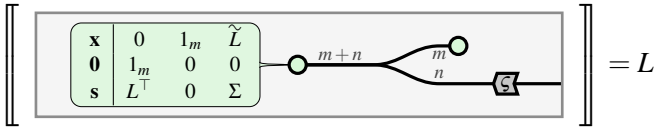
By replacing the transpose in the scalable notation for ZX with the with the transpose conjugate in the scalable notation for δZX, we can represent every shifted affine Lagrangian relation as a δZX diagram in graph-like form:

Theorem 10. *δZX is universal with respect to the interpretation into $\mathsf{sALR}_{\mathsf{fd}}$, meaning that all shifted Lagrangian relations between shifted symplectic vector spaces of the form $\mathbb{F}_p(\delta)^n$ can be represented by δZX diagrams.*

Proof. Because $\mathsf{sALR}_{\mathsf{fd}}$ and $\delta\mathsf{ZX}$ are compact closed, it suffices to prove the claim for states. Given any state in $\delta\mathsf{ZX}$, the corresponding shifted affine Lagrangian subspace has as a unique generating tableaux in canonical form:

$$L = \ker\left(\left[\begin{array}{cc|cc} I_m & \widetilde{L} & \Sigma & 0 \\ 0 & 0 & -L^\top & I_{n-m} \end{array}\right]\left[\begin{array}{c|c} \varsigma & 0 \\ \hline 0 & \varsigma \end{array}\right]\right) + \mathbf{a}$$

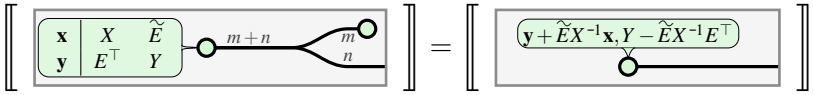
Taking $(\mathbf{x},\mathbf{s}) = \varsigma\mathbf{a}$, this corresponds to the following scalable $\delta\mathsf{ZX}$ diagram:



□

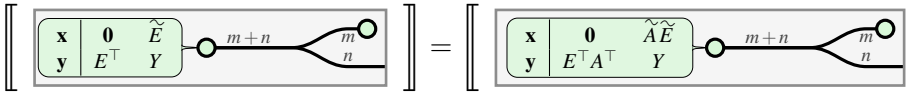
A generalised form of Schur-complementation and thus, local complementation and pivoting, for delayed graph-like diagrams can be easily be shown to be sound with respect to the interpretation in $\mathsf{sALR}_{\mathsf{fd}}$:

Lemma 14. *Given matrices $X : m \to m$, $Y : n \to n$, $E : n \to m$ over $\mathbb{F}_p$ and vectors $\mathbf{x} \in \mathbb{F}_p^m$, $\mathbf{y} \in \mathbb{F}_p^n$, such that $X^\top = \widetilde{X}$ and $Y^\top = \widetilde{Y}$ are shifted-symmetric and X is invertible:*



Similarly for the generalised anti-diagonal row reduction:

Lemma 15. *Take a matrix $E : n \to m$, a shifted-symmetric matrix $Y = \widetilde{Y} : n \to n$ and vectors $\mathbf{x} \in \mathbb{F}_p^m$, $\mathbf{y} \in \mathbb{F}_p^n$. Given any invertible matrix A:*



The equation given in lemma 15 can easily be shown to hold in $\delta\mathsf{ZX}$. We conjecture that the same is true for lemma 14. If this is true, then it follows immediately that:

Conjecture 1. *The interpretation $[\![-]\!] : \delta\mathsf{ZX} \to \mathsf{sALR}_{\mathsf{fd}}$ is a $\dagger$-CC equivalence, so that the delayed stabiliser ZX-calculus is sound, universal, and complete for shifted affine Lagrangian relations.*

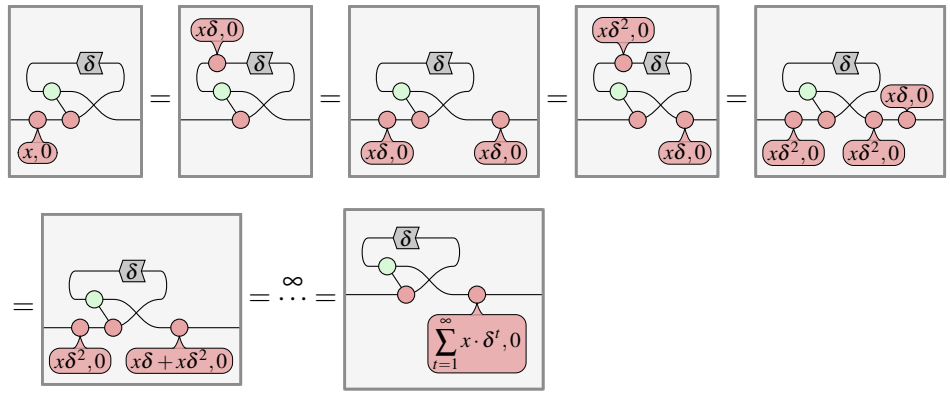
Proving that the generalised form of Schur complementation can be derived from the equational theory of $\delta\mathsf{ZX}$ should involve carefully following the proof of Booth et al. [7, Thm. 45], replacing the transpose with the conjugate transpose throughout.

6.5 Examples

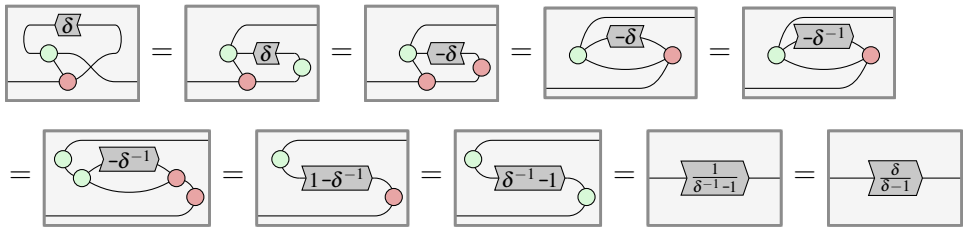
We end with examples demonstrating how the delayed stabiliser ZX-calculus can be used to reason about quantum convolutional codes and lattice codes.

First, we recall the delayed controlled-X gate from examples 10 and 13.

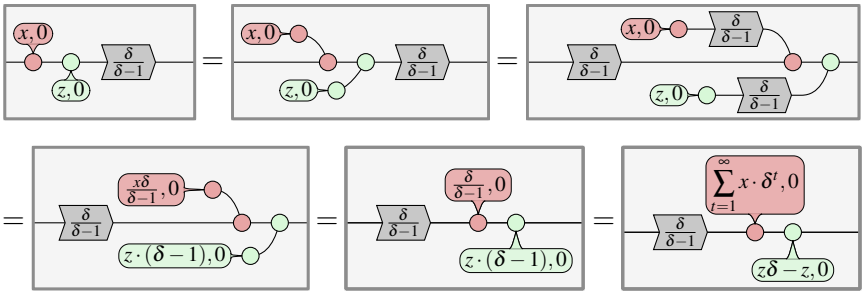
Example 15. *If we want to push the Pauli-X gate past the delayed controlled-X gate, we need infinite time, as it propagates infinitely far into the future:*



However, using the axioms of δZX, the delayed controlled-X gate can be simplified into a rational multiplier:

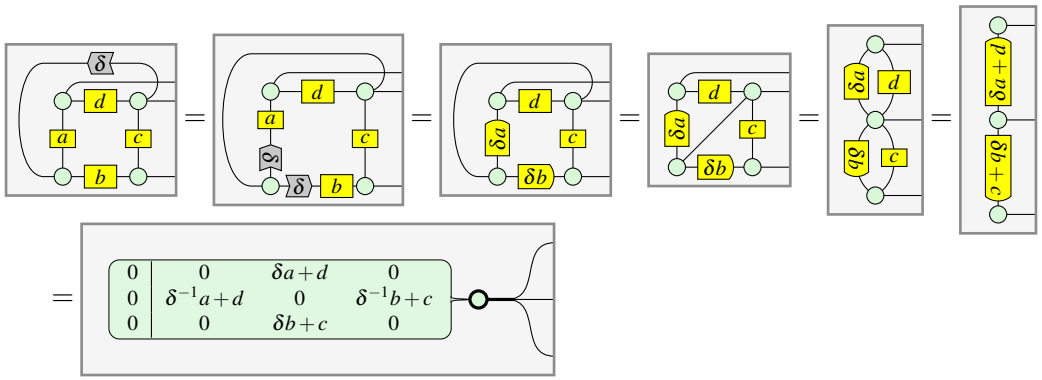


Which allows us to easily compute its stabilisers using only finite reasoning:

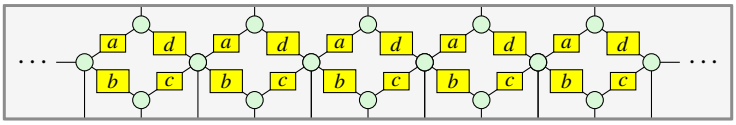


Next, we return to the delayed graph state from examples 9 and 12:

Example 16. *The tableaux of a delayed graph-state becomes explicit after reduction to the normal form:*

