Denotational semantics for stabiliser quantum programs

Robert I. Booth
University of Oxford
United Kingdom

Cole Comfort
Université Paris-Saclay, CNRS, ENS Paris-Saclay, Inria,
CentraleSupélec, Laboratoire Méthodes Formelles
France

The stabiliser fragment of quantum theory is a foundational building block for quantum error correction, and hence for the fault-tolerant compilation of quantum programs. In this article, we develop a sound, universal, and complete denotational semantics for stabiliser operations, including measurement, classically controlled Pauli operators, and affine classical computation; supporting an explicit treatment of quantum error-correcting codes. We interpret stabiliser operations as affine relations over finite fields, yielding a semantics that reflects the algebraic structure underlying stabiliser quantum error correction. Because stabiliser quantum mechanics has a well-behaved algebraic structure, our relational semantics is conceptually transparent and computationally tractable when compared to standard denotational models for general quantum programs. We demonstrate the resulting semantics by describing a small assembly language for stabiliser programs with fully-abstract denotational semantics.

1 Introduction

The problem of compiling quantum algorithms into fault-tolerant hardware-level instructions is now central to large-scale quantum computing [3, 5, 16, 2, 6, 4, 11]. This raises key challenges, including certifying the fault-tolerance of a given implementation and optimising compiled code while preserving fault-tolerance [13, 20, 21, 8, 7, 10, 15, 14, 17]. Addressing these challenges requires a compositional account of stabiliser programs and of the quantum error-correcting codes they manipulate.

Stabiliser codes are the most prevalent and well-studied quantum error-correcting codes, see for example the breadth of research following Gottesman's original work on the subject [9]. Therefore, the programs that manipulate them are central to these questions. Unlike general-purpose quantum programs, stabiliser programs can be simulated efficiently on a probabilistic classical computer [1]. Although the stabiliser formalism is mathematically well understood, there is still no denotational semantics tailored for the quantum programs used to implement quantum error-correction protocols: encoding circuits, syndrome-extraction routines, and recovery procedures. Existing denotational semantics for general-purpose quantum programming languages [22] are sufficiently expressive, but restricting them to the stabiliser fragment does not yield a tractable or algebraically transparent account of these protocols.

Recent work has introduced a denotational semantics for Clifford operators [12]. However, stabiliser quantum error correction does not only concern Clifford operators. These protocols also involve preparing stabiliser states, discarding, measurement, classical post-processing, and classically controlled Pauli corrections.

We construct a semantics for stabiliser quantum programs specifically tailored to quantum error-correction protocols. We do so incrementally: we begin with the purely quantum stabiliser fragment, then add discarding, and finally measurements and classical control, following Selinger [18, 19]. Figure 1 summarises this progression and the corresponding semantic descriptions. Our syntax and semantics are restricted to odd prime dimensions, being further restricted to qubit CSS stabiliser codes in dimension 2.

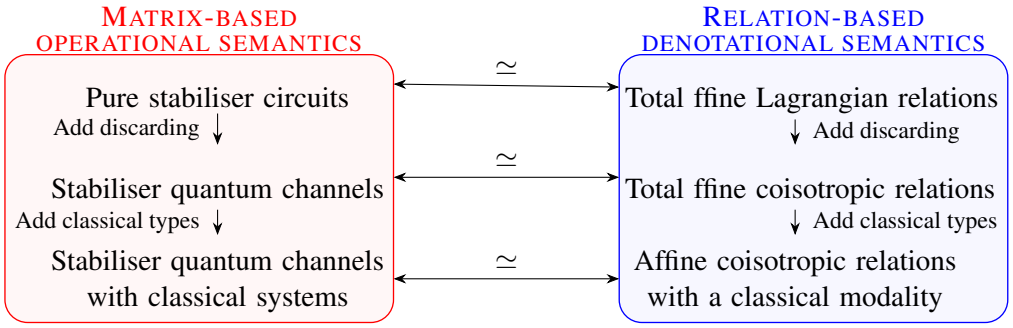

Figure 1: A semantic roadmap for stabiliser QEC.

The top row describes pure Clifford evolution on stabiliser states. Adding discarding gives the mixed setting in which measurement can be represented, so syndrome extraction and error detection enter the picture. Adding classical types then makes those outcomes explicit classical systems, so one can distinguish classical from quantum operations, process syndromes classically, and feed them into the classically controlled Pauli corrections used for recovery.

2 A Language for Stabiliser QEC

We construct an assembly language SPL for stabiliser QEC. Its commands are generated by the following grammar for:

$$c,d \;::=\; c \fatsemi d \mid \mathbf{init}\ \underline{x} \mid \underline{\mathbf{y}} \mathrel{+}= A * \underline{\mathbf{x}} \mid \mathbf{disc}\ \underline{x} \mid \mathbf{qinit}_P\ \underline{x} \mid \underline{\mathbf{x}} \mathrel{*}= U \mid \mathbf{meas}_P\ \underline{x} \mid \mathbf{ctrl}_P\ \underline{x}\ \underline{y} \mid \mathbf{skip}.$$

Here $A : \mathbb{Z}_p^n \to \mathbb{Z}_p^m$ is an affine transformation, U is a Clifford operator, and $P \in \{Z, X\}$ is a Pauli operator. **init** initialises a classical register to 0, $\underline{\mathbf{y}} \mathrel{+}= A * \underline{\mathbf{x}}$ applies affine classical processing, $\mathbf{qinit}_P$ prepares the ground quantum state in the Pauli P-basis, $\underline{\mathbf{x}} \mathrel{*}= U$ applies a Clifford operator, $\mathbf{meas}_P\ \underline{x}$ measures a quantum register in the Pauli P-basis , $\mathbf{ctrl}_P\ \underline{x}\ \underline{y}$ applies a classically controlled Pauli correction in the Pauli-P basis, **disc** $\underline{x}$ discards a classical register, and **skip** is the identity command.

These are exactly the primitives used in stabiliser QEC protocols: **init** allocates fresh classical registers, $\mathbf{qinit}_P$ and Clifford gates build encoded blocks and syndrome-extraction gadgets, $\mathbf{meas}_P$ turns check outcomes into classical syndrome registers, affine classical commands process those syndromes, $\mathbf{ctrl}_P$ applies the recovery Pauli selected by the syndrome, and **disc** removes measured ancillas once they are no longer needed. Together they provide the assembly-level operations used to encode data, measure for errors, and correct against them.

Programs are generated by the following formation rules:

$$\frac{\Gamma \vdash c \triangleright \Delta \quad \Delta \vdash d \triangleright \Sigma}{\Gamma \vdash c \fatsemi d \triangleright \Sigma} \qquad \frac{}{\underline{x} : \mathtt{qpit}, \Gamma \vdash \mathbf{meas}_P\ \underline{x} \triangleright \underline{x} : \mathtt{pit}, \Gamma}$$

$$\frac{}{\underline{\mathbf{x}} : \mathtt{pit}^n, \underline{\mathbf{y}} : \mathtt{pit}^m, \Gamma \vdash \underline{\mathbf{y}} \mathrel{+}= A * \underline{\mathbf{x}} \triangleright \underline{\mathbf{x}} : \mathtt{pit}^n, \underline{\mathbf{y}} : \mathtt{pit}^m, \Gamma} \qquad \frac{}{\Gamma \vdash \mathbf{init}\ \underline{x} \triangleright \underline{x} : \mathtt{pit}, \Gamma}$$

$$\frac{}{\Gamma \vdash \mathbf{skip} \triangleright \Gamma} \qquad \frac{}{\underline{x} : \mathtt{pit}, \Gamma \vdash \mathbf{disc}\ \underline{x} \triangleright \Gamma} \qquad \frac{}{\underline{\mathbf{x}} : \mathtt{qpit}^n, \Gamma \vdash \underline{\mathbf{x}} \mathrel{*}= U \triangleright \underline{\mathbf{x}} : \mathtt{qpit}^n, \Gamma}$$

$$\frac{}{\Gamma \vdash \mathbf{qinit}_P\ \underline{x} \triangleright \underline{x} : \mathtt{qpit}, \Gamma} \qquad \frac{}{\underline{x} : \mathtt{pit}, \underline{y} : \mathtt{qpit}, \Gamma \vdash \mathbf{ctrl}_P\ \underline{x}\ \underline{y} \triangleright \underline{x} : \mathtt{pit}, \underline{y} : \mathtt{qpit}, \Gamma}$$

Here $n \in \mathbb{N}^{>0}$, $\tau \in \mathsf{Ty}$, and $\underline{\mathbf{x}} : \tau^n$ is shorthand for $\{\underline{x1} : \tau, \cdots, \underline{xn} : \tau\}$ where $\underline{\mathbf{x}} = (\underline{x1}, \cdots, \underline{xn}) \in \mathsf{Reg}^n$.

We construct an operational semantics for this language using standard techniques, which can be found in the book Ying [22]). Operationally, a well-formed program acts on a configuration consisting of a stabiliser state together with a collection of classical registers, and evaluation produces measurement branching according to the Born rule. Observational equivalence then identifies well-formed programs that yield the same measurement statistics.

Denotationally, well-formed programs are interpreted in the category $\mathsf{AR}_{\mathsf{qc}}$ of quantum-classical affine relations. Omitting contexts, the base terms are interpreted as follows:

$$[\![c \fatsemi d]\!] = [\![d]\!] \circ [\![c]\!] \quad [\![U]\!] = \mathrm{graph}(S_U) \quad [\![\mathbf{disc}]\!] = \{(a,*) \mid a \in \mathbb{Z}_p\} \quad [\![\mathbf{init}]\!] = \{(*,a) \mid a = 0 \in \mathbb{Z}_p\}$$
$$[\![\mathbf{qinit}_Z]\!] = \{(*,(x,z)) \mid x = 0,\ z \in \mathbb{Z}_p\} \quad [\![\mathbf{qinit}_X]\!] = \{(*,(x,z)) \mid x \in \mathbb{Z}_p,\ z = 0\}$$
$$[\![A*]\!] = \{(\mathbf{a},\mathbf{b}) \mid \mathbf{b} = M\mathbf{a} + \mathbf{c}\} \quad [\![\mathbf{meas}_Z]\!] = \{((x,z),a) \mid a = x \in \mathbb{Z}_p\} \quad [\![\mathbf{meas}_X]\!] = \{((x,z),a) \mid a = z \in \mathbb{Z}_p\}$$
$$[\![\mathbf{ctrl}_X]\!] = \{((a,x,z),(a,x+a,z)) \mid a,x,z \in \mathbb{Z}_p\} \quad [\![\mathbf{ctrl}_Z]\!] = \{((a,x,z),(a,x,z+a)) \mid a,x,z \in \mathbb{Z}_p\}$$

In this semantics, quantum states are represented by vectors in $\mathbb{Z}_p^2$, whereas classical states are represented by scalars in $\mathbb{Z}_p$. Here S_U is the affine symplectomorphism associated to the Clifford operator U. If $P = X^u Z^v$, then the control register stores the value a, and $\mathbf{ctrl}_P$ applies P^a by adding $a(u,v)$ to the symplectic coordinates of the target. In particular, $\mathbf{ctrl}_X$ shifts the x-coordinate and $\mathbf{ctrl}_Z$ shifts the z-coordinate. Thus measurement and classically controlled correction are interpreted in the same semantic category as the purely quantum fragment.