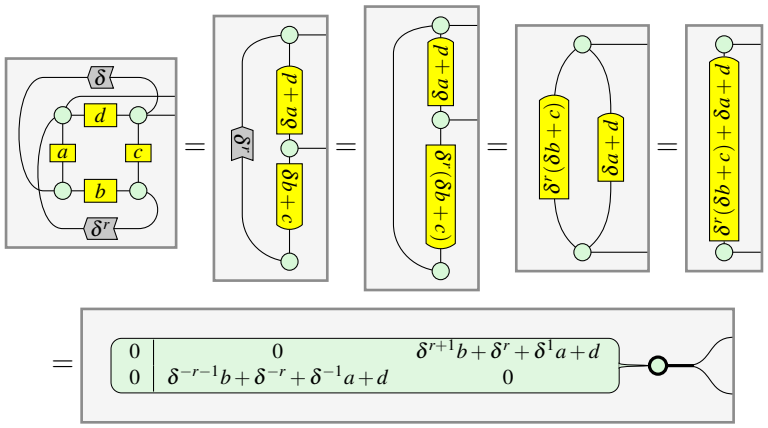


This unrolls to an infinite graph state built from tiling square plaquettes in a line:

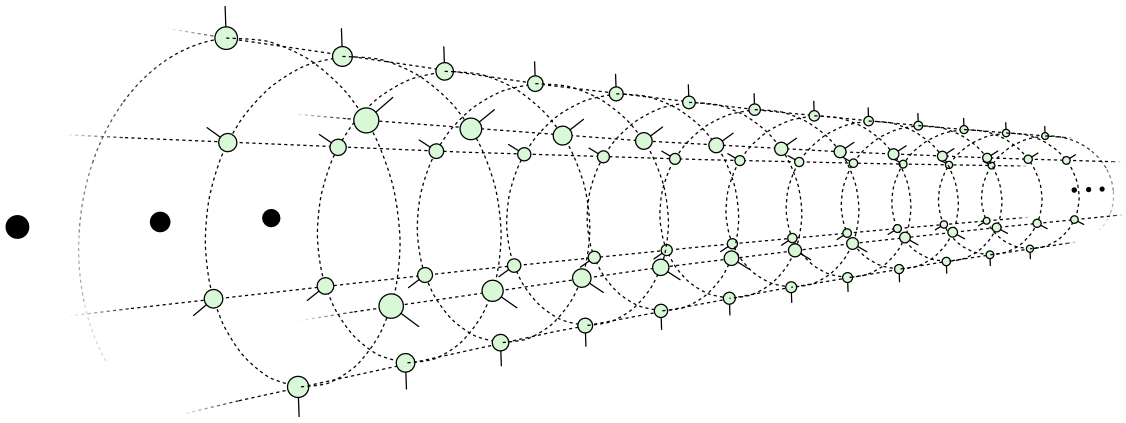


By delaying taking multiple delayed traces of a graph state, we obtain a surface code. For example, we can add another delayed edge to the previous example:

Example 17. *Consider the following delayed graph state for $r \in \mathbb{N}$:*



This unrolls to an infinite surface code with square plaquettes wrapped around a cylinder with "radius" r. For example given $r = 6$ and $a = b = c = d = 1$, this can be visualised as follows, where the dotted lines denote edges connected by an H-box:



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A Proofs of Section 2

Proposition 2. *Given an affine Lagrangian subspace $R \subseteq (\mathbb{F}_p^{2n}, \omega_n)$, there exist $m \in \mathbb{N}$, a matrix $L : n-m \to m$, a symmetric matrix $\Sigma^\top = \Sigma : n-m \to n-m$, a permutation $\varsigma : n \to n$, and $\mathbf{a} \in \mathbb{F}_p^{2n}$ such that,*

$$R = \ker\left(\left[\begin{array}{cc|cc} I_m & L & \Sigma & 0 \\ 0 & 0 & -L^\top & I_{n-m} \end{array}\right]\left[\begin{array}{c|c} \varsigma & 0 \\ \hline 0 & \varsigma \end{array}\right]\right) + \mathbf{a}.$$

*Call this presentation of R the **canonical form** of the tableau associated to R. Given any affine Lagrangian subspace, there is a unique canonical tableau. In case $m = 0$, say that R is the tableau associated to a **graph-state**.*

Proof of Proposition 2. Write $R = S + \mathbf{a}$, where $S \subseteq \mathbb{F}_p^{2n}$ is the linear part of R. Because R is affine Lagrangian, S is Lagrangian. Since ω_n is non-degenerate and $S = S^{\omega_n}$, we have $\dim(S) = 2n - \dim(S)$, hence $\dim(S) = n$. Therefore there is a full-rank matrix $H = \left[\begin{array}{c|c} X & Z \end{array}\right] \in \mathbb{F}_p^{n\times 2n}$. with $S = \ker(H)$. Applying elementary row-operations, we can reduce this matrix to the following form, while preserving the kernel, reordering the columns with a symplectic permutation matrix ς:

$$H \sim \left[\begin{array}{cc|cc} I_m & A & B & 0 \\ 0 & 0 & C & I_{n-m} \end{array}\right]\left[\begin{array}{c|c} \varsigma & 0 \\ \hline 0 & \varsigma \end{array}\right].$$

The Lagrangian condition on S forces $C = -A^\top$, and $B^\top = B$. Setting $\Sigma \coloneqq B$ and $L = A$:

$$S = \ker\left(\left[\begin{array}{cc|cc} I_m & L & \Sigma & 0 \\ 0 & 0 & -L^\top & I_{n-m} \end{array}\right]\left[\begin{array}{c|c} \varsigma & 0 \\ \hline 0 & \varsigma \end{array}\right]\right),$$

To prove uniqueness, suppose two canonical tableaux define the same linear subspace S, with data $(m, L, \Sigma, \varsigma)$ and $(m', L', \Sigma', \varsigma')$, and let the corresponding full-rank matrices be $H = [X \mid Z]$ and $H' = [X' \mid Z']$. Since $\ker(H) = \ker(H') = S$ and both have rank n, there is some $U \in \mathrm{GL}_n(\mathbb{F}_p)$ such that $H' = UH$. Hence $X' = UX$, so that $m' = \mathrm{rank}(X') = \mathrm{rank}(UX) = \mathrm{rank}(X) = m$, thus m is unique.

With m fixed, the canonical reduction has fixed pivot ordering, so L, Σ, ς are determined by S. The affine part is unique since R fixes the coset $\mathbf{a} + S$. □

B Proofs of Section 3

Proposition 5. Proj(CPM) *is a discard bicategory with respect to the trace and the* ***purification order****. Given projective completely positive maps* $\Phi, \Psi : \mathscr{H} \to \mathscr{K}$ *in* Proj(CPM)*,* $\Phi \preceq_{\mathsf{P}} \Psi$ *in case there exists some Hilbert space* $\mathscr{E}$ *and maps* $\Psi_0 : \mathscr{H} \to \mathscr{K} \otimes \mathscr{E}$ *and* $\Phi_0 : \mathscr{E} \to \mathbb{C}$ *in* Proj(CPM) *such that* $\Psi = \mathrm{Tr}_{\mathscr{E}}(\Psi_0)$ *and* $\Phi = (1_{\mathscr{K}} \otimes \Phi_0) \circ \Psi_0$*.*

Moreover, Proj(CPM) *is a discard bicategory with respect to the trace.*

Proof of Proposition 5. By Carette et al. the purification order is a preorder enrichment on the full subcategory of CPM whose objects are of the form $\in^n$, for all $n \in \mathbb{N}$ [10, Lem. 6]. However, their argument holds for all finite dimensional Hilbert spaces so that the purification order is a preorder enrichment on CPM. The quotient by scalars makes the purification order a poset-enrichment on Proj(CPM).

Next, we show that Proj(CPM) is a discard bicategory with respect to the trace. Take any effect $\Xi : \mathscr{H} \to \mathbb{C}$ in Proj(CPM). Let $Phi_0 = \Xi$ and $Psi_0 = 1_{\mathscr{H}}$, then $Tr_{\mathscr{H}}(\Psi_0) = Tr_{\mathscr{H}}$ and $(1_{\mathscr{C}} \otimes \Phi_0) \circ \Psi_0 = \Xi$, and it follows that $\Xi \preceq_{\mathsf{P}} \mathrm{Tr}_{\mathscr{H}}$. □

Corollary 1. Proj($\mathsf{Stab}^{\dagger\otimes}$) *is a discard bicategory with respect to the purification order and the trace, and the canonical functor* Proj($\mathsf{Stab}^{\dagger\otimes}$) $\to$ Proj(CPM) *preserves the order.*

Proof of Corollary 1. Since any purification in Proj($\mathsf{Stab}^{\dagger\otimes}$) is a purification in Proj(CPM) the canonical functor Proj($\mathsf{Stab}^{\dagger\otimes}$) $\to$ Proj(CPM) preserves the order.

By forward reference to theorem 5, we know that the purification order on Proj($\mathsf{Stab}^{\dagger\otimes}$) is actually a poset-enrichment. □

Theorem 4. $[\Phi] \preceq_{\mathsf{P}} [\Psi]$ *in* Proj(CPM) *if and only if there is some* $\lambda \in \mathbb{R}^+$ *such that* $\Phi \preceq_{\mathsf{L}} \lambda\Psi$ *in* CPM*.*

Proof of Theorem 4. Let $[\Psi]$ denote an equivalence class in Proj(CPM) with chosen representative Ψ in CPM. Suppose that $[\Phi] \preceq_{\mathsf{P}} [\Psi]$, then there exists $[\Phi_0] : E \to I$ and $[\Psi_0] : X \to Y \otimes E$ such that $[\Phi] = (1 \otimes [\Phi_0]) \circ [\Psi_0]$ and $[\Psi] = \mathrm{Tr}_E([\Psi_0])$. Then there exists scalars $a, b \in \mathbb{R}^+$ such that in CPM:

$$\Phi = a(1 \otimes \Phi_0) \circ \Psi_0 \quad \Psi = b\,\mathrm{Tr}_E(\Psi_0)$$

Now we normalize the effect Φ_0. Let $c = \frac{1}{\|\Phi_0\|}$ then $c\Phi_0$ is a completely positive trace non-increasing map and we have $\Phi = \frac{a}{c}(1 \otimes c\Phi_0) \circ \Psi_0$. Then let $\lambda = \frac{a}{bc}$:

$$\begin{aligned} \lambda\Psi - \Phi &= \frac{ba}{bc}\mathrm{Tr}_E(\Psi_0) - \frac{a}{c}(1 \otimes c\Phi_0) \circ \Psi_0 \\ &= \frac{a}{c}(1 \otimes (\mathrm{Tr}_E - c\Phi_0)) \circ \Psi_0 \end{aligned}$$

Since $c\Phi_0$ is trace non-increasing, $f_0 = (\mathrm{Tr}_E - c\Phi_0)$ is a completely positive map. Then since $\frac{a}{c} \geq 0$ and Ψ_0 is completely positive, $\lambda\Psi - \Phi$ is also completely positive. Therefore $\Phi \preceq_{\mathsf{L}} \lambda\Psi$.