This allows us to state the two main results of the paper:

Theorem 1 (Full abstraction)**.** *Two programs are observationally equivalent if and only if they correspond to the same relation in* $\mathsf{AR}_{\mathsf{qc}}$.

Theorem 2 (Full definability)**.** *Every morphism in* $\mathsf{AR}_{\mathsf{qc}}$ *is the denotation of a* SPL *program.*

Definability says that the semantic model contains no spurious behaviours: every morphism in $\mathsf{AR}_{\mathsf{qc}}$ can be realised by a program built from initialisation, Clifford evolution, measurement, affine classical post-processing, controlled Pauli correction, and discarding. Together with full abstraction, this shows that $\mathsf{AR}_{\mathsf{qc}}$ provides exactly the right semantic universe for stabiliser QEC at this assembly level.

For qubits, the same language and semantics restrict to CSS stabiliser operations: this language has all of the same terms as the odd prime dimensional case, except where $U \in \{Z, X, C^X\}$ ranges over a generating set for the CSS Clifford operators, where Z and X are the Pauli gates and C^X is the controlled-X gate.

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References

[1] Scott Aaronson & Daniel Gottesman (2004): *Improved simulation of stabilizer circuits*. *Physical Review A—Atomic, Molecular, and Optical Physics* 70(5), p. 052328.

[2] Rajeev Acharya, Dmitry A. Abanin, Laleh Aghababaie-Beni, Igor Aleiner, Trond I. Andersen, Markus Ansmann, Frank Arute, Kunal Arya, Abraham Asfaw, Nikita Astrakhantsev, Juan Atalaya, Ryan Babbush, Dave Bacon, Brian Ballard, Joseph C. Bardin, Johannes Bausch, Andreas Bengtsson, Alexander Bilmes, Sam Blackwell, Sergio Boixo, Gina Bortoli, Alexandre Bourassa, Jenna Bovaird, Leon Brill, Michael Broughton, David A. Browne, Brett Buchea, Bob B. Buckley, David A. Buell, Tim Burger, Brian Burkett, Nicholas Bushnell, Anthony Cabrera, Juan Campero, Hung-Shen Chang, Yu Chen, Zijun Chen, Ben Chiaro, Desmond Chik, Charina Chou, Jahan Claes, Agnetta Y. Cleland, Josh Cogan, Roberto Collins, Paul Conner, William Courtney, Alexander L. Crook, Ben Curtin, Sayan Das, Alex Davies, Laura De Lorenzo, Dripto M. Debroy, Sean Demura, Michel Devoret, Agustin Di Paolo, Paul Donohoe, Ilya Drozdov, Andrew Dunsworth, Clint Earle, Thomas Edlich, Alec Eickbusch, Aviv Moshe Elbag, Mahmoud Elzouka, Catherine Erickson, Lara Faoro, Edward Farhi, Vinicius S. Ferreira, Leslie Flores Burgos, Ebrahim Forati, Austin G. Fowler, Brooks Foxen, Suhas Ganjam, Gonzalo Garcia, Robert Gasca, Élie Genois, William Giang, Craig Gidney, Dar Gilboa, Raja Gosula, Alejandro Grajales Dau, Dietrich Graumann, Alex Greene, Jonathan A. Gross, Steve Habegger, John Hall, Michael C. Hamilton, Monica Hansen, Matthew P. Harrigan, Sean D. Harrington, Francisco J. H. Heras, Stephen Heslin, Paula Heu, Oscar Higgott, Gordon Hill, Jeremy Hilton, George Holland, Sabrina Hong, Hsin-Yuan Huang, Ashley Huff, William J. Huggins, Lev B. Ioffe, Sergei V. Isakov, Justin Iveland, Evan Jeffrey, Zhang Jiang, Cody Jones, Stephen Jordan, Chaitali Joshi, Pavol Juhas, Dvir Kafri, Hui Kang, Amir H. Karamlou, Kostyantyn Kechedzhi, Julian Kelly, Trupti Khaire, Tanuj Khattar, Mostafa Khezri, Seon Kim, Paul V. Klimov, Andrey R. Klots, Bryce Kobrin, Pushmeet Kohli, Alexander N. Korotkov, Fedor Kostritsa, Robin Kothari, Borislav Kozlovskii, John Mark Kreikebaum, Vladislav D. Kurilovich, Nathan Lacroix, David Landhuis, Tiano Lange-Dei, Brandon W. Langley, Pavel Laptev, Kim-Ming Lau, Loïck Le Guevel, Justin Ledford, Joonho Lee, Kenny Lee, Yuri D. Lensky, Shannon Leon, Brian J. Lester, Wing Yan Li, Yin Li, Alexander T. Lill, Wayne Liu, William P. Livingston, Aditya Locharla, Erik Lucero, Daniel Lundahl, Aaron Lunt, Sid Madhuk, Fionn D. Malone, Ashley Maloney, Salvatore Mandrà, James Manyika, Leigh S. Martin, Orion Martin, Steven Martin, Cameron Maxfield, Jarrod R. McClean, Matt McEwen, Seneca Meeks, Anthony Megrant, Xiao Mi, Kevin C. Miao, Amanda Mieszala, Reza Molavi, Sebastian Molina, Shirin Montazeri, Alexis Morvan, Ramis Movassagh, Wojciech Mruczkiewicz, Ofer Naaman, Matthew Neeley, Charles Neill, Ani Nersisyan, Hartmut Neven, Michael Newman, Jiun How Ng, Anthony Nguyen, Murray Nguyen, Chia-Hung Ni, Murphy Yuezhen Niu, Thomas E. O'Brien, William D. Oliver, Alex Opremcak, Kristoffer Ottosson, Andre Petukhov, Alex Pizzuto, John Platt, Rebecca Potter, Orion Pritchard, Leonid P. Pryadko, Chris Quintana, Ganesh Ramachandran, Matthew J. Reagor, John Redding, David M. Rhodes, Gabrielle Roberts, Eliott Rosenberg, Emma Rosenfeld, Pedram Roushan, Nicholas C. Rubin, Negar Saei, Daniel Sank, Kannan Sankaragomathi, Kevin J. Satzinger, Henry F. Schurkus, Christopher Schuster, Andrew W. Senior, Michael J. Shearn, Aaron Shorter, Noah Shutty, Vladimir Shvarts, Shraddha Singh, Volodymyr Sivak, Jindra Skruzny, Spencer Small, Vadim Smelyanskiy, W. Clarke Smith, Rolando D. Somma, Sofia Springer, George Sterling, Doug Strain, Jordan Suchard, Aaron Szasz, Alex Sztein, Douglas Thor, Alfredo Torres, M. Mert Torunbalci, Abeer Vaishnav, Justin Vargas, Sergey Vdovichev, Guifre Vidal, Benjamin Villalonga, Catherine Vollgraff Heidweiller, Steven Waltman, Shannon X. Wang, Brayden Ware, Kate Weber, Travis Weidel, Theodore White, Kristi Wong, Bryan W. K. Woo, Cheng Xing, Z. Jamie Yao, Ping Yeh, Bicheng Ying, Juhwan Yoo, Noureldin Yosri, Grayson Young, Adam Zalcman, Yaxing Zhang, Ningfeng Zhu, Nicholas Zobrist & Google Quantum AI and Collaborators (2025): *Quantum error correction below the surface code threshold*. *Nature* 638(8052), pp. 920–926, doi:10.1038/s41586-024-08449-y. Publisher: Nature Publishing Group.

[3] Rajeev Acharya, Igor Aleiner, Richard Allen, Trond I. Andersen, Markus Ansmann, Frank Arute, Kunal Arya, Abraham Asfaw, Juan Atalaya, Ryan Babbush, Dave Bacon, Joseph C. Bardin, Joao Basso, Andreas Bengtsson, Sergio Boixo, Gina Bortoli, Alexandre Bourassa, Jenna Bovaird, Leon Brill, Michael Broughton, Bob B. Buckley, David A. Buell, Tim Burger, Brian Burkett, Nicholas Bushnell, Yu Chen, Zijun Chen, Ben Chiaro, Josh Cogan, Roberto Collins, Paul Conner, William Courtney, Alexander L. Crook, Ben Curtin, Dripto M. Debroy, Alexander Del Toro Barba, Sean Demura, Andrew Dunsworth, Daniel Eppens, Catherine Erickson, Lara Faoro, Edward Farhi, Reza Fatemi, Leslie Flores Burgos, Ebrahim Forati, Austin G. Fowler, Brooks Foxen, William Giang, Craig Gidney, Dar Gilboa, Marissa Giustina, Alejandro Grajales Dau,

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[4] Michael E. Beverland, Prakash Murali, Matthias Troyer, Krysta M. Svore, Torsten Hoefler, Vadym Kliuchnikov, Guang Hao Low, Mathias Soeken, Aarthi Sundaram & Alexander Vaschillo: *Assessing requirements to scale to practical quantum advantage*, doi:10.48550/arXiv.2211.07629. arXiv:2211.07629 [quant-ph].

[5] Sergey Bravyi, Andrew W. Cross, Jay M. Gambetta, Dmitri Maslov, Patrick Rall & Theodore J. Yoder (2024): *High-threshold and low-overhead fault-tolerant quantum memory*. Nature 627(8005), pp. 778–782, doi:10.1038/s41586-024-07107-7. Publisher: Nature Publishing Group.

[6] Earl T. Campbell, Barbara M. Terhal & Christophe Vuillot: *Roads towards fault-tolerant universal quantum computation* 549(7671), pp. 172–179. doi:10.1038/nature23460. Available at `https://www.nature.com/articles/nature23460`. Publisher: Nature Publishing Group.

[7] Kean Chen, Yuhao Liu, Wang Fang, Jennifer Paykin, Xin-Chuan Wu, Albert Schmitz, Steve Zdancewic & Gushu Li: *Verifying Fault-Tolerance of Quantum Error Correction Codes*, doi:10.48550/arXiv.2501.14380. arXiv:2501.14380 [quant-ph].

[8] Wang Fang & Mingsheng Ying: *Symbolic Execution for Quantum Error Correction Programs* 8, pp. 189:1040–189:1065. doi:10.1145/3656419. Available at `https://dl.acm.org/doi/10.1145/3656419`.

[9] Daniel Gottesman (2009): *An Introduction to Quantum Error Correction and Fault-Tolerant Quantum Computation*. arXiv:arXiv:0904.2557.

[10] Qifan Huang, Li Zhou, Wang Fang, Mengyu Zhao & Mingsheng Ying: *Efficient Formal Verification of Quantum Error Correcting Programs* 9, pp. 1068–1093. doi:10.1145/3729293. Available at `https://dl.acm.org/doi/10.1145/3729293`.

[11] Alexandru Paler, Ilia Polian, Kae Nemoto & Simon J Devitt (2017): *Fault-tolerant, high-level quantum circuits: form, compilation and description*. Quantum Science and Technology 2(2), p. 025003, doi:10.1088/2058-9565/aa66eb. Available at `http://dx.doi.org/10.1088/2058-9565/aa66eb`.

[12] Jennifer Paykin & Sam Winnick (2024): *Qudit Quantum Programming with Projective Cliffords*. arXiv preprint arXiv:2407.16801.

[13] Robert Rand, Aarthi Sundaram, Kartik Singhal & Brad Lackey: *Gottesman Types for Quantum Programs*. p. 12.

[14] Benjamin Rodatz, Boldizsár Poór & Aleks Kissinger: *Fault Tolerance by Construction*. arXiv:2506.17181.

[15] Benjamin Rodatz, Boldizsár Poór & Aleks Kissinger: *Floquetifying stabiliser codes with distance-preserving rewrites*. arXiv:2410.17240.

[16] C. Ryan-Anderson, J. G. Bohnet, K. Lee, D. Gresh, A. Hankin, J. P. Gaebler, D. Francois, A. Chernoguzov, D. Lucchetti, N. C. Brown, T. M. Gatterman, S. K. Halit, K. Gilmore, J. A. Gerber, B. Neyenhuis, D. Hayes & R. P. Stutz (2021): *Realization of Real-Time Fault-Tolerant Quantum Error Correction*. *Physical Review X* 11(4), p. 041058, doi:10.1103/PhysRevX.11.041058. Publisher: American Physical Society.

[17] Maximilian Rüsch, Benjamin Rodatz & Aleks Kissinger: *Completeness for Fault Equivalence of Clifford ZX Diagrams*. arXiv:2510.08477.

[18] Peter Selinger: *Dagger Compact Closed Categories and Completely Positive Maps: (Extended Abstract)*. *Proceedings of the 3rd International Workshop on Quantum Programming Languages (QPL 2005)* 170, pp. 139–163, doi:10.1016/j.entcs.2006.12.018. Available at `https://www.sciencedirect.com/science/article/pii/S1571066107000606`.

[19] Peter Selinger: *Idempotents in Dagger Categories*. 210, pp. 107–122, doi:10.1016/j.entcs.2008.04.021. Available at `https://doi.org/10.1016/j.entcs.2008.04.021`.

[20] Aarthi Sundaram, Robert Rand, Kartik Singhal & Brad Lackey: *Hoare meets Heisenberg: A Lightweight Logic for Quantum Programs*, doi:10.48550/arXiv.2101.08939. arXiv:2101.08939 [quant-ph].

[21] Anbang Wu, Gushu Li, Hezi Zhang, Gian Giacomo Guerreschi, Yuan Xie & Yufei Ding: *QECV: Quantum Error Correction Verification*, doi:10.48550/arXiv.2111.13728. arXiv:2111.13728 [quant-ph].

[22] Mingsheng Ying (2016): *Foundations of Quantum Programming*. Elsevier Inc., doi:10.1016/C2014-0-02660-3.

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Robert I. Booth
University of Oxford, United Kingdom

Cole Comfort
Université Paris-Saclay, CNRS, ENS Paris-Saclay, Inria, CentraleSupélec, Laboratoire Méthodes Formelles

Abstract

The stabiliser fragment of quantum theory is a foundational building block for quantum error correction, and hence for the fault-tolerant compilation of quantum programs. In this article, we develop a sound, universal, and complete denotational semantics for stabiliser operations, including measurement, classically controlled Pauli operators, and affine classical computation; supporting an explicit treatment of quantum error-correcting codes. We interpret stabiliser operations as *affine relations* over finite fields, yielding a semantics that reflects the algebraic structure underlying stabiliser quantum error correction. Because stabiliser quantum mechanics has a well-behaved algebraic structure, our relational semantics is conceptually transparent and computationally tractable when compared to standard denotational models for general quantum programs. We demonstrate the resulting semantics by describing a small, low-level assembly language for stabiliser programs with fully-abstract denotational semantics.

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Keywords and phrases quantum programming languages, quantum error correction, denotational semantics, categorical semantics, stabiliser theory, symplectic linear algebra

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1 Introduction

With increasing experimental demonstrations of error-correction in real quantum systems [3, 7, 32, 2], the problem of compiling quantum algorithms into fault-tolerant hardware-level instructions is now a central problem in the design of scalable quantum processors [10, 5, 26]. This raises key challenges, including certifying the fault-tolerance of a given implementation, and optimising compiled code while preserving fault-tolerance. For fault-tolerant compilation to scale, we need a better understanding of the compositional structure of fault-tolerance, including the semantics of quantum error-correcting programs.

Stabiliser codes are the most prevalent and well-studied quantum error-correcting codes, hence the programs that manipulate them are central for these challenges [17]. Unlike general-purpose quantum programs, stabiliser quantum programs can be simulated efficiently on a probabilistic classical computer [1]. While the mathematical theory underlying the stabiliser formalism is well established, so far, it has not been used to provide a denotational semantics for stabiliser programs.

Quantum error correction (QEC) makes essential use of measurements to detect when an error has occurred, and applies corrections conditioned on their outcomes. A language for QEC must therefore be able to express both measurements and classically-controlled corrections, and its denotational semantics must be able to model them. While denotational semantics for *general-purpose* quantum programming languages are well-studied [40], they are inherently intractable due to the exponential scaling of classical simulation. Simply

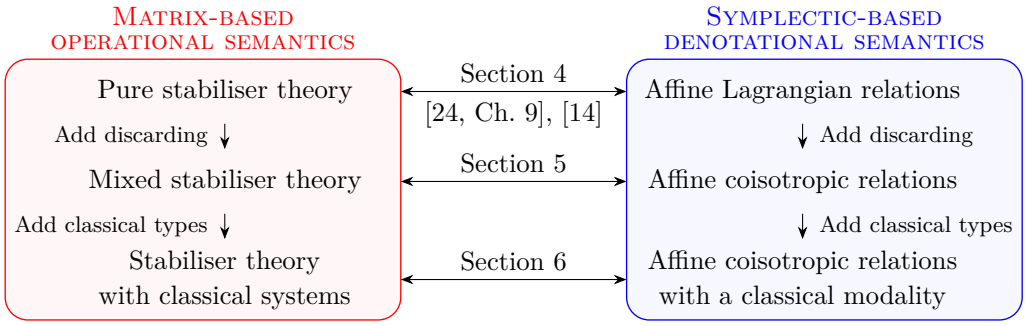

Figure 1 Structure of the article. The horizontal arrows represent categorical equivalences which we systematically prove section by section. The vertical arrows represent the categorical constructions of [35, 36] which are used to construct each successive category from the previous one.

restricting these semantics to the stabiliser fragment does not yield a tractable or algebraically transparent model of program behaviour. The mathematical structure underlying stabiliser programs, particularly including measurements and classical control, remains poorly understood. Furthermore, these general-purpose programming languages and semantics have not been designed with quantum error correction in mind.

Quantum error correction and fault-tolerant compilation have only become a subject of interest to the PL community in recent years, with a focus on verification [28, 38, 39, 16, 11, 21] and program optimisation [30, 29, 33]. The structure of the stabiliser theory is intimately linked with *symplectic linear algebra*, which provides a convenient and tractable setting for representing and reasoning about stabiliser operations. Paykin and Winnick leverage this connection to obtain a type system and λ-calculus for stabiliser programs [27], with a key caveat: their language is unable to express state initialisation, measurement, or classical control. Incorporating these essential primitives in a compositional way requires a denotational semantics for stabiliser programs that has not yet been developed.

1.1 Contributions

The main technical result of this paper is that stabiliser quantum programs with measurement and classical control admit a fully abstract denotational semantics in terms of affine relations over finite fields (Theorem 37). Moreover, this semantics is fully definable in the following sense: we identify a subcategory of affine relations that is exactly the image of the denotational semantics, so that a relation is definable if and only if it lies in this subcategory (Theorem 34).

To establish Theorems 34 and 37, we prove the following results:

- Completely positive stabiliser maps between quantum systems are equivalent to affine coisotropic relations (Proposition 24).
- Completely positive trace-preserving stabiliser maps between quantum systems are equivalent to the *total* affine coisotropic relations (Proposition 25 and Theorem 26).
- Completely positive trace-preserving stabiliser maps between quantum-classical systems are represented by adjoining a *classical modality* $!(-)$ to total affine coisotropic relations (Definition 32 and Propositions 33 and 35).

Taken together, these results characterise the symplectic structure of stabiliser computation with measurement and classical control, providing a basis for future work on abstractions, verification, and semantics-preserving optimisation for fault-tolerant stabiliser programs.

1.2 Proof strategy and organisation

Quantum processes can be stratified into three increasingly expressive settings [8]:

1. *Pure quantum mechanics*: complex linear maps between finite-dimensional vector spaces;
2. *Mixed quantum mechanics*: completely positive maps between matrix algebras;
3. *Quantum mechanics with classical systems*: completely positive maps between finite-dimensional C^*-algebras.

The transition from pure to mixed quantum mechanics adds discarding via the linear-algebraic trace. The transition from mixed quantum mechanics to the classical–quantum setting adds explicit classical systems as types, rather than representing classical data implicitly as subsystems of quantum systems.

In this article we do not consider all quantum processes, but restrict throughout to stabiliser processes. For *pure* stabiliser processes, a symplectic representation is already available; we recall it in Section 4. This provides the purely quantum core of our denotational semantics, and we extend it to incorporate discarding and explicit classical systems and operations such as measurements and classical control.

Following Selinger, given a †-compact closed category of "pure" processes, there are categorical constructions that extend it with discarding [35] and classical systems [36]. We apply these constructions in parallel to stabiliser quantum processes and to their symplectic representation, showing that their equivalence lifts along both extensions, see Figure 1. To keep the denotational semantics algebraically simple, we show that at each stage the symplectic semantics admits an equivalent and more cleanly algebraic form. The main work consists of isolating the correct categorical notions. This abstract approach reduces the technical core of the article to a small number of clean, conceptual verifications. However, this requires considerable exposition.

2 Background: dagger compact-closed categories

In this section, we review the basic theory of symmetric monoidal categories and †-compact closed categories. These will serve as the mathematical structures with which we model theories of *processes*, and allowing us to formally state our denotational semantics.