Now for the second implication, suppose $\Phi \preceq_{\mathsf{L}} \lambda\Psi$ in CPM. Then $\Xi = \lambda\Psi - \Phi$ is a completely positive map. Consider the qubit space $\mathbb{C}^2$ with basis $|0\rangle, |1\rangle$ and define $\Psi_0 : \mathscr{H} \to \mathscr{K} \otimes \mathbb{C}^2$ and $\Phi_0 : mathbbC^2 \to \mathbb{C}$ by:

$$\Psi_0 = \Phi \otimes |0\rangle\langle 0| + \Xi \otimes |1\rangle\langle 1| \qquad \Phi_0(\rho) = \langle 0|\rho|0\rangle$$

for any qubit state ρ. Then we have $\mathrm{Tr}_{\mathbb{C}^2}([\Psi_0]) = [\mathrm{Tr}_{\mathbb{C}^2}(\Psi_0)] = [\Phi + \Xi] = [\lambda\Psi] = [\Psi]$ and $(1 \otimes [\Phi_0]) \circ [\Psi_0] = [(1 \otimes \Phi_0) \circ \Psi_0] = [\Phi]$ from which it follows immediately that $[\Phi] \preceq_{\mathsf{P}} [\Psi]$ in Proj(CPM). □

Theorem 5. *There is an equivalence of †-CC discard bicategories* $\mathsf{Proj}(\mathsf{Stab}^{\shortmid\!\shortmid\!\shortmid}) \cong \mathsf{ALR}_{\mathsf{fd}}^{\shortmid\!\shortmid\!\shortmid}$.

Proof of Theorem 5. First, by Booth and Comfort, there is a †-CC equivalence $F : \mathsf{Proj}(\mathsf{Stab}^{\shortmid\!\shortmid\!\shortmid}) \cong \mathsf{ALR}_{\mathsf{fd}}^{\shortmid\!\shortmid\!\shortmid}$ extending the †-CC equivalence $\mathsf{Proj}(\mathsf{Stab}) \simeq \mathsf{ALR}_{\mathsf{fd}}$, sending the evaluation map $\mathrm{ev}_{\mathscr{H}_p^{\otimes e}}$ in $\mathsf{Proj}(\mathsf{Stab}^{\shortmid\!\shortmid\!\shortmid})$ to the relation $\mathtt{disc}_{(\mathbb{F}_p^{2n},\omega_e)}$ in $\mathsf{ALR}_{\mathsf{fd}}^{\shortmid\!\shortmid\!\shortmid}$. It is immediate that the discard is preserved; therefore, it remains to prove that the poset-enrichment is preserved and reflected. Because both categories are compact closed, it suffices to prove the claim on effects $X \to I$.

First we prove the preservation of the poset-enrichment. Take two effects $\Phi, \Psi : \mathscr{H}_p^{\otimes n} \to \mathbb{C}$ in $\mathsf{Proj}(\mathsf{Stab}^{\mathtt{disc}})$, such that $\Phi \preceq_{\mathsf{P}} \Psi$ is witnessed by $\Psi_0 : \mathscr{H}_p^{\otimes n} \to \mathscr{H}_p^{\otimes e}$ and $\Phi_0 : \mathscr{H}_p^{\otimes e} \to \mathbb{C}$. Then

$$F(\Phi) = F(\Phi_0) \circ F(\Psi_0) \subseteq \mathtt{disc}_{(\mathbb{F}_p^{2e},\omega_e)} \circ F(\Psi_0) = F(\mathrm{ev}_{\mathscr{H}_p^{\otimes e}}) \circ F(\Psi_0) = F(\mathrm{ev}_{\mathscr{H}_p^{\otimes e}} \circ \Psi_0) = F(\Psi)$$

Next, we show that the poset-enrichment is reflected. Take two effects $R, S : (\mathbb{F}_p^{2n}, \omega_n) \to \{\bullet\}$ in $\mathsf{ALR}_{\mathsf{fd}}^{\shortmid\!\shortmid\!\shortmid}$ such that $R \subseteq S$. By lemma 6 there exists some $e \in \mathbb{N}$ and coisometry $V : (\mathbb{F}_p^{2n}, \omega_n) \to (\mathbb{F}_p^{2e}, \omega_e)$ such that $S = \mathtt{disc}_{(\mathbb{F}_p^{2e},\omega_e)} \circ V$.

Let G denote the inverse of F, when restricted to the objects of the form $(\mathbb{F}_p^{2n}, \omega_n)$ in $\mathsf{ALR}_{\mathsf{fd}}^{\dagger}$, for all $n \in \mathbb{N}$. Take $\Phi_0 = G(R \circ V^\dagger)$ and $\Psi_0 = G(V)$. On the one hand,

$$\mathrm{ev}_{\mathscr{H}_p^{\otimes e}} \circ \Psi_0 = G(\mathtt{disc}_{(\mathbb{F}_p^{2e},\omega_e)}) \circ G(V) = G(\mathtt{disc}_{(\mathbb{F}_p^{2e},\omega_e)} \circ V) = G(S)$$

On the other hand, because $R \subseteq S$ we have $(V^\dagger \circ V) \circ R = \{(\mathbf{v}, \mathbf{v}) \mid \forall (\mathbf{v}, \bullet) \in S\} \circ R = R$, therefore,

$$\Phi_0 \circ \Psi_0 = G(R \circ V^\dagger) \circ G(V) = G(R \circ V^\dagger \circ V) = G(R)$$

□

Theorem 6. *There is an equivalence of †-CC discard bicategories* $\mathsf{ZX}^{\shortmid\!\shortmid\!\shortmid} \simeq \mathsf{ALR}_{\mathsf{fd}}^{\shortmid\!\shortmid\!\shortmid} \simeq \mathsf{Proj}(\mathsf{Stab}^{\shortmid\!\shortmid\!\shortmid})$.

Proof of Theorem 6. The fact that this is a †-CC discard equivalence follows from [7, Cor. 54]. To show that this equivalence preserves and reflects the preorder, first observe that the linear Lagrangian relations is poset enriched with respect to the discrete preorder, as all linear Lagrangian relations of the same type have the same dimension. For affine Lagrangian relations, we have almost the same story, except that all relations contain the empty set. Because affine coisotropic relations can be obtained by composing affine Lagrangian relations with the discard relation $\shortmid\!\shortmid\!\shortmid_E$, the poset-enrichment with respect to subspace inclusion is determined by the poset $\mathsf{ACR}_{\mathsf{fd}}(E, I)$, for which the discard relation is the top element. □

C Proofs of Section 4

Theorem 7. *The data in definition 22 makes* $\mathrm{Obs}(\mathscr{C})$ *a discard bicategory.*

Proof of Theorem 7. Write

$$J \coloneqq \{(\ell, r) \in \mathbb{Z}^2 \mid \ell \leq r\}, \quad X_{\ell,r} \coloneqq \bigotimes_{k=\ell}^{r} X_k$$

(and similarly $Y_{\ell,r}, Z_{\ell,r}$). Indices are pairs $(\ell, r) \in J$.

(1) Structural maps. Identities, associators, unitors, symmetry, and discards are defined componentwise from $\mathscr{C}$, hence are monotone sequences.

(2) Approximation is a preorder. Reflexivity is immediate ($L = R = 0$). For transitivity, assume $\mathbf{U} \subseteq \mathbf{V} \subseteq \mathbf{W}$, and fix $(\ell, r) \in J$. Choose (L_0, R_0) witnessing $\mathbf{V} \subseteq \mathbf{W}$ at (ℓ, r), and (L_1, R_1) witnessing $\mathbf{U} \subseteq \mathbf{V}$ at $(\ell - L_0, r + R_0)$. Composing witnesses gives $(L_0 + L_1, R_0 + R_1)$, so $\mathbf{U} \subseteq \mathbf{W}$.

(3) Composition and tensor on classes. Pointwise composition and pointwise tensor of monotone sequences are monotone. Approximation is stable under whiskering:

$$\mathbf{U} \subseteq \mathbf{U}' \Rightarrow \mathbf{V} \circ \mathbf{U} \subseteq \mathbf{V} \circ \mathbf{U}', \qquad \mathbf{V} \subseteq \mathbf{V}' \Rightarrow \mathbf{V} \circ \mathbf{U} \subseteq \mathbf{V}' \circ \mathbf{U},$$

and similarly for $\otimes$. Hence composition and tensor descend to observational equivalence classes.

(4) Poset enrichment and discard bicategory axioms. Quotienting by mutual approximation gives the partial order

$$[\mathbf{V}] \subseteq [\mathbf{U}] \iff \mathbf{V} \subseteq \mathbf{U}.$$

By (3), composition and tensor are monotone for this order. All category/symmetric-monoidal/discard axioms hold pointwise, since they hold in $\mathscr{C}$.

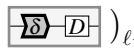
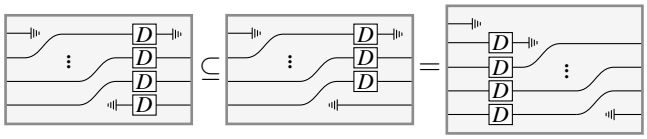
Therefore $\mathrm{Obs}(\mathscr{C})$ is a discard bicategory. □

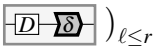
Proposition 7. *The unrolling of a stateful ZX-diagram is a †-CC functor* $\texttt{Unroll} : \mathrm{St}(\mathsf{ZX}) \to \mathrm{Obs}(\mathsf{ZX}^{\oplus})$.

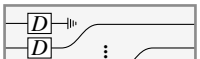
Proof of Proposition 7. First observe that the action of `Unroll` on the generators of $\mathrm{St}(\mathsf{ZX})$ produces monotone sequences, therefore `Unroll` produces monotone sequences.

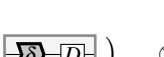
It remains to that unrolling modulo observational equivalence preserves the equations of $\mathrm{St}(\mathsf{ZX})$. For the equational theory of ZX, this holds pointwise, except for the zero rule. However, if one stage of the unrolling is zero, then by looking ahead far enough and recalling that zero is the bottom in the poset enrichment, it is equivalent to a monotone sequence where every stage is zero.

Next, we prove that the sliding rule is preserved. Take any ZX-diagram D consisting only of spiders with trivial affine phases. Then

$(\texttt{disc} \otimes 1^{\otimes(r-\ell+1)}) \circ \texttt{Unroll}\big($  $\big)_{\ell-1\le r} =$ 

$= \texttt{disc} \otimes \texttt{Unroll}\big($  $\big)_{\ell\le r}$

$(1^{\otimes(r-\ell+1)} \otimes \texttt{disc}) \circ \texttt{Unroll}\big($  $\big)_{\ell\le r+1} =$

$= \texttt{Unroll}\big($  $\big)_{\ell\le r} \otimes \texttt{disc}$

Because both sequences approximate each other, they are observationally equivalent, so that the sliding rule is preserved. The preservation of all of the other structure holds pointwise. □

Proposition 11 (Functoriality of $\mathtt{Lim}$)**.** *There is a †-CC discard functor* $\mathtt{Lim} : \mathsf{Obs}(\mathsf{ALR}^{\mathsf{ill}}_{\mathsf{fd}}) \to \mathsf{AR}_\infty$ *given on morphisms by*

$$\mathtt{Lim}([\mathbf{R}]) \coloneqq \left\{ (\mathbf{x},\mathbf{y}) \in \prod_{k\in\mathbb{Z}} X_k \times \prod_{k\in\mathbb{Z}} Y_k \;\middle|\; \forall \ell \le r \in \mathbb{Z} :\ \big((x_\ell,\dots,x_r),(y_\ell,\dots,y_r)\big) \in R_{\ell,r} \right\}.$$

Proof. Write $X_{\ell,r} \coloneqq \prod_{k=\ell}^{r} X_k$ (and similarly for $Y_{\ell,r}, Z_{\ell,r}$). The index set is

$$\{\,(\ell,r) \in \mathbb{Z}^2 \mid \ell \le r\,\},$$

and each index (ℓ,r) corresponds to the finite interval $[\ell,r] \subseteq \mathbb{Z}$. When we use inclusion, we use interval inclusion $[\ell,r] \subseteq [\ell',r']$, and let $\pi_{\ell,r}^{\ell',r'} : X_{\ell',r'} \to X_{\ell,r}$, $\pi_{\ell,r}^{\infty} : \prod_{k\in\mathbb{Z}} X_k \to X_{\ell,r}$ be the evident projections.

Define $\mathtt{Lim}$ on objects by

$$\mathbf{X} = (X_k)_{k\in\mathbb{Z}} \mapsto \prod_{k\in\mathbb{Z}} X_k,$$

and on monotone sequences $\mathbf{R} : \mathbf{X} \to \mathbf{Y}$ by

$$\mathtt{Lim}(\mathbf{R}) = \bigcap_{\ell\le r} (\pi_{\ell,r}^{\infty})^{-1}(R_{\ell,r}) \subseteq \Big(\prod_{k\in\mathbb{Z}} X_k\Big) \times \Big(\prod_{k\in\mathbb{Z}} Y_k\Big).$$

To show this is a valid functor we need it to preserve identities, discards, monoidal unit, the poset structure on monotone sequences, composition, and monoidal product.

Preservation of monoidal unit and discards $\mathtt{Lim}$ strictly preserves the monoidal unit since $\prod_{k\in\mathbb{Z}}\{*\} \cong \{*\}$. Also

$$\mathtt{Lim}(\mathtt{disc}_{\mathbf{X}}) = \bigcap_{\ell\le r} (\pi_{\ell,r}^{\infty})^{-1}(X_{\ell,r}) = \prod_{k\in\mathbb{Z}} X_k.$$

So $\mathtt{Lim}$ preserves discards.

Preservation of identities We show $1_{\prod_{k\in\mathbb{Z}} X_k} = \mathtt{Lim}(1_{\mathbf{X}})$. First, for each (ℓ,r), $\pi_{\ell,r}^{\infty}(1_{\prod X_k}) \subseteq 1_{X_{\ell,r}}$, hence

$$1_{\prod X_k} \subseteq \bigcap_{\ell\le r} (\pi_{\ell,r}^{\infty})^{-1}(1_{X_{\ell,r}}) = \mathtt{Lim}(1_{\mathbf{X}}).$$

Conversely, if $(x,y) \in \mathtt{Lim}(1_{\mathbf{X}})$, then $\pi_{\ell,r}^{\infty}(x) = \pi_{\ell,r}^{\infty}(y)$ for all $\ell \le r$. For each $k \in \mathbb{Z}$, taking $(\ell,r) = (k,k)$ gives $x_k = y_k$, so $x = y$. Hence $\mathtt{Lim}(1_{\mathbf{X}}) \subseteq 1_{\prod X_k}$.

Preservation of poset and well-definedness on equivalence classes Suppose $\mathbf{R} \subseteq \mathbf{S}$. For each pair (ℓ,r), choose (ℓ',r') with $[\ell,r] \subseteq [\ell',r']$ witnessing approximation:

$$\pi_{\ell,r}^{\ell',r'} R_{\ell',r'} \subseteq S_{\ell,r}.$$

Since $\pi_{\ell,r}^{\infty} = \pi_{\ell,r}^{\ell',r'} \circ \pi_{\ell',r'}^{\infty}$,

$$(\pi_{\ell',r'}^{\infty})^{-1}(R_{\ell',r'}) \subseteq (\pi_{\ell,r}^{\infty})^{-1}(S_{\ell,r}).$$

Intersecting over all (ℓ,r) yields $\mathtt{Lim}(\mathbf{R}) \subseteq \mathtt{Lim}(\mathbf{S})$. Therefore $\mathtt{Lim}$ preserves the poset and is well-defined on observational classes.

Since each X_k and Y_k is a finite-dimensional vector space over $\mathbb{F}_p$, they are compact Hausdorff spaces (with the discrete topology). Equip $\prod_{k\in\mathbb{Z}} X_k$ and $\prod_{k\in\mathbb{Z}} Y_k$ with the product topology; by Tychonoff's theorem these products are compact Hausdorff, and all coordinate projections are continuous. Each $R_{\ell,r} \subseteq X_{\ell,r} \times Y_{\ell,r}$ is affine; inverse images along coordinate projections are affine; intersections of affine subsets are affine (possibly empty). So $\mathtt{Lim}(\mathbf{R}) \in \mathsf{AR}_\infty$.

Preservation of monoidal product For composable object families $\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W}$, and $\mathbf{R}:\mathbf{X}\to\mathbf{Y}$, $\mathbf{S}:\mathbf{Z}\to\mathbf{W}$,

$$(\mathbf{R}\otimes\mathbf{S})_{\ell,r}=R_{\ell,r}\oplus S_{\ell,r},$$

since in $\mathsf{Obs}(\mathsf{ALR}^{\Vdash}_{\mathsf{fd}})$ the tensor $\otimes$ is defined componentwise from $\mathsf{ALR}^{\Vdash}_{\mathsf{fd}}$, whose monoidal product is $\oplus$.

$$\mathtt{Lim}(\mathbf{R}\otimes\mathbf{S})=\bigcap_{\ell\le r}\Big((\pi^{\infty,X}_{\ell,r})^{-1}(R_{\ell,r})\oplus(\pi^{\infty,Z}_{\ell,r})^{-1}(S_{\ell,r})\Big)=\mathtt{Lim}(\mathbf{R})\oplus\mathtt{Lim}(\mathbf{S}).$$

Hence `Lim` preserves tensor.