We work throughout in the strict setting for notational convenience; by coherence, this entails no loss of generality, and equalities should be read as canonical coherence isomorphisms.

▶ **Definition 1.** *A **symmetric monoidal category** (SMC) is a category $\mathcal{C}$ equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called the monoidal product, and an object $I \in \mathcal{C}$, called the monoidal unit, so that for all objects X, Y, Z, $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$ and $I \otimes X = X = X \otimes I$. Moreover, we require that there is a natural isomorphism:*

$$\mathtt{swap}_{X,Y} : X \otimes Y \to Y \otimes X \quad \textit{such that} \quad \mathtt{swap}_{X,Y} = \mathtt{swap}_{Y,X}^{-1}.$$

▶ **Definition 2.** *A **†-compact-closed category** (†-CCC) is a SMC, equipped with an identity on objects functor $\dagger : \mathcal{C} \to \mathcal{C}^{\mathsf{op}}$ such that $\mathtt{swap}_{X,Y}^{\dagger} = \mathtt{swap}_{Y,X}$. Moreover, we require that every object X has a* dual *X^*, such that $(X \otimes Y)^* = Y^* \otimes X^*$ and $(X^*)^* = X$. Finally we ask that for all objects X there are maps $\cup_X : I \to X \otimes X^*$ and $\cap_X : X^* \otimes X \to I$, satisfying*

$$(1_X \otimes \cap_X) \circ (\cup_X \otimes 1_X) = 1_X, \quad (\cap_X \otimes 1_{X^*}) \circ (1_{X^*} \otimes \cup_X) = 1_{X^*}, \quad \textit{and} \quad \cap_A^{\dagger} = \mathtt{swap}_{A,A^*} \circ \cup_A.$$

*A map f is an **isometry** if $f^\dagger \circ f = 1$, **coisometry** if $f \circ f^\dagger = 1$, and **unitary** if $f^\dagger = f^{-1}$.*

Functors between such categories are implicitly understood to preserve relevant structure.

The isometries in the following †-CCC capture *pure state quantum mechanics*.

▶ **Example 3.** The †-CCC FHilb has finite-dimensional Hilbert spaces as objects and linear maps as morphisms. The monoidal structure is given by the tensor product, the dagger is given by the Hermitian adjoint. The dual is given by the dual vector space, where the cup and cap are given by evaluation and the adjoint of evaluation.

Note that FHilb is equivalent to the category of complex matrices.

More generally, in order to also account for classical systems and operations, our operational semantics will have configurations which are states in the following SMC:

▶ **Example 4.** The SMC QChan of **quantum channels** has finite-dimensional C^*-algebras as objects and completely positive trace-preserving maps as morphisms.

It is not important to understand the technical theory of C^*-algebras. It is however important to understand that QChan represents exactly the quantum operations which can be performed between quantum and classical systems. In Sections 5 and 6, we will build up to an equivalent definition of QChan starting from pure state quantum mechanics formulated in FHilb.

On the other hand, our denotational semantics is built from the following †-CCC:

▶ **Example 5.** Given a field $\mathbb{K}$, †-CCC AR of **affine relations** over $\mathbb{K}$ has finite-dimensional $\mathbb{K}$ vector spaces as objects. Morphisms $V \to W$ are given by (possibly empty) affine subspaces $S \subseteq V \times W$, which are composed using *relational composition*

$$(x, z) \in S \circ R \iff \exists y : (x, y) \in R \wedge (y, z) \in S.$$

The monoidal structure is given by the direct sum, and the dagger is given by the converse $(y, x) \in R^\dagger \iff (x, y) \in R$. Finally, the cup and cap are given by the diagonal subspaces

$$\cup_V \coloneqq \left\{ \left(\{\bullet\}, \begin{bmatrix} x \\ x \end{bmatrix} \right) : x \in V \right\} \quad \text{and} \quad \cap_V \coloneqq \left\{ \left(\begin{bmatrix} x \\ x \end{bmatrix}, \{\bullet\} \right) : x \in V \right\}.$$

In Sections 4, 5 and 6, we refine AR to exactly capture our denotational semantics.

3 A minimal assembly language for stabiliser QEC

In this article, we study a variant of QPL for odd prime dimensional stabiliser and qubit CSS stabiliser quantum operations, the imperative language SPL (*Stabiliser Programming Language*). This language captures the primitive operations needed to perform quantum error correction, including measurements and classical control. Compared to QPL [34], our quantum operations are now restricted to stabiliser operations and our classical operations are restricted to *affine* transformations. We have implemented SPL and its denotational semantics in Python [13].

Notation. Throughout, let p be a prime number; where $\mathbb{Z}_p \coloneqq \mathbb{Z}/p\mathbb{Z}$ is the field of integers modulo p. Call an element of $\mathbb{Z}_p$ a *pit* and an element of $\mathbb{Z}_2$ a *bit*. Fix the p-dimensional Hilbert space $\mathcal{H}_p \coloneqq \mathrm{Span}\{|n\rangle \mid n \in \mathbb{Z}_p\}$, representing a normalised element of $\mathcal{H}_p$ by $|\varphi\rangle \in \mathcal{H}_p$, which we shall call a *qupit pure quantum state*, or a *qubit* quantum state for $p = 2$. We write the tensor product of pure states as $|\varphi, \psi\rangle \coloneqq |\varphi\rangle \otimes |\psi\rangle$, the Hermitian adjoint as $\langle\varphi| \coloneqq |\varphi\rangle^\dagger$, the inner product as $\langle\varphi|\psi\rangle$, and the outer product as $|\varphi\rangle\langle\psi|$.

Kind	Term	Operation
Structural:	**skip**	Do nothing.
	$c \fatsemi d$	Sequential composition of subterms.
Quantum:	$\mathbf{qinit}_P\ \underline{\mathtt{x}}$	Initialise $\underline{\mathtt{x}}$ as the qupit $\lvert 0\rangle$ (if P=Z) or $H\lvert 0\rangle$ (if P=X).
	$\underline{\mathbf{x}} \mathrel{*}= U$	Apply the Clifford operator U to the qupit on $\underline{\mathbf{x}}$.
Classical:	$\mathbf{init}\ \underline{\mathtt{x}}$	Initialise $\underline{\mathtt{x}}$ as the pit 0.
	$\underline{\mathbf{y}} \mathrel{+}= A \mathrel{*} \underline{\mathbf{x}}$	Apply A to the pit $\underline{\mathbf{x}}$ and add result to the pit on $\underline{\mathbf{y}}$.
	$\mathbf{disc}\ \underline{\mathtt{x}}$	Discard the pit on $\underline{\mathtt{x}}$.
Quantum-classical:	$\mathbf{meas}_P\ \underline{\mathtt{x}}$	Measure the qupit on $\underline{\mathtt{x}}$ in the P-basis.
	$\mathbf{ctrl}_P\ \underline{\mathtt{x}}\ \underline{\mathtt{y}}$	Apply the Pauli operator P to the qupit on $\underline{\mathtt{y}}$, classically controlled by the pit on $\underline{\mathtt{x}}$.

Figure 2 Operations of SPL**.**

$\{\underline{\mathtt{in}} : \mathtt{qpit}\} \vdash \mathbf{qinit}_Z\ \underline{\mathtt{x}} \fatsemi \mathbf{qinit}_Z\ \underline{\mathtt{out}} \fatsemi \underline{\mathtt{x}} \mathrel{*}= H \fatsemi (\underline{\mathtt{x}}, \underline{\mathtt{out}}) \mathrel{*}= C^X \fatsemi$ `% prepare Bell pair`
$(\underline{\mathtt{in}}, \underline{\mathtt{x}}) \mathrel{*}= C^X \fatsemi \underline{\mathtt{in}} \mathrel{*}= H \fatsemi \mathbf{meas}_Z\ \underline{\mathtt{in}} \fatsemi \mathbf{meas}_Z\ \underline{\mathtt{x}} \fatsemi$ `% Bell measurement`
$\mathbf{ctrl}_Z\ \underline{\mathtt{in}}\ \underline{\mathtt{out}} \fatsemi \mathbf{ctrl}_X\ \underline{\mathtt{x}}\ \underline{\mathtt{out}} \fatsemi$ `% phase correction`
$\mathbf{disc}\ \underline{\mathtt{in}} \fatsemi \mathbf{disc}\ \underline{\mathtt{x}} \triangleright \{\underline{\mathtt{out}} : \mathtt{qpit}\}$ `% discard ancillae`

Figure 3 Quantum teleportation in SPL**.**

3.1 Syntax

The terms of SPL are generated from the following grammar with respect to some fixed, linearly ordered index set of registers Reg:

$$c, d \ ::= \ c \fatsemi d \mid \mathbf{init}\ \underline{\mathtt{x}} \mid \underline{\mathbf{y}} \mathrel{+}= A \mathrel{*} \underline{\mathbf{x}} \mid \mathbf{disc}\ \underline{\mathtt{x}} \mid \mathbf{qinit}_P\ \underline{\mathtt{x}} \mid \underline{\mathbf{x}} \mathrel{*}= U \mid \mathbf{meas}_P\ \underline{\mathtt{x}} \mid \mathbf{ctrl}_P\ \underline{\mathtt{x}}\ \underline{\mathtt{y}} \mid \mathbf{skip}.$$

The elements $\underline{\mathtt{x}}, \underline{\mathtt{y}} \in \mathsf{Reg}$, are registers and $\underline{\mathbf{x}} \in \mathsf{Reg}^n, \underline{\mathbf{y}} \in \mathsf{Reg}^m$ are lists of registers for all $n, m \in \mathbb{N}$. $A : \mathbb{Z}_p^n \to \mathbb{Z}_p^m$ is any $\mathbb{Z}_p$-affine transformation. $P \in \{Z, X\}$ ranges over the Pauli operators (Definition 9). In odd prime dimensions $U \in \{Z, X, C^X, H, P\}$ ranges over a generating set for the *Clifford operators* (Definition 11); whereas for $p = 2$, $U \in \{Z, X, C^X\}$ ranges over a generating set for the *CSS Clifford operators* (Definition 12). We give the interpretations of terms as quantum operations in Figure 2.