Preservation of composition: laxness Let $\mathbf{R}:\mathbf{X}\to\mathbf{Y}$, $\mathbf{S}:\mathbf{Y}\to\mathbf{Z}$. If $(x,z)\in\mathtt{Lim}(\mathbf{S})\circ\mathtt{Lim}(\mathbf{R})$, choose y with $(x,y)\in\mathtt{Lim}(\mathbf{R})$, $(y,z)\in\mathtt{Lim}(\mathbf{S})$. Then for every $\ell\le r$, $(\pi^{\infty}_{\ell,r}x,\pi^{\infty}_{\ell,r}y)\in R_{\ell,r}$ and $(\pi^{\infty}_{\ell,r}y,\pi^{\infty}_{\ell,r}z)\in S_{\ell,r}$, so $(\pi^{\infty}_{\ell,r}x,\pi^{\infty}_{\ell,r}z)\in(S\circ R)_{\ell,r}$. Thus $(x,z)\in\mathtt{Lim}(\mathbf{S}\circ\mathbf{R})$, giving

$$\mathtt{Lim}(\mathbf{S})\circ\mathtt{Lim}(\mathbf{R})\subseteq\mathtt{Lim}(\mathbf{S}\circ\mathbf{R}).$$

Preservation of composition: oplaxness Now suppose $(x,z)\in\mathtt{Lim}(\mathbf{S}\circ\mathbf{R})$. For each $\ell\le r$, define

$$W_{\ell,r}:=\{\,y_{\ell,r}\in Y_{\ell,r}\mid(\pi^{\infty}_{\ell,r}x,y_{\ell,r})\in R_{\ell,r},\ (y_{\ell,r},\pi^{\infty}_{\ell,r}z)\in S_{\ell,r}\,\}.$$

Each $W_{\ell,r}$ is nonempty, and since $Y_{\ell,r}$ is finite (hence discrete), $W_{\ell,r}$ is closed. For $[\ell,r]\subseteq[\ell',r']$, monotonicity implies $\pi^{\ell',r'}_{\ell,r}(W_{\ell',r'})\subseteq W_{\ell,r}$, hence

$$(\pi^{\ell',r'}_{\ell,r})^{-1}(W_{\ell,r})\supseteq W_{\ell',r'}.$$

Define

$$C_{\ell,r}:=(\pi^{\infty}_{\ell,r})^{-1}(W_{\ell,r})\subseteq\prod_{k\in\mathbb{Z}}Y_k.$$

Then each $C_{\ell,r}$ is nonempty and closed, and $[\ell,r]\subseteq[\ell',r']$ implies $C_{\ell',r'}\subseteq C_{\ell,r}$. Given finitely many pairs (ℓ_i,r_i), take $(\ell^*,r^*)=(\min_i\ell_i,\max_i r_i)$. Then $C_{\ell^*,r^*}\subseteq\bigcap_i C_{\ell_i,r_i}$, so the family $\{C_{\ell,r}\}_{\ell\le r}$ has the finite intersection property. Since $\prod_{k\in\mathbb{Z}}Y_k$ is compact in the product topology, $\bigcap_{\ell\le r}C_{\ell,r}\neq\varnothing$. Choose y in this intersection. Then for every $\ell\le r$, $(\pi^{\infty}_{\ell,r}x,\pi^{\infty}_{\ell,r}y)\in R_{\ell,r}$ and $(\pi^{\infty}_{\ell,r}y,\pi^{\infty}_{\ell,r}z)\in S_{\ell,r}$, so $(x,y)\in\mathtt{Lim}(\mathbf{R})$, $(y,z)\in\mathtt{Lim}(\mathbf{S})$, and therefore

$$\mathtt{Lim}(\mathbf{S}\circ\mathbf{R})\subseteq\mathtt{Lim}(\mathbf{S})\circ\mathtt{Lim}(\mathbf{R}).$$

So `Lim` preserves composition and tensor, and preserves identity, discard and monoidal unit. Dagger, cups and caps are componentwise, so `Lim` preserves the †-CC structure. □

Lemma 16 (Stabilisation)**.** *Every monotone sequence* $\mathbf{R}:\mathbf{X}\to\mathbf{Y}$ *in* $\mathsf{ALR}^{\Vdash}_{\mathsf{fd}}$ *is observationally equivalent to a unique causal sequence* $\mathrm{Fix}(\mathbf{R})$.

Proof. Fix (ℓ,r) with $\ell\le r$. For each (ℓ',r') with $[\ell,r]\subseteq[\ell',r']$, set

$$B^{\ell',r'}_{\ell,r}:=(\mathtt{disc}_{Y_{\ell',\ell-1}}\oplus 1_{Y_{\ell,r}}\oplus\mathtt{disc}_{Y_{r+1,r'}})\circ R_{\ell',r'}\circ(\mathtt{codisc}_{X_{\ell',\ell-1}}\oplus 1_{X_{\ell,r}}\oplus\mathtt{codisc}_{X_{r+1,r'}}).$$

By monotonicity, enlarging the interval $[\ell',r']$ can only refine this morphism. Since all hom-posets $\mathsf{ALR}^{\Vdash}_{\mathsf{fd}}(X_{\ell,r},Y_{\ell,r})$ are Artinian, the family $\{B^{\ell',r'}_{\ell,r}\}_{[\ell,r]\subseteq[\ell',r']}$ stabilises at some pair $\rho(\ell,r)=(\rho_-(\ell,r),\rho_+(\ell,r))$ with $\rho_-(\ell,r)\le\ell\le r\le\rho_+(\ell,r)$. Define

$$\mathrm{Fix}(\mathbf{R})_{\ell,r}:=B^{\rho_-(\ell,r),\rho_+(\ell,r)}_{\ell,r}.$$

We show that $\mathrm{Fix}(\mathbf{R})$ is causal. Let $[\ell,r] \subseteq [a,b]$. Then

$$(\mathtt{disc} \oplus 1 \oplus \mathtt{disc}) \circ \mathrm{Fix}(\mathbf{R})_{a,b} \circ (\mathtt{codisc} \oplus 1 \oplus \mathtt{codisc}) = B_{\ell,r}^{\rho_-(a,b),\rho_+(a,b)}.$$

Since $\rho_-(a,b) \le a \le \ell \le r \le b \le \rho_+(a,b)$, and the family for (ℓ,r) stabilised at $\rho(\ell,r)$, we get $B_{\ell,r}^{\rho_-(a,b),\rho_+(a,b)} = B_{\ell,r}^{\rho_-(\ell,r),\rho_+(\ell,r)} = \mathrm{Fix}(\mathbf{R})_{\ell,r}$. So $\mathrm{Fix}(\mathbf{R})$ is causal.

We show that $\mathrm{Fix}(\mathbf{R})$ is observationally equivalent to $\mathbf{R}$. For each (ℓ,r), monotonicity at $[\ell,r] \subseteq [\rho_-(\ell,r),\rho_+(\ell,r)]$ gives

$$(\mathtt{disc} \oplus 1 \oplus \mathtt{disc}) \circ R_{\rho_-(\ell,r),\rho_+(\ell,r)} \subseteq R_{\ell,r} \oplus \mathtt{disc}.$$

Precomposing the corresponding codiscard factors and using $\mathtt{disc} \circ \mathtt{codisc} \subseteq 1_I$ yields $\mathrm{Fix}(\mathbf{R})_{\ell,r} \subseteq R_{\ell,r}$, hence $\mathrm{Fix}(\mathbf{R}) \subseteq \mathbf{R}$.

For the converse, using $1 \subseteq \mathtt{codisc} \circ \mathtt{disc}$ on the extra input factors gives

$$(\mathtt{disc} \oplus 1 \oplus \mathtt{disc}) \circ R_{\rho_-(\ell,r),\rho_+(\ell,r)} \subseteq \mathrm{Fix}(\mathbf{R})_{\ell,r} \oplus \mathtt{disc},$$

so $\mathbf{R} \subseteq \mathrm{Fix}(\mathbf{R})$. Thus $\mathbf{R}$ and $\mathrm{Fix}(\mathbf{R})$ are observationally equivalent.

We show uniqueness. Let $\mathbf{S}$ be a causal sequence observationally equivalent to $\mathbf{R}$. Then $\mathbf{S}$ is observationally equivalent to $\mathrm{Fix}(\mathbf{R})$. Since $\mathbf{S}$ and $\mathrm{Fix}(\mathbf{R})$ are causal, approximation implies pointwise comparison, hence $\mathbf{S} = \mathrm{Fix}(\mathbf{R})$ componentwise. □

Lemma 17 (Projection of Causal Limits)**.** *Let* $\mathbf{R} : \mathbf{X} \to \mathbf{Y}$ *be a causal sequence in* $\mathrm{Obs}(\mathsf{ALR}_{\mathsf{fd}}^{\mathbb{N}})$*. For every* $\ell \le r$*,*

$$\pi_{\ell,r}^{\infty}\big(\mathtt{Lim}(\mathbf{R})\big) = R_{\ell,r}.$$

Proof. The inclusion $\pi_{\ell,r}^{\infty}(\mathtt{Lim}(\mathbf{R})) \subseteq R_{\ell,r}$ is immediate since $\mathtt{Lim}(\mathbf{R}) \subseteq (\pi_{\ell,r}^{\infty})^{-1}(R_{\ell,r})$.

For the reverse inclusion, fix (ℓ_0,r_0) with $\ell_0 \le r_0$. If $R_{\ell_0,r_0} = \varnothing$ there is nothing to show. Choose $u \in R_{\ell_0,r_0}$. For each (a,b) with $[\ell_0,r_0] \subseteq [a,b]$, set

$$C_{a,b} \coloneqq (\pi_{a,b}^{\infty})^{-1}(R_{a,b}) \cap (\pi_{\ell_0,r_0}^{\infty})^{-1}(\{u\}) \subseteq \Big(\prod_{k\in\mathbb{Z}} X_k\Big) \times \Big(\prod_{k\in\mathbb{Z}} Y_k\Big).$$

Since $\mathbf{R}$ is causal, $\pi_{\ell_0,r_0}^{a,b}(R_{a,b}) = R_{\ell_0,r_0}$, so each $C_{a,b}$ is nonempty. Each $C_{a,b}$ is closed since it is an intersection of inverse images of subsets under continuous projections.

Take finitely many pairs (a_i,b_i) with $[\ell_0,r_0] \subseteq [a_i,b_i]$, and define $(a^*,b^*) = (\min_i a_i, \max_i b_i)$. Then $C_{a^*,b^*} \subseteq \bigcap_i C_{a_i,b_i}$, so $\{C_{a,b}\}_{[\ell_0,r_0]\subseteq[a,b]}$ has the finite intersection property. Since the ambient product is compact in the product topology, $\bigcap_{[\ell_0,r_0]\subseteq[a,b]} C_{a,b} \neq \varnothing$. Choose w in this intersection. Then $w \in \mathtt{Lim}(\mathbf{R})$ and $\pi_{\ell_0,r_0}^{\infty}(w) = u$, so $u \in \pi_{\ell_0,r_0}^{\infty}(\mathtt{Lim}(\mathbf{R}))$. □

Proposition 12 (Faithfulness of Lim)**.** *The functor* $\mathtt{Lim} : \mathrm{Obs}(\mathsf{ALR}_{\mathsf{fd}}^{\mathbb{N}}) \to \mathsf{AR}_{\infty}$ *is faithful.*

Proof. Suppose $\mathtt{Lim}([\mathbf{R}]) = \mathtt{Lim}([\mathbf{S}])$. By lemma 16, let $\mathrm{Fix}(\mathbf{R})$ and $\mathrm{Fix}(\mathbf{S})$ be the unique causal representatives of these classes. Since Lim is defined on observational classes, $\mathtt{Lim}(\mathrm{Fix}(\mathbf{R})) = \mathtt{Lim}(\mathrm{Fix}(\mathbf{S}))$. Applying lemma 17 for each $\ell \le r$,

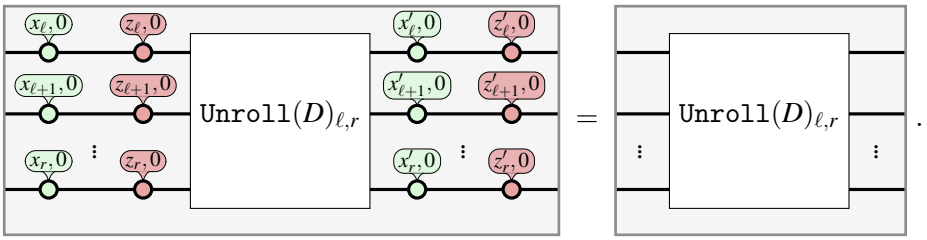
$$\mathrm{Fix}(\mathbf{R})_{\ell,r} = \pi_{\ell,r}^{\infty}\big(\mathtt{Lim}(\mathrm{Fix}(\mathbf{R}))\big) = \pi_{\ell,r}^{\infty}\big(\mathtt{Lim}(\mathrm{Fix}(\mathbf{S}))\big) = \mathrm{Fix}(\mathbf{S})_{\ell,r}.$$

Hence $\mathrm{Fix}(\mathbf{R}) = \mathrm{Fix}(\mathbf{S})$, so $[\mathbf{R}] = [\mathbf{S}]$. Therefore Lim is faithful. □

Theorem 8. *There is a faithful †-CC functor* $\mathtt{Lim} : \mathsf{Obs}(\mathsf{ALR}_{\mathsf{fd}}^{\oplus}) \rightarrowtail \mathsf{AR}_\infty$ *sending morphisms* $[\mathbf{R}] : \mathbf{X} \to \mathbf{Y}$ *to the relation* $\left\{(\mathbf{x},\mathbf{y}) \;\middle|\; \forall \ell \leq r \in \mathbb{Z} :\ \big((x_\ell,\ldots,x_r),(y_\ell,\ldots,y_r)\big) \in R_{\ell,r}\right\} \subseteq \prod_{j\in\mathbb{Z}} X_j \times \prod_{j\in\mathbb{Z}} Y_j$.

Proof of Theorem 8. Functoriality and †-CC preservation are proposition 11. Faithfulness is proposition 12. □

Corollary 2. *This induces a †-CC functor* $\mathtt{StabGrp} : \mathrm{St}(\mathsf{ZX}) \to \mathsf{Obs}(\mathsf{ZX}^{\oplus}) \simeq \mathsf{Obs}(\mathsf{ALR}_{\mathsf{fd}}^{\oplus}) \to \mathsf{AR}_\infty$, *such that given any stateful ZX-diagram* $D : n \to m$, *a sequence* $\left(\left[\begin{smallmatrix}\mathbf{x}\\ \mathbf{z}\end{smallmatrix}\right], \left[\begin{smallmatrix}\mathbf{x}'\\ \mathbf{z}'\end{smallmatrix}\right]\right) \in ((\mathbb{F}_p^n)^{\mathbb{Z}})^2 \times ((\mathbb{F}_p^m)^{\mathbb{Z}})^2$ *of Pauli operators is an element of* $\mathtt{StabGrp}(D)$ *if and only if for all* $\ell \leq r \in \mathbb{Z}$*:*



Where the thick spiders denote multi-quopit Pauli operators in the scalable notation for ZX *(see section 2.4 for reference). This means that* $\mathtt{StabGrp}(D)$ *is the space of Pauli operators which stabilise each stage of the unrolling of the stateful ZX-diagram D.*

Proof of Corollary 2. Let $E : \mathsf{ZX}^{\oplus} \to \mathsf{ALR}_{\mathsf{fd}}^{\oplus}$ be the poset-enriched †-CC discard equivalence of theorem 6. Define

$$\mathsf{Obs}(E) : \mathsf{Obs}(\mathsf{ZX}^{\oplus}) \to \mathsf{Obs}(\mathsf{ALR}_{\mathsf{fd}}^{\oplus})$$

by

$$\mathsf{Obs}(E)(\mathbf{X}) \coloneqq (E(X_k))_{k\in\mathbb{Z}}, \qquad (\mathsf{Obs}(E)(\mathbf{U}))_{\ell,r} \coloneqq E(U_{\ell,r}).$$

This is a †-CC discard functor, componentwise. In $\mathsf{Obs}(\mathsf{ZX}^{\oplus})$, the component monoidal product is inherited from $\mathsf{ZX}^{\oplus}$, namely $\otimes$; in $\mathsf{Obs}(\mathsf{ALR}_{\mathsf{fd}}^{\oplus})$, it is inherited from $\mathsf{ALR}_{\mathsf{fd}}^{\oplus}$, namely $\oplus$.

Monotonicity. Let $\mathbf{U} : \mathbf{X} \to \mathbf{Y}$ be monotone in $\mathsf{Obs}(\mathsf{ZX}^{\oplus})$. For each pair (ℓ, r), right monotonicity gives

$$(1 \otimes \mathtt{disc}) \circ U_{\ell,r+1} \subseteq U_{\ell,r} \otimes \mathtt{disc},$$

and similarly on the left boundary. Applying E, and using that E preserves order, composition, and sends $\otimes$ to $\oplus$, gives

$$(1 \oplus \mathtt{disc}) \circ E(U_{\ell,r+1}) \subseteq E(U_{\ell,r}) \oplus \mathtt{disc},$$

and similarly on the left. Hence $\mathsf{Obs}(E)(\mathbf{U})$ is monotone.

Well-defined on observational classes. Suppose $\mathbf{U} \subseteq \mathbf{V}$. Fix (ℓ, r), and choose $L, R \in \mathbb{N}$ such that

$$(\mathtt{disc} \otimes 1 \otimes \mathtt{disc}) \circ U_{\ell-L,r+R} \subseteq \mathtt{disc} \otimes V_{\ell,r} \otimes \mathtt{disc}.$$

Applying E gives

$$(\mathtt{disc} \oplus 1 \oplus \mathtt{disc}) \circ E(U_{\ell-L,r+R}) \subseteq \mathtt{disc} \oplus E(V_{\ell,r}) \oplus \mathtt{disc}.$$

So $\mathsf{Obs}(E)(\mathbf{U}) \subseteq \mathsf{Obs}(E)(\mathbf{V})$ with the same lookahead; hence $\mathsf{Obs}(E)$ descends to observational equivalence classes.

Functoriality. Identities, composition, tensor, dagger, cups, caps, and discards are preserved pairwise by E, so $\mathrm{Obs}(E)$ is a †-CC discard functor.

Equivalence. We show $\mathrm{Obs}(E)$ is essentially surjective, faithful, and full.

Essential surjectivity is immediate.

Faithfulness. Suppose $\mathrm{Obs}(E)([\mathbf{U}]) = \mathrm{Obs}(E)([\mathbf{V}])$. Then $\mathrm{Obs}(E)(\mathbf{U})$ and $\mathrm{Obs}(E)(\mathbf{V})$ are observationally equivalent. To show $\mathbf{U} \subseteq \mathbf{V}$, fix a pair (ℓ, r), and choose $L, R \in \mathbb{N}$ such that

$$(\mathtt{disc} \oplus 1 \oplus \mathtt{disc}) \circ E(U_{\ell-L,r+R}) \subseteq \mathtt{disc} \oplus E(V_{\ell,r}) \oplus \mathtt{disc}.$$

Since E preserves composition, tensor, and discard, this is $E(u) \subseteq E(v)$, where

$$u \coloneqq (\mathtt{disc} \otimes 1 \otimes \mathtt{disc}) \circ U_{\ell-L,r+R}, \qquad v \coloneqq \mathtt{disc} \otimes V_{\ell,r} \otimes \mathtt{disc}.$$

Because E reflects the hom-poset order, $u \subseteq v$, so $\mathbf{U} \subseteq \mathbf{V}$. The converse is identical, hence $[\mathbf{U}] = [\mathbf{V}]$.