SPL is equipped with a *linear* type system which enforces single-use of quantum data, presented in the *leftover typing* style, where typing returns a residual context [23, 41]. SPL has classical and quantum types $\mathsf{Ty} \ ::= \ \mathtt{pit} \mid \mathtt{qpit}$. Typed environments are partial functions $\Gamma : \mathsf{Reg} \to \mathsf{Ty}$ which bind registers to be either qupits or pits, and which we will represent by $\{\underline{\mathtt{x}} : \tau,\ \underline{\mathtt{y}} : \sigma,\ \underline{\mathtt{z}} : \mu,\ \cdots\}$ for $\underline{\mathtt{x}}, \underline{\mathtt{y}}, \underline{\mathtt{z}}, \cdots \in \mathsf{Reg}$ and $\tau, \sigma, \mu, \cdots \in \mathsf{Ty}$. We impose that the domain $\mathrm{dom}(\Gamma)$ of Γ, i.e. the set of bound registers $\{\underline{\mathtt{x}}, \underline{\mathtt{y}}, \underline{\mathtt{z}}, \cdots\}$, is *finite*, and that every register has at most one type. Judgments are triples $\Gamma \vdash t \triangleright \Delta$ consisting of a term t and typed environments Γ, Δ. A judgment $\Gamma \vdash t \triangleright \Delta$ is **well-formed** if it is derivable from the formation rules given in Figure 4. See Figure 3 for a simple example of an SPL program.

3.2 Operational semantics

In Appendix A, we give SPL a conventional operational semantics that follows the literature on quantum programming languages, and summarise the details here. In any quantum programming language, measurements are inherently probabilistic, hence properly describing

$$\frac{\Gamma \vdash c \triangleright \Delta \quad \Delta \vdash d \triangleright \Sigma}{\Gamma \vdash c \fatsemi d \triangleright \Sigma} \qquad \frac{}{\underline{x} : \mathtt{qpit}, \Gamma \vdash \mathbf{meas}_P\ \underline{x} \triangleright \underline{x} : \mathtt{pit}, \Gamma}$$

$$\frac{}{\underline{\mathbf{x}} : \mathtt{pit}^n, \underline{\mathbf{y}} : \mathtt{pit}^m, \Gamma \vdash \underline{\mathbf{y}} \mathrel{+}= A * \underline{\mathbf{x}} \triangleright \underline{\mathbf{x}} : \mathtt{pit}^n, \underline{\mathbf{y}} : \mathtt{pit}^m, \Gamma} \qquad \frac{}{\Gamma \vdash \mathbf{init}\ \underline{x} \triangleright \underline{x} : \mathtt{pit}, \Gamma}$$

$$\frac{}{\Gamma \vdash \mathbf{skip} \triangleright \Gamma} \qquad \frac{}{\underline{x} : \mathtt{pit}, \Gamma \vdash \mathbf{disc}\ \underline{x} \triangleright \Gamma} \qquad \frac{}{\underline{\mathbf{x}} : \mathtt{qpit}^n, \Gamma \vdash \underline{\mathbf{x}} \mathrel{*}= U \triangleright \underline{\mathbf{x}} : \mathtt{qpit}^n, \Gamma}$$

$$\frac{}{\Gamma \vdash \mathbf{qinit}_P\ \underline{x} \triangleright \underline{x} : \mathtt{qpit}, \Gamma} \qquad \frac{}{\underline{x} : \mathtt{pit}, \underline{y} : \mathtt{qpit}, \Gamma \vdash \mathbf{ctrl}_P\ \underline{x}\ \underline{y} \triangleright \underline{x} : \mathtt{pit}, \underline{y} : \mathtt{qpit}, \Gamma}$$

Figure 4 Formation rules for SPL. New variables are always assumed to be fresh. $n \in \mathbb{N}^{>0}$, $\tau \in \mathsf{Ty}$, and $\underline{\mathbf{x}} : \tau^n$ is shorthand for $\{\underline{x1} : \tau, \cdots, \underline{xn} : \tau\}$ such that $\underline{\mathbf{x}} = (\underline{x1}, \cdots, \underline{xn}) \in \mathsf{Reg}^n$.

their operational behaviour requires care. Measurements in SPL are *destructive*: they consume a qupit to produce a pit.

▶ **Definition 6.** *Given an environment* Γ*, a* ***state*** *is a pair* $(\sigma, |\psi\rangle)$ *consisting of a classical store* $\sigma : \Gamma^{-1}(\mathtt{pit}) \to \mathbb{Z}_p$ *and a* normalised *vector* $|\psi\rangle \in \bigotimes_{\Gamma^{-1}(\mathtt{qpit})} \mathcal{H}_p$*. We identify states* $(\sigma, |\psi\rangle)$ *and* $(\sigma, |\psi'\rangle)$ *when there is a complex phase* $\theta \in [0, 2\pi)$ *such that* $|\psi\rangle = \exp(i\theta)\,|\psi'\rangle$*.*

▶ **Definition 7.** *A* ***configuration*** *is a pair* $\left[\Gamma \vdash t \triangleright \Delta \;\middle|\; \sigma, |\psi\rangle\right]$*, or* $\left[t \;\middle|\; \sigma, |\psi\rangle\right]$ *for short, consisting of a well-formed judgement* $\Gamma \vdash t \triangleright \Delta$ *and a state* $(\sigma, |\psi\rangle)$ *of* Γ*.*

The operational semantics of SPL is given a big-step reduction $\left[t \;\middle|\; \sigma, \psi\right] \Downarrow \mu$ (defined in Appendix A, Figure 6), where μ is a probability distribution over states for the output environment, arising from measurement branching via the *Born rule*. This operational semantics is *intensional*, in the sense that it distinguishes computational behaviours that are physically indistinguishable. This is because different distributions over the quantum states can represent the same physical state. Nevertheless, by postcomposing with terms which only output classical data, we can rely on the standard definition of observational equivalence for probabilistic operational semantics:

▶ **Definition 8.** *Let* Θ *be a* classical *environment containing no quantum registers, i.e.* $\Theta^{-1}(\mathtt{qpit}) = \varnothing$*. Well-formed* SPL *terms* $\Sigma \vdash c \triangleright \Delta$ *and* $\Sigma \vdash d \triangleright \Delta$ *are* ***observationally equivalent*** *if for any well-formed terms* $\Gamma \vdash a \triangleright \Sigma$ *and* $\Delta \vdash b \triangleright \Theta$*,* $\left[a \fatsemi c \fatsemi b \;\middle|\; \sigma, |\psi\rangle\right] \Downarrow \mu$ *implies* $\left[a \fatsemi d \fatsemi b \;\middle|\; \sigma, |\psi\rangle\right] \Downarrow \mu$*, and vice-versa.*

3.3 Denotational semantics

We give a denotational semantics for SPL in terms of affine relations over $\mathbb{Z}_p$. On types, define $[\![\mathtt{pit}]\!] \coloneqq \mathbb{Z}_p$ and $[\![\mathtt{qpit}]\!] \coloneqq \mathbb{Z}_p^2$, so that the denotation of a typed environment is given by the direct sum $[\![\Gamma]\!] \coloneqq \bigoplus_{\underline{X} \in \mathrm{dom}(\Gamma)} [\![\Gamma(\underline{x})]\!]$. For an ordered list $\underline{\mathbf{u}} \subseteq \mathrm{dom}(\Gamma)$, denote the complement by $\underline{\mathbf{u}}^c \subseteq \mathrm{dom}(\Gamma)$. Using the canonical isomorphism $[\![\Gamma]\!] \cong [\![\Gamma|_{\underline{\mathbf{u}}}]\!] \oplus [\![\Gamma|_{\underline{\mathbf{u}}^c}]\!]$, denote the projections by $\pi_{\underline{\mathbf{u}}} : [\![\Gamma]\!] \to [\![\Gamma|_{\underline{\mathbf{u}}}]\!]$ and $\pi_{\underline{\mathbf{u}}^c} : [\![\Gamma]\!] \to [\![\Gamma|_{\underline{\mathbf{u}}^c}]\!]$. Given contexts Γ, Δ, ordered lists $\underline{\mathbf{u}} \subseteq \mathrm{dom}(\Gamma)$, $\underline{\mathbf{v}} \subseteq \mathrm{dom}(\Delta)$ such that $\Gamma|_{\underline{\mathbf{u}}^c} = \Delta|_{\underline{\mathbf{v}}^c}$, and an affine relation $R : [\![\Gamma|_{\underline{\mathbf{u}}}]\!] \to [\![\Delta|_{\underline{\mathbf{v}}}]\!]$, define:

$$R_{\underline{\mathbf{u}} \to \underline{\mathbf{v}}} \coloneqq \left\{(\gamma, \delta) \in [\![\Gamma]\!] \times [\![\Delta]\!] \;\middle|\; (\pi_{\underline{\mathbf{u}}}(\gamma), \pi_{\underline{\mathbf{v}}}(\delta)) \in R \text{ and } \pi_{\underline{\mathbf{u}}^c}(\gamma) = \pi_{\underline{\mathbf{v}}^c}(\delta)\right\} : [\![\Gamma]\!] \to [\![\Delta]\!].$$

When $\mathrm{dom}(\Gamma) = \mathrm{dom}(\Delta)$ and $\underline{\mathbf{u}} = \underline{\mathbf{v}}$, write $R_{\underline{\mathbf{u}}} \coloneqq R_{\underline{\mathbf{u}} \to \underline{\mathbf{u}}}$. Finally, the denotations of terms are given in Figure 5.

Despite the algebraically simple nature of the denotational semantics, proving full-abstraction is non-trivial. To prove this, we construct a subcategory $\mathsf{AR}_{\mathsf{qc}} \subset \mathsf{AR}$ which captures exactly the denotations of SPL, culminating in Section 6 with the results:

▶ **Theorem 34** (Full definability). *For all typed environments* Γ *and* Δ *and morphisms* $f : [\![\Gamma]\!] \to [\![\Delta]\!]$ *in* $\mathsf{AR}_{\mathsf{qc}}$*, there exists a well-formed judgement* $\Gamma \vdash c \triangleright \Delta$ *such that* $[\![c]\!] = f$*.*

▶ **Theorem 37** (Full abstraction). *Well-formed judgements* c *and* d *are observationally equivalent if and only if their denotations are equal* $[\![c]\!] = [\![d]\!]$*.*

Section 4, Section 5, and Section 6 are devoted to constructing $\mathsf{AR}_{\mathsf{qc}}$ in such a way that Theorem 34 and Theorem 37 follow effortlessly. This pursuit requires a detailed exposition of various aspects of quantum mechanics and category theory. The following sections can be seen to serve as a certificate of correctness for these theorems.

Term	Denotation
Structural operations:	
$[\![\mathbf{skip}]\!]$	$:= 1_{[\![\Gamma]\!]}$
$[\![c \fatsemi d]\!]$	$:= [\![d]\!] \circ [\![c]\!]$
Quantum operations:	
$[\![\mathbf{qinit}_Z\ \underline{\mathtt{u}}]\!]$	$:= \left\{ \left(\bullet, \left[\begin{smallmatrix} 0 \\ z \end{smallmatrix}\right] \right) \in \{\bullet\} \times \mathbb{Z}_p^2 \right\}_{\varnothing \to \underline{\mathtt{u}}}$
$[\![\mathbf{qinit}_X\ \underline{\mathtt{u}}]\!]$	$:= \left\{ \left(\bullet, \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \right) \in \{\bullet\} \times \mathbb{Z}_p^2 \right\}_{\varnothing \to \underline{\mathtt{u}}}$
$[\![\underline{\mathtt{u}} \mathrel{*}= X]\!]$	$:= \left\{ \left(\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right], \left[\begin{smallmatrix} x+1 \\ z \end{smallmatrix}\right] \right) \,\middle|\, \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] \in \mathbb{Z}_p^2 \right\}_{\underline{\mathtt{u}}}$
$[\![\underline{\mathtt{u}} \mathrel{*}= Z]\!]$	$:= \left\{ \left(\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right], \left[\begin{smallmatrix} x \\ z+1 \end{smallmatrix}\right] \right) \,\middle|\, \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] \in \mathbb{Z}_p^2 \right\}_{\underline{\mathtt{u}}}$
$[\![(\underline{\mathtt{u}}, \underline{\mathtt{v}}) \mathrel{*}= C^X]\!]$	$:= \left\{ \left(\left(\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right], \left[\begin{smallmatrix} x' \\ z' \end{smallmatrix}\right] \right), \left(\left[\begin{smallmatrix} x \\ z-z' \end{smallmatrix}\right], \left[\begin{smallmatrix} x+x' \\ z' \end{smallmatrix}\right] \right) \right) \,\middle|\, \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right], \left[\begin{smallmatrix} x' \\ z' \end{smallmatrix}\right] \in \mathbb{Z}_p^2 \right\}_{\underline{\mathtt{u}}, \underline{\mathtt{v}}}$
$[\![\underline{\mathtt{u}} \mathrel{*}= H]\!]$	$:= \left\{ \left(\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right], \left[\begin{smallmatrix} z \\ -x \end{smallmatrix}\right] \right) \,\middle|\, \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] \in \mathbb{Z}_p^2 \right\}_{\underline{\mathtt{u}}}$
$[\![\underline{\mathtt{u}} \mathrel{*}= P]\!]$	$:= \left\{ \left(\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right], \left[\begin{smallmatrix} x \\ z-x \end{smallmatrix}\right] \right) \,\middle|\, \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] \in \mathbb{Z}_p^2 \right\}_{\underline{\mathtt{u}}}$
	odd prime p only
Classical operations:	
$[\![\mathbf{disc}\ \underline{\mathtt{u}}]\!]$	$:= \left\{ (c, \bullet) \in \mathbb{Z}_p \times \{\bullet\} \right\}_{\underline{\mathtt{u}} \to \varnothing}$
$[\![\mathbf{init}\ \underline{\mathtt{u}}]\!]$	$:= \left\{ (\bullet, 0) \in \{\bullet\} \times \mathbb{Z}_p \right\}_{\varnothing \to \underline{\mathtt{u}}}$
$[\![\underline{\mathbf{v}} \mathrel{+}= A \mathrel{*} \underline{\mathbf{u}}]\!]$	$:= \left\{ \left(\left[\begin{smallmatrix} \mathbf{u} \\ \mathbf{v} \end{smallmatrix}\right], \left[\begin{smallmatrix} \mathbf{u} \\ \mathbf{v}+A(\mathbf{u}) \end{smallmatrix}\right] \right) \right\}_{\underline{\mathbf{u}}, \underline{\mathbf{v}}}$
Quantum-classical operations:	
$[\![\mathbf{meas}_X\ \underline{\mathtt{u}}]\!]$	$:= \left\{ \left(\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right], z \right) \in \mathbb{Z}_p^2 \times \mathbb{Z}_p \right\}_{\underline{\mathtt{u}}}$
$[\![\mathbf{meas}_Z\ \underline{\mathtt{u}}]\!]$	$:= \left\{ \left(\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right], x \right) \in \mathbb{Z}_p^2 \times \mathbb{Z}_p \right\}_{\underline{\mathtt{u}}}$
$[\![\mathbf{ctrl}_X\ \underline{\mathtt{u}}\ \underline{\mathtt{v}}]\!]$	$:= \left\{ \left((c, \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right]), (c, \left[\begin{smallmatrix} x+c \\ z \end{smallmatrix}\right]) \right) \mid c \in \mathbb{Z}_p,\ \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] \in \mathbb{Z}_p^2 \right\}_{\underline{\mathtt{u}}, \underline{\mathtt{v}}}$
$[\![\mathbf{ctrl}_Z\ \underline{\mathtt{u}}\ \underline{\mathtt{v}}]\!]$	$:= \left\{ \left((c, \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right]), (c, \left[\begin{smallmatrix} x \\ z+c \end{smallmatrix}\right]) \right) \mid c \in \mathbb{Z}_p,\ \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] \in \mathbb{Z}_p^2 \right\}_{\underline{\mathtt{u}}, \underline{\mathtt{v}}}$

Figure 5 Denotation of SPL terms into AR

4 The pure stabiliser theory

In this section, we expose the theory underlying the purely quantum operations in SPL, called the *pure stabiliser theory*. The operations in the pure stabiliser theory consist of the Pauli operators, Clifford operators and the initialisation of quantum registers.

This theory is based, fundamentally, on an equivalence between two †-CCCs of pure stabiliser quantum processes:

1. a *concrete* †-CCC, Stab, given by restricting the objects and morphisms in FHilb,
2. and an *abstract* †-CCC, ALR, described in terms of affine subspaces over finite fields.

The first will form the basis of our operational semantics; whereas the second will form the basis of our denotational semantics.

Note that this representation does not fully carry through for qubits. Therefore, we are forced to restrict the qubit stabiliser theory to what is known as the *CSS stabiliser theory* [37, 9]. Although this may initially seem like a significant restriction, some of the most important stabiliser codes, such as the surface code and toric codes are CSS codes [6, 15, 22]. To avoid repeatedly making case distinctions between even and odd dimensions, we omit the CSS qualifier, and *all results for qubits are understood to be restricted in this way.*

4.1 The stabiliser formalism

We first recall the Hilbert space exposition of the pure stabiliser theory.

▶ **Definition 9.** *Let* $\chi(x) \coloneqq \exp(i2\pi x/p)$. *The odd prime* ***qupit Pauli group*** $\mathcal{P}_p \subseteq \mathcal{U}(\mathcal{H}_p)$ *is generated by the operators* $Z\left|x\right\rangle \coloneqq \chi(x)\left|x\right\rangle$ *and* $X\left|x\right\rangle \coloneqq \left|x+1\right\rangle$. *Similarly, for* $p=2$, *the* ***real qubit Pauli group*** *is generated by* X *and* Z.

Henceforth, by *qubit Pauli group* we mean the *real* qubit Pauli group. The *stabiliser formalism* of quantum error correction begins with the following observation:

▶ **Lemma 10.** *Take a maximal Abelian subgroup* $S \subseteq \mathcal{P}_p^{\otimes n}$ *such that* $\chi(a)1_{\mathcal{H}_p}^{\otimes n} \in S$ *if and only if* $a \equiv 0 \mod p$. *Up to global phase* $\exp(2\pi i\theta)$, S *uniquely determines a pure quantum state* $\left|S\right\rangle \in \mathcal{H}_p^{\otimes n}$ *such that* $s\left|S\right\rangle = \left|S\right\rangle$ *for all* $s \in S$, *called a* ***stabiliser state*** *associated to the* ***stabiliser group*** S.

The stabiliser-state-preserving unitaries can also be captured group theoretically:

▶ **Definition 11.** *The* ***Clifford group*** *is the unitary normaliser of* $\mathcal{P}_p^{\otimes n}$*:*

$$\mathcal{C}\ell_p^n \coloneqq \{U \in \mathcal{U}(\mathcal{H}_p^{\otimes n}) \mid \forall P \in \mathcal{P}_p^n, UPU^\dagger \in \mathcal{P}_p^n\} \subseteq \mathcal{U}(\mathcal{H}_p^{\otimes n}).$$

The odd-prime-dimensional Clifford group is generated by the Pauli X operator, the classically-controlled-X gate C^X, the phase shift gate S, the Fourier transform (or Hadamard) H, and `swap`; where stabiliser states are generated by applying Cliffords to $\left|0\right\rangle^{\otimes n}$:

$$C^X(\left|x,y\right\rangle) \coloneqq \left|x,y+x\right\rangle, \quad S\left|x\right\rangle \coloneqq \chi\left(\tfrac{x(x-1)}{2}\right)\left|x\right\rangle, \quad \text{and} \quad H\left|x\right\rangle \coloneqq \tfrac{1}{\sqrt{p}}\sum_{y=0}^{p-1}\chi(xy)\left|y\right\rangle.$$

We define restricted notions of these structures for qubits:

▶ **Definition 12.** *An* n*-qubit stabiliser state is CSS if its stabiliser group is generated by Paulis of the form* $\pm\bigotimes_{j=1}^n Z_j^{b_j}$ *and* $\pm\bigotimes_{j=1}^n X_j^{b_j}$. *A qubit Clifford operator is CSS if it preserves stabiliser states.*

The CSS Clifford group is generated by the Pauli operators, C^X, and `swap`; and the CSS stabiliser states are generated by by applying CSS Cliffords to $|0\rangle^{\otimes n} \otimes (H|0\rangle)^{\otimes m}$. Henceforth, take qubit stabiliser states and Clifford operators to denote their CSS counterparts.

All of these different operations together into a †-CCC:

▶ **Definition 13.** *The †-CCC,* Stab*, of (pure)* ***qupit stabiliser maps*** *is the †-compact-closed subcategory of* FHilb *generated by the qupit stabiliser states and Clifford operators as well as the scalars $1/\sqrt{p}$ and $\sqrt{p}$ under tensor product, composition and the Hermitian adjoint.*

We add the scalars $1/\sqrt{p}$ and $\sqrt{p}$ to obtain a compact closed category. The evolution of **pure stabiliser quantum processes** is given by the isometries in Stab.

4.2 The symplectic representation

The stabiliser formalism can be reformulated using finite-dimensional symplectic linear algebra over $\mathbb{Z}_p$. We recall the essential definitions and statements here. Throughout this section we fix the standard symplectic vector space $(\mathbb{Z}_p^{2n}, \omega_n)$, where $\omega_n\big((\mathbf{x},\mathbf{z}),(\mathbf{x}',\mathbf{z}')\big) \coloneqq \mathbf{x}^\mathsf{T}\mathbf{z}' - \mathbf{z}^\mathsf{T}\mathbf{x}'$. For $p = 2$ we require all subspaces to be split,[1] and all linear maps to be block diagonal.

Paulis are vectors.