Fullness. Let $[\mathbf{W}] : \mathrm{Obs}(E)(\mathbf{X}) \to \mathrm{Obs}(E)(\mathbf{Y})$ in $\mathrm{Obs}(\mathsf{ALR}^{\dagger\oplus}_{\mathsf{fd}})$, with representative $\mathbf{W}$. For each pair (ℓ, r), fullness of E gives

$$W_{\ell,r} : \bigoplus_{k=\ell}^{r} E(X_k) \to \bigoplus_{k=\ell}^{r} E(Y_k),$$

$$U_{\ell,r} : \bigotimes_{k=\ell}^{r} X_k \to \bigotimes_{k=\ell}^{r} Y_k \quad \text{with} \quad E(U_{\ell,r}) = W_{\ell,r}$$

after the canonical identifications $E(\bigotimes_{k=\ell}^{r} X_k) \cong \bigoplus_{k=\ell}^{r} E(X_k)$ and $E(\bigotimes_{k=\ell}^{r} Y_k) \cong \bigoplus_{k=\ell}^{r} E(Y_k)$. Let $\mathbf{U} = \{U_{\ell,r}\}_{\ell \le r}$. Monotonicity of $\mathbf{W}$, together with order reflection of E, implies that $\mathbf{U}$ is monotone. Therefore $[\mathbf{U}]$ is a morphism in $\mathrm{Obs}(\mathsf{ZX}^{\dagger\oplus})$, and by construction $\mathrm{Obs}(E)([\mathbf{U}]) = [\mathbf{W}]$. So $\mathrm{Obs}(E)$ is full.

Therefore $\mathrm{Obs}(E)$ is an equivalence $\mathrm{Obs}(\mathsf{ZX}^{\dagger\oplus}) \simeq \mathrm{Obs}(\mathsf{ALR}^{\dagger\oplus}_{\mathsf{fd}})$.

Define $\mathtt{Rel} \coloneqq \mathtt{Lim} \circ \mathrm{Obs}(E) \circ \mathtt{Unroll}$. By propositions 7 and 11, $\mathtt{Rel}$ is †-CC. For $D : n \to m$, the displayed condition is exactly the defining condition of $\mathtt{Lim}$ applied to $\mathrm{Obs}(E)(\mathtt{Unroll}(D))$, so it characterises membership in $\mathtt{StabGrp}(D)$. □

D Proofs of Section 5

Proposition 8. *Given a shifted affine Lagrangian subspace $R \subseteq (\mathbb{F}_p(\delta)^{2n}, \widetilde{\omega}_n)$, there exist $m \in \mathbb{N}$, a matrix $L : n-m \to m$, a sesqui-symmetric matrix $\Sigma^\top = \widetilde{\Sigma} : n-m \to n-m$, a permutation $\varsigma : n \to n$, and $\mathbf{a} \in \mathbb{F}_p(\delta)^{2n}$ such that,*

$$R = \ker\left(\left[\begin{array}{cc|cc} I_m & \widetilde{L} & \Sigma & 0 \\ 0 & 0 & -L^\top & I_{n-m} \end{array}\right] \left[\begin{array}{c|c} \varsigma & 0 \\ \hline 0 & \varsigma \end{array}\right] \right) + \mathbf{a}.$$

*Call this presentation of R the **canonical form** of the **tableau** associated to R. Given any shifted affine Lagrangian subspace, there is a unique canonical tableau. In case $m = 0$, say that R is the tableau associated to a **shifted graph-state**.*

Proof of Proposition 8. Write $R = S + \mathbf{a}$, where $S \subseteq \mathbb{F}_p(\delta)^{2n}$ is the linear part of R. Because R is shifted affine Lagrangian, S is shifted Lagrangian. Let $\widetilde{S} \coloneqq \{\tilde{s} \mid s \in S\} \subseteq \mathbb{F}_p(\delta)^{2n}$. Since $\widetilde{\omega}_n(x,s) = \omega_n(x,\tilde{s})$, we have $S^{\widetilde{\omega}_n} = \widetilde{S}^{\omega_n}$. The involution $s \mapsto \tilde{s}$ is bijective, therefore $\dim_{\mathbb{F}_p(\delta)}(\widetilde{S}) = \dim_{\mathbb{F}_p(\delta)}(S)$. Because

ω_n is non-degenerate as a sesquilinear form, $\dim_{\mathbb{F}_p(\delta)}(\widetilde{S}^{\omega_n}) = 2n - \dim_{\mathbb{F}_p(\delta)}(\widetilde{S})$. Because S is shifted Lagrangian, $S = S^{\widetilde{\omega}_n} = \widetilde{S}^{\omega_n}$, hence $\dim_{\mathbb{F}_p(\delta)}(S) = 2n - \dim_{\mathbb{F}_p(\delta)}(S)$, therefore $\dim_{\mathbb{F}_p(\delta)}(S) = n$. Therefore there is a full-rank matrix $H = \left[\begin{array}{c|c} X & Z \end{array}\right] \in \mathbb{F}_p(\delta)^{n\times 2n}$. with $S = \ker(H)$. Applying elementary row-operations, we can reduce this matrix to the following form, while preserving the kernel, reordering the columns with a shifted symplectic permutation matrix ς:

$$H \sim \left[\begin{array}{cc|cc} I_m & A & B & 0 \\ 0 & 0 & C & I_{n-m} \end{array}\right] \left[\begin{array}{c|c} \varsigma & 0 \\ \hline 0 & \varsigma \end{array}\right].$$

The shifted Lagrangian condition on S forces $A = \widetilde{L}$, $C = -L^\top$, and $B^\top = \widetilde{B}$. Setting $\Sigma \coloneqq B$ and $L = \widetilde{A}$:

$$S = \ker\left(\left[\begin{array}{cc|cc} I_m & \widetilde{L} & \Sigma & 0 \\ 0 & 0 & -L^\top & I_{n-m} \end{array}\right] \circ \left[\begin{array}{c|c} \varsigma & 0 \\ \hline 0 & \varsigma \end{array}\right]\right),$$

Uniqueness follows by the same argument as in the unshifted case. □

Theorem 9. *There is a †-CC oplax normal functor $\Gamma : \mathsf{sALR}_{\mathsf{fd}} \to \mathsf{AR}_\infty$, which factors through $\mathrm{Obs}(\mathsf{ACR}_{\mathsf{fd}})$, sending objects $(\mathbb{F}_p^{2n}(\delta), \widetilde{\omega}_n) \mapsto (\mathbb{F}_p^{\mathbb{Z}})^{2n}$ and sending morphisms $R \subseteq (\mathbb{F}_p^{2n}(\delta), -\widetilde{\omega}_n) \times (\mathbb{F}_p^{2m}(\delta), \widetilde{\omega}_m)$ to the $\mathbb{F}_p$-affine subspace of $(\mathbb{F}_p^{\mathbb{Z}})^{2n} \times (\mathbb{F}_p^{\mathbb{Z}})^{2m}$ given on elements by computing the geometric series expansion pointwise and then regarding formal Laurent series as $\mathbb{Z}$ indexed sequences over $\mathbb{F}_p$.*

Proof of Theorem 9. Let $\gamma : \mathbb{F}_p(\delta) \to \mathbb{F}_p((\delta))$ be geometric-series expansion. For $h \in \mathbb{F}_p((\delta))$, write $[\delta^k](\sum_{t\in\mathbb{Z}} h_t \delta^t) \coloneqq h_k$. For each $n \in \mathbb{N}$ and each pair $(\ell, r) \in \mathbb{Z}^2$ with $\ell \le r$, define the truncation map

$$\tau^n_{\ell,r} : \mathbb{F}_p(\delta)^{2n} \to \prod_{t=\ell}^{r} \mathbb{F}_p^{2n}, \qquad \tau^n_{\ell,r}(v) \coloneqq \big([\delta^\ell]\gamma(v), \ldots, [\delta^r]\gamma(v)\big).$$

Step 1: construct the observational functor. We first prove there is an oplax †-CC functor $\Gamma_{\mathrm{obs}} : \mathsf{sALR}_{\mathsf{fd}} \to \mathrm{Obs}(\mathsf{ALR}^{\#}_{\mathsf{fd}})$ given on objects by the constant family $\Gamma_{\mathrm{obs}}\big((\mathbb{F}_p(\delta)^{2n}, \widetilde{\omega}_n)\big)_k \coloneqq \mathbb{F}_p^{2n}$, and on morphisms $R : n \to m$ by $(\Gamma_{\mathrm{obs}}(R))_{\ell,r} \coloneqq \left\{ \big(\tau^n_{\ell,r}(x), \tau^m_{\ell,r}(y)\big) \;\middle|\; (x,y) \in R \right\}$.