We equip $\mathbb{Z}_p \oplus \mathbb{Z}_p^{2n}$ with the group law $(a,\mathbf{v}) * (b,\mathbf{w}) \coloneqq (a + b + \gamma(\mathbf{v},\mathbf{w}), \mathbf{v}+\mathbf{w})$, where γ is chosen so that there is a group isomorphism $\mathbb{Z}_p \oplus \mathbb{Z}_p^{2n} \cong \mathcal{P}_p^n$ given by

$$\pi(a,(\mathbf{x},\mathbf{z})) \coloneqq \Big\{ \chi\big(a + \tfrac{1}{2}\,\mathbf{z}^\mathsf{T}\mathbf{x}\big) \bigotimes_{k=1}^{n} X^{x_k} Z^{z_k} \text{ if } p \text{ is odd; else } (-1)^a \bigotimes_{k=1}^{n} X^{x_k} Z^{z_k} \text{ if } p = 2.$$

Stabilisers are affine Lagrangian subspaces.

Given a linear subspace S of a symplectic vector space (V,ω), its **symplectic complement** is the space $S^\omega \coloneqq \{\mathbf{v} \in V \mid \forall \mathbf{s} \in S,\ \omega(\mathbf{v},\mathbf{s}) = 0\}$. A linear subspace $S \subseteq (V,\omega)$ is: **isotropic** if $S \subseteq S^\omega$, **coisotropic** if $S^\omega \subseteq S$, **Lagrangian** if $S = S^\omega$. An affine subspace is isotropic, coisotropic, Lagrangian, or split if its linear component[2] is.

▶ **Proposition 14** ([18, Lem. 8])**.** *Nonempty affine Lagrangian subspaces $L + \mathbf{a} \subseteq (\mathbb{Z}_p^{2n}, \omega_n)$ are in bijection with stabiliser subgroups of $\mathcal{P}_p^n$ via $L + \mathbf{a} \longmapsto \{\pi(\omega_n(\mathbf{a},\mathbf{b}),\mathbf{b}) \mid \mathbf{b} \in L\}$.*

Cliffords are affine symplectic maps.

Clifford operators Clifford operators act on Paulis by conjugation, hence induce affine maps $\varphi(\mathbf{v}) = S\mathbf{v} + \mathbf{t}$ on $\mathbb{Z}_p^{2n}$ with $\omega_n(S\mathbf{v}, S\mathbf{w}) = \omega_n(\mathbf{v},\mathbf{w})$; for qubits we restrict to the block diagonal case $S(\mathbb{Z}_p^n \oplus 0) = \mathbb{Z}_p^n \oplus 0$ and $S(0 \oplus \mathbb{Z}_p^n) = 0 \oplus \mathbb{Z}_p^n$.

▶ **Proposition 15** ([18, Lem. 4])**.** *The Clifford group modulo global phase is isomorphic to the group of affine symplectic automorphisms of $(\mathbb{Z}_p^{2n}, \omega_n)$.*

For example

$$[\![X]\!](x,z) = (x{+}1, z), \quad [\![Z]\!](x,z) = (x, z{+}1), \quad [\![C^X]\!]\big((x,z),(x',z')\big) = \big((x, z{-}z'),(x{+}x', z')\big);$$

and for odd p $\qquad [\![S]\!](x,z) = (x+z, z), \quad [\![H]\!](x,z) = (z,-x).$

[1] A linear subspace $S \subseteq (\mathbb{Z}_p^{2n}, \omega_n)$ is **split** if $S = (S \cap (\mathbb{Z}_p^n \oplus 0)) + (S \cap (0 \oplus \mathbb{Z}_p^n))$.

[2] The **linear component** of a non-empty affine subspace $S \subseteq \mathbb{Z}_p^n$ is the linear subspace $\{\mathbf{x} - \mathbf{y} \mid \mathbf{x},\mathbf{y} \in S\} \subseteq \mathbb{Z}_p^n$.

Affine Lagrangian relations.

Affine Lagrangian subspaces are closed under relational composition [19, Eq. 9.6]. This yields a †-compact category ALR whose morphisms $(\mathbb{Z}_p^{2n}, \omega_n) \to (\mathbb{Z}_p^{2m}, \omega_m)$ are affine Lagrangian subspaces of $(\mathbb{Z}_p^{2n} \oplus \mathbb{Z}_p^{2m}, (-\omega_n) \oplus \omega_m)$.

▶ **Theorem 16** ([14, 24]). *There is an equivalence* $\mathsf{ALR} \simeq \mathsf{Stab}$ *modulo nonzero scalars.*

5 Mixed states and stabiliser codes

Pure state quantum mechanics is not expressive enough to encode quantum information redundantly, a feature which is *essential* for quantum error correction. Even worse, in pure state quantum mechanics, it is not even possible to discard information, which we have taken as a primitive operation in SPL. In this subsection we address these problems using the machinery of *mixed state quantum mechanics* and *stabiliser codes*.

Recall that a stabiliser group S uniquely determines a one dimensional subspace $\mathrm{Span}\{|S\rangle\} \subseteq \mathcal{H}_p^{\otimes n}$, hence a rank-one projector $|S\rangle\langle S| : \mathcal{H}_p^{\otimes n} \to \mathcal{H}_p^{\otimes n}$, called a *pure quantum state*. Note that using the internal hom, this can be regarded as an element of $\mathcal{H} \otimes \mathcal{H}^*$. As opposed to a stabiliser subgroup, a **stabiliser code** is a *non-maximal* subgroup of the Pauli group, and determines a higher-dimensional subspace of $\mathcal{H}_p^{\otimes n}$, called the **codespace**:

$$\mathcal{H}_G \coloneqq \mathrm{Span}\{|\varphi\rangle \in \mathcal{H}_p^{\otimes n} : \forall P \in G, P\,|\varphi\rangle = |\varphi\rangle\} \subseteq \mathcal{H}_p^{\otimes n}.$$

The projector onto the codespace $\mathcal{H}_G \subseteq \mathcal{H}_p^{\otimes n}$ is given by the operator

$$\Pi_G \coloneqq \frac{1}{|G|} \sum_{P \in G} P : \mathcal{H}_p^{\otimes n} \to \mathcal{H}_p^{\otimes n}.$$

Given an orthonormal basis $B = \{|\varphi_j\rangle\}_j$ for $\mathcal{H}_G$, the normalised state $\rho_G \coloneqq \Pi_G / \mathrm{Tr}(\Pi_G)$ can be written as a uniform mixture $\frac{1}{\dim(\mathcal{H}_G)} \sum |\varphi_j\rangle\langle\varphi_j|$ of pure states $|\varphi_j\rangle\langle\varphi_j|$ in $\mathcal{H} \otimes \mathcal{H}^*$. In quantum mechanics, such probabilistic mixtures of pure states are called *mixed states*; therefore, we can treat stabiliser codes within the framework of **mixed state quantum mechanics**.

Mixed state quantum mechanics is not only concerned with states, but also the *processes*; i.e. the linear maps $\mathcal{H} \otimes \mathcal{H}^* \to \mathcal{S} \otimes \mathcal{S}^*$, which can be implemented physically. Given a †-compact closed category $\mathcal{C}$, we recall the construction of the †-symmetric monoidal category $\mathrm{CPTP}(\mathcal{C})$ which captures mixed state quantum mechanics when restricted to $\mathcal{C} \coloneqq \mathsf{FHilb}$:

▶ **Definition 17** ([35, Def. 4.18]). *Let $\mathcal{C}$ be a †-compact closed category. The †-compact closed category $\mathrm{CP}(\mathcal{C})$ of abstract* ***completely positive maps*** *has the same objects as $\mathcal{C}$. The morphisms $X \to Y$ are given by maps of the following form in $\mathcal{C}$*

$$[S, g] \coloneqq (1_Y \otimes 1_{Y^*}) \circ (g \otimes (g^*)^\dagger) \circ (1_X \otimes \cup_S \otimes 1_{X^*}) : X \otimes X^* \to Y \otimes Y^*$$

for some $g : X \otimes S \to Y$ in $\mathcal{C}$, where $g^* \coloneqq (1_{Y^*} \otimes \cup_Y) \circ g \circ (\cup_X \otimes 1_{X^*})$ is the transpose of g.

Applying the following functor yields an abstract notion of pure states:

▶ **Lemma 18** ([35, Rem. 4.19]). *There is a functor* $\mathsf{CP}[-] : \mathcal{C} \to \mathrm{CP}(\mathcal{C})$ *sending $f \mapsto f \otimes (f^*)^\dagger$.*

We need to restrict to a subcategory whose morphisms are "normalised" to capture mixed-state quantum mechanics:

▶ **Definition 19** ([35, Def. 4.6]). *Define the symmetric monoidal subcategory* $\mathrm{CPTP}(\mathcal{C}) \subseteq \mathrm{CP}(\mathcal{C})$ *of abstract* ***completely positive trace-preserving maps*** *to have morphisms satisfying* $\mathrm{Tr}_X = \mathrm{Tr}_Y \circ f$*, where* $\mathrm{Tr}_Y \coloneqq \cap_{Y^*} \circ \mathtt{swap}_{Y,Y^*}$ *denotes the trace.*

We interpret the abstract trace as an operation which *discards* information. For $\mathcal{C} \coloneqq \mathsf{FHilb}$, the abstract trace becomes the linear-algebraic trace, meaning that $\mathrm{CPTP}(\mathsf{FHilb})$ reproduces the standard setting for mixed state quantum mechanics between quantum systems:

▶ **Proposition 20** ([35, Ex. 4.21]). $\mathrm{CPTP}(\mathsf{FHilb})$ *is equivalent to the category* CPTP *of completely positive trace-preserving maps between quantum systems.*

Stabiliser quantum operations as affine coisotropic relations

We now apply this construction in the novel setting of the stabiliser theory. We treat Stab and ALR in parallel: in the Hilbert-space model, "mixedness" is captured by completely positive maps; in the symplectic model, it will be captured by relaxing Lagrangian relations to coisotropic ones (where $R^\omega \subseteq R$).

▶ **Definition 21.** *The categories* $\mathsf{LR}, \mathsf{CR}, \mathsf{ACR}$ *are variants of* ALR *whose morphisms are respectively, linear Lagrangian, linear coisotropic and affine coisotropic subspaces.*

We will repeatedly pass from a relation $R : V \to W$ to its set-theoretic image in W. To express this internally as composition, we single out the "total state" $\mathrm{im}_V : I \to V$:

▶ **Definition 22.** *Denote the coisotropic relation which picks out the entire space as follows*

$$\mathrm{im}_{(V,\omega_V)} \coloneqq \{(\{\bullet\}, \mathbf{v}) \mid \forall \mathbf{v} \in V\} : (\mathbb{Z}_p^0, \omega_0) \rightarrowtail (V, \omega_V).$$

We use the notation $\mathrm{im}_{(V,\omega_V)}$ because postcomposition with an affine Lagrangian, or affine coisotropic relation $R : (V, \omega_V) \to (W, \omega_W)$ is identified with the set-theoretic image:

$$R \circ \mathrm{im}_{(V,\omega_V)} = \{(\{\bullet\}, \mathbf{w}) \mid \exists \mathbf{v} : (\mathbf{v}, \mathbf{w}) \in R\} = \mathbb{Z}_p^0 \times \mathrm{im}(R) \cong \mathrm{im}(R).$$

Indeed, this observation leads to a symplectic form of Stinespring dilation:

▶ **Lemma 23.** *Every non-empty affine coisotropic subspace of* $(\mathbb{Z}_p^{2n}, \omega_n)$ *of dimension* $2n-k$ *is the image of an affine Lagrangian isometry* $(\mathbb{Z}_p^{2(n-k)}, \omega_{n-k}) \rightarrowtail (\mathbb{Z}_p^{2n}, \omega_n)$.