We check this lands in $\mathsf{ALR}^{\#}_{\mathsf{fd}}$. If $R = \varnothing$, all components are empty. If $R \neq \varnothing$, write $R = L + \mathbf{a}$, where L is shifted Lagrangian in $((\mathbb{F}_p(\delta)^{2n} \times \mathbb{F}_p(\delta)^{2m}), \Psi)$, with $\Psi \coloneqq -\widetilde{\omega}_n \times \widetilde{\omega}_m$. Fix (ℓ, r). Set

$$W_{\ell,r} \coloneqq \prod_{t=\ell}^{r} \mathbb{F}_p^{2n} \times \prod_{t=\ell}^{r} \mathbb{F}_p^{2m}, \qquad \Psi_{\ell,r} \coloneqq -\bigoplus_{t=\ell}^{r} \omega_n \times \bigoplus_{t=\ell}^{r} \omega_m,$$

$$M_{\ell,r} \coloneqq (\tau^n_{\ell,r} \times \tau^m_{\ell,r})(L).$$

so that

$$(\Gamma_{\mathrm{obs}}(R))_{\ell,r} = M_{\ell,r} + (\tau^n_{\ell,r} \times \tau^m_{\ell,r})(\mathbf{a}).$$

Given $f = \sum_{t=\ell}^{r} f_t \delta^t$ and $g = \sum_{s\in\mathbb{Z}} g_s \delta^s$, it follows that $[\delta^0]\widetilde{\omega}_n(f,g) = \sum_{t=\ell}^{r} \omega_n(f_t, g_t)$. Applying this to Ψ, we have $\Psi_{\ell,r}\big(w, (\tau^n_{\ell,r} \times \tau^m_{\ell,r})(u)\big) = [\delta^0]\Psi\big(\iota_{\ell,r}(w), u\big)$, where $\iota_{\ell,r}\big((w_t)_{t=\ell}^{r}\big) \coloneqq \sum_{t=\ell}^{r} w_t \delta^t$. Hence $w \in M_{\ell,r}^{\Psi_{\ell,r}}$ implies $[\delta^0]\Psi(\iota_{\ell,r}(w), u) = 0$ for all $u \in L$. Since $\delta^k u \in L$ for all $k \in \mathbb{Z}$, the same identity with $\delta^k u$ shows $[\delta^k]\Psi(\iota_{\ell,r}(w), u) = 0$ for all k, so $\Psi(\iota_{\ell,r}(w), u) = 0$ for all $u \in L$. Thus $\iota_{\ell,r}(w) \in L^\Psi = L$, hence $w \in M_{\ell,r}$. Therefore $M_{\ell,r}^{\Psi_{\ell,r}} \subseteq M_{\ell,r}$: each $(\Gamma_{\mathrm{obs}}(R))_{\ell,r}$ is affine coisotropic. Because each $W_{\ell,r}$ is finite-dimensional over $\mathbb{F}_p$, each component lies in $\mathsf{ALR}^{\#}_{\mathsf{fd}}$.

All components are projections of the same affine subspace $\gamma(R)$, so $\Gamma_{\text{obs}}(R)$ is causal, hence monotone. Thus Γ_{obs} is well-defined on morphisms.

Now check oplaxness. Let $R : n \to m$, $S : m \to p$, and fix (ℓ, r). If $S \circ R = \varnothing$, inclusion is trivial. Otherwise, take $(u, w) \in (\Gamma_{\text{obs}}(S \circ R))_{\ell,r}$. Then for some $(x, z) \in S \circ R$,

$$u = \tau^n_{\ell,r}(x), \qquad w = \tau^p_{\ell,r}(z).$$

Choose $y \in \mathbb{F}_p(\delta)^{2m}$ with $(x, y) \in R$, $(y, z) \in S$. Then

$$\big(u, \tau^m_{\ell,r}(y)\big) \in (\Gamma_{\text{obs}}(R))_{\ell,r}, \qquad \big(\tau^m_{\ell,r}(y), w\big) \in (\Gamma_{\text{obs}}(S))_{\ell,r},$$

so

$$(\Gamma_{\text{obs}}(S \circ R))_{\ell,r} \subseteq (\Gamma_{\text{obs}}(S))_{\ell,r} \circ (\Gamma_{\text{obs}}(R))_{\ell,r}.$$

Hence $\Gamma_{\text{obs}}(S \circ R) \subseteq \Gamma_{\text{obs}}(S) \circ \Gamma_{\text{obs}}(R)$. Identity and the †-CC structure are preserved componentwise.

Step 2: apply compactness. Finally define $\Gamma \coloneqq \texttt{Lim} \circ \Gamma_{\text{obs}}$.

By compactness (proposition 11), $\texttt{Lim} : \text{Obs}(\mathsf{ALR}^{\mathbb{N}}_{\mathsf{fd}}) \to \mathsf{AR}_\infty$ is †-CC. Therefore Γ is a †-CC oplax functor. On objects,

$$\Gamma\big((\mathbb{F}_p(\delta)^{2n}, \widetilde{\omega}_n)\big) \cong \prod_{k \in \mathbb{Z}} \mathbb{F}_p^{2n} = (\mathbb{F}_p^{\mathbb{Z}})^{2n},$$

and on morphisms this is exactly the affine relation obtained by coefficient extraction after geometric-series expansion, i.e. the one described in the statement. □

Proposition 10. *For all $D \in \text{St}(\mathsf{ZX})$, $\Gamma(\texttt{Ext}(D)) \subseteq \texttt{StabGrp}(D)$.*

Proof of Proposition 10. A simple diagram chase shows that the two paths agree strictly on affine generators, linear generators, and the delay. Therefore, by oplaxness of Γ, the triangle commutes laxly. □

E Proofs of Section 6

Lemma 7. *The ring of self-conjugate Laurent polynomials $h(\delta) = h(1/\delta) \in \mathbb{F}_p[\delta, 1/\delta]$ is isomorphic to the ring $\mathbb{F}_p[\delta + 1/\delta] \subset \mathbb{F}_p[\delta, 1/\delta]$ of polynomials in $\delta + 1/\delta$.*

Proof of Lemma 7. First, we prove that if $h(\delta) \in \mathbb{F}_p[\delta + 1/\delta]$, then $h(\delta) = h(1/\delta) \in \mathbb{F}_p[\delta, 1/\delta]$. Since, $\widetilde{\delta + 1/\delta} = 1/\delta + \delta = \delta + 1/\delta$ every element of $\mathbb{F}_p[\delta + 1/\delta]$ is self-symmetric.

For the converse direction, we prove that $(\delta^j + \delta^{-j}) \in \mathbb{F}_p[\delta + 1/\delta]$ for all $j \in \mathbb{N}$, which implies that every self-conjugate Laurent polynomial $h(\delta) = h_0 + \sum_{j=1}^n h_j(\delta + \delta^{-j}) \in \mathbb{F}_p[\delta + 1/\delta]$.

We prove that $(\delta^j + \delta^{-j}) \in \mathbb{F}_p[\delta + 1/\delta]$ by induction on $j \in \mathbb{N}$. For the base case of $j = 0$:

$$\delta^0 + \delta^{-0} = 2(\delta + 1/\delta)^0 \in \mathbb{F}_p[\delta + 1/\delta]$$

For the inductive hypothesis, take some $0 < j \in \mathbb{N}$ such that for every $m \in \mathbb{N}$, such that $m < j$, $(\delta^m + \delta^{-m}) \in \mathbb{F}_p[\delta + 1/\delta]$. From the binomial theorem, we have:

$$(\delta + \delta^{-1})^j = \sum_{k=0}^{j} \binom{j}{k} \delta^{j-2k} = \binom{j}{0}\delta^j + \sum_{k=1}^{\lfloor j/2 \rfloor} \binom{j}{k} \delta^{j-2k} + \sum_{k=\lfloor j/2 \rfloor + 1}^{j-1} \binom{j}{k} \delta^{j-2k} + \binom{j}{j}\delta^{-j} \tag{5}$$

$$= \delta^j + \delta^{-j} + \sum_{k=1}^{\lfloor j/2 \rfloor} \binom{j}{k} \left(\delta^{j-2k} + \delta^{-(j-2k)}\right) \tag{6}$$

Therefore, rearranging terms:

$$\delta^j + \delta^{-j} = (\delta + \delta^{-1})^j - \sum_{k=1}^{\lfloor j/2 \rfloor} \binom{j}{k} \left(\delta^{j-2k} + \delta^{-(j-2k)}\right) \tag{7}$$

It is clear that $(\delta + \delta^{-1})^j = (\delta + 1/\delta)^j \in \mathbb{F}_p[\delta + 1/\delta]$. Moreover all of the other summands are also in $\mathbb{F}_p[\delta + 1/\delta]$ by the inductive hypothesis. Therefore, because $\mathbb{F}_p[\delta + 1/\delta]$ is a ring, $\delta^j + \delta^{-j} \in \mathbb{F}_p[\delta + 1/\delta]$, as desired. □

Lemma 8. *The field of self-conjugate rational functions is isomorphic to the field of fractions $\mathbb{F}_p(\delta + 1/\delta) \subset \mathbb{F}_p(\delta)$ of the ring of self-conjugate Laurent polynomials $\mathbb{F}_p[\delta + 1/\delta] \subset \mathbb{F}_p[\delta, 1/\delta]$ in $\delta + 1/\delta$.*

Proof of Lemma 8. First, remark that 0 is both a self-conjugate and in $\mathbb{F}_p(\delta + 1/\delta)$. Take a non-zero self-conjugate rational function

$$\frac{f(\delta)}{g(\delta)} = \widetilde{\left(\frac{f(\delta)}{g(\delta)}\right)} = \frac{f(1/\delta)}{g(1/\delta)} \in \mathbb{F}_p(\delta)$$

for $f(\delta), g(\delta) \in \mathbb{F}_p[\delta]$. Then:

$$\frac{f(\delta)}{g(\delta)} = \widetilde{\left(\frac{f(\delta)}{g(\delta)}\right)} = \frac{f(1/\delta)}{g(1/\delta)} = \frac{f(1/\delta)g(\delta)}{g(1/\delta)g(\delta)}$$

Remark that $g(1/\delta)g(\delta)$ and $f(1/\delta)g(\delta)$ are Laurent polynomials. Moreover,

$$\widetilde{g(1/\delta)g(\delta)} = g(\delta)g(1/\delta) = g(1/\delta)g(\delta) \in \mathbb{F}_p[\delta + 1/\delta]$$

by lemma 7. Similarly, since $f(\delta) = g(\delta)f(1/\delta)/g(1/\delta)$, another application of lemma 7 gives:

$$\widetilde{f(1/\delta)g(\delta)} = f(\delta)g(1/\delta) = \frac{f(1/\delta)g(\delta)g(1/\delta)}{g(1/\delta)} = f(1/\delta)g(\delta) \in \mathbb{F}_p[\delta + 1/\delta]$$

Therefore, every self-conjugate rational function is a fraction of self-conjugate Laurent polynomials in $\mathbb{F}_p[\delta + 1/\delta]$ as desired. □