Proof. We prove the claim for linear Lagrangian coisometries and coisotropic linear subspaces, after which the affine generalisation follows immediately.

Take a linear coisotropic subspace $S \subseteq (\mathbb{Z}_p^{2n}, \omega_n)$ of dimension $2n-k$. Then its symplectic complement $S^{\omega_n} \subseteq (\mathbb{Z}_p^{2n}, \omega_n)$ is an isotropic subspace of dimension k. Consider a basis of S^{ω_n} which by [31, Exercise 7.2.2-(a)] extends to a symplectic basis of $(\mathbb{Z}_p^{2n}, \omega_n)$. By [31, Corollary 7.1.5], this yields a symplectomorphism $\varphi : (\mathbb{Z}_p^{2n}, \omega_n) \xrightarrow{\cong} (\mathbb{Z}_p^{2n}, \omega_n)$ such that $\varphi(S^\omega) = \{(\mathbf{x}, \mathbf{0}_{2n-k}) \mid \mathbf{x} \in \mathbb{Z}_p^k\}$. In case S is split, then φ can be chosen to be block diagonal. Remark that $D \coloneqq \{(\mathbf{x}, \mathbf{0}_k) \mid \mathbf{x} \in \mathbb{Z}_p^k\} \subseteq (\mathbb{Z}_p^{2k}, \omega_k)$ is a Lagrangian subspace, inducing a Lagrangian coisometry $E \coloneqq D \times \mathbb{Z}_p^0 : (\mathbb{Z}_p^{2k}, \omega_k) \twoheadrightarrow (\mathbb{Z}_p^0, \omega_0)$. Therefore the following composite of Lagrangian coisometries is itself a Lagrangian coisometry:

$$C \coloneqq (E \times 1_{(\mathbb{Z}_p^{2(n-k)}, \omega_{n-k})}) \circ \mathrm{Gr}(\varphi) : (\mathbb{Z}_p^{2n}, \omega_n) \twoheadrightarrow (\mathbb{Z}_p^{2(n-k)}, \omega_{n-k}).$$

Taking the kernel of this Lagrangian coisometry, we have:

$$\begin{aligned}
\mathbb{Z}_p^0 \times \ker C &= C^\dagger(\{\mathbf{0}_{2(n-k)}\}) = (\mathrm{Gr}(\varphi)^\dagger \circ (E^\dagger \times 1_{(\mathbb{Z}_p^{2(n-k)}, \omega_{n-k})}))(\{\mathbf{0}_{2(n-k)}\}) \\
&= \mathrm{Gr}(\varphi)^\dagger \circ \{(\{\bullet\}, (\mathbf{d}, \mathbf{0}_{2(n-k)})) \mid \mathbf{d} \in D\} = \mathrm{Gr}(\varphi)^\dagger \circ \{(\{\bullet\}, (\mathbf{x}, \mathbf{0}_{2n-k})) \mid \mathbf{x} \in \mathbb{Z}_p^k\} \\
&= \{(\{\bullet\}, \varphi^{-1}(D \times \{\mathbf{0}_{2n-k}\}))\} = \{(\{\bullet\}, \mathbf{s}) \mid \mathbf{s} \in S^{\omega_n}\} = \mathbb{Z}_p^0 \times S^{\omega_n}.
\end{aligned}$$

Finally, by taking the double symplectic complement of $\mathbb{Z}_p^0 \times S$, we have the desired result

$$\begin{aligned}\mathbb{Z}_p^0 \times S &= ((\mathbb{Z}_p^0 \times S)^{-\omega_0 \times \omega_n})^{-\omega_0 \times \omega_n} = (\mathbb{Z}_p^0 \times \ker C)^{-\omega_0 \times \omega_n} \\ &= \mathbb{Z}_p^0 \times \mathrm{im}(C^\dagger) = C^\dagger \circ \mathrm{im}_{(\mathbb{Z}_p^{2(n-k)}, \omega_{n-k})}\,. \end{aligned}$$

◀

Using Lemma 23, we find that the "mixed symplectic" processes are affine coisotropic:

▶ **Proposition 24.** *There is an isomorphism* $\mathrm{CP}(\mathsf{ALR}) \cong \mathsf{ACR}$.

Proof. Define a functor $F : \mathrm{CP}(\mathsf{ALR}) \to \mathsf{ACR}$ by $[f, (S, \omega_S)] \mapsto f \circ (1 \oplus \mathrm{im}_{(S,\omega_S)})$. We prove the claim for the linear case, after which the affine case follows immediately.

Restricting F to the linear subcategories yields a †-compact functor $F : \mathrm{CP}(\mathsf{LR}) \to \mathsf{CR}$, which is identity-on-objects. Since both $\mathrm{CP}(\mathsf{LR})$ and CR are †-compact closed, it suffices to show that the induced map on states is a bijection. Surjectivity follows from Lemma 23. For injectivity, take Lagrangian relations $L : (S, \omega_S) \to (V, \omega_V)$ and $M : (T, \omega_T) \to (V, \omega_V)$ such that $\mathrm{im}(L) \neq \mathrm{im}(M)$.Then $\mathbf{v} \in \mathrm{im}(L)$ iff $(\mathbf{v}, \mathbf{v}) \in L \circ (L^*)^\dagger$, i.e. iff there exists $\mathbf{s} \in S$ with $(\mathbf{s}, \mathbf{v}) \in L$ and $(\mathbf{s}, \mathbf{v}) \in (L^*)^\dagger$. However, by assumption there is some $\mathbf{v} \in V$ such that $\mathbf{v} \in \mathrm{im}(L)$ and $\mathbf{v} \notin \mathrm{im}(M)$, so that $[L, (S, \omega_S)] \neq [M, (T, \omega_T)]$ in $\mathrm{CP}(\mathsf{LR})$. This proves that $F : \mathrm{CP}(\mathsf{LR}) \to \mathsf{CR}$ is injective on states, hence a †-compact isomorphism.

Finally, every affine Lagrangian relation $L + \mathbf{a} \subseteq V \times W$ is obtained from its linear part L by composing with translations on V and W; likewise every affine coisotropic relation is obtained from its linear part by composing with translations. Since translations are invertible in ALR (hence are sent by CP to invertible morphisms), the linear isomorphism extends along translations, yielding an equivalence $\mathrm{CP}(\mathsf{ALR}) \simeq \mathsf{ACR}$. ◀

Moreover, trace preservation also has a relational analogue:

▶ **Proposition 25.** $\mathrm{CPTP}(\mathsf{ALR})$ *is equivalent to the category* TACR *of* ***total*** *affine coisotropic relations, meaning that for all* $x \in X$*, there exists some* $y \in Y$ *such that* $(x, y) \in R$.

Proof. Consider the equivalence $F : \mathrm{CP}(\mathsf{ALR}) \simeq \mathsf{ACR}$ of Proposition 24. Write $\mathrm{im}_X : I \to X$ for the total relation $\{(\{\bullet\}, x) \mid x \in X\}$ and $\mathrm{im}_X^\dagger : X \to I$ for its dagger (discard). For a relation $R : X \to Y$ in ACR, we have $\mathrm{im}_Y^\dagger \circ R = \{(x, \bullet) \mid \exists y \in Y : (x, y) \in R\} \subseteq X \times I$, so $\mathrm{im}_Y^\dagger \circ R = \mathrm{im}_X^\dagger$ holds if and only if for every $x \in X$ there exists $y \in Y$ with $(x, y) \in R$.

Under F, the abstract trace Tr_X in $\mathrm{CP}(\mathsf{ALR})$ is sent to the discard relation $\mathrm{im}_X^\dagger$ in ACR. Therefore, transporting the trace-preservation condition $\mathrm{Tr}_Y \circ f = \mathrm{Tr}_X$ along F identifies $\mathrm{CPTP}(\mathsf{ALR})$ with the subcategory $\mathsf{TACR} \subseteq \mathsf{ACR}$ of total affine coisotropic relations. ◀

Both the standard and the symplectic notions of mixed stabiliser quantum processes coincide *exactly*, without even having to quotient by scalars:

▶ **Theorem 26.** *There is an equivalence* $\mathrm{CPTP}(\mathsf{Stab}) \simeq \mathsf{TACR}$.

Proof. Consider the equivalence $\mathsf{ALR} \cong \mathsf{Stab}$ modulo scalars of Theorem 16 and apply the CP construction. Transporting along Proposition 24 yields a symmetric monoidal functor

$$\mathrm{CPTP}(\mathsf{Stab}) \longrightarrow \mathrm{CPTP}(\mathsf{ALR}) \cong \mathsf{TACR},$$

where the final isomorphism is Proposition 25.

It remains to prove faithfulness. Since $\mathsf{ALR} \simeq \mathsf{Stab}$ holds modulo scalars, if two morphisms $f, g : X \to Y$ in $\mathrm{CPTP}(\mathsf{Stab})$ have the same image in $\mathrm{CPTP}(\mathsf{ALR})$ then $g = \lambda \cdot f$ for some nonzero scalar λ. Using trace preservation and centrality of scalars,

$$\mathrm{Tr}_Y \circ g \;=\; \mathrm{Tr}_Y \circ (\lambda \cdot f) \;=\; \lambda \cdot (\mathrm{Tr}_Y \circ f) \;=\; \lambda \cdot \mathrm{Tr}_X\,.$$

But also $\mathrm{Tr}_Y \circ g = \mathrm{Tr}_X$, hence $\lambda = 1$ and therefore $f = g$. Thus the functor is faithful. ◀

Note that CPTP(Stab) is generated by the Clifford operators, stabiliser states and the quantum trace on $\mathcal{H}_p$, where $[\![\mathrm{Tr}_{\mathcal{H}_p}]\!] = \left\{ \left(\left[\begin{smallmatrix} x \\ z \end{smallmatrix} \right], \bullet \right) \mid \left[\begin{smallmatrix} x \\ z \end{smallmatrix} \right] \in \mathbb{Z}_p^2 \right\} : (\mathbb{Z}_p^2, \omega_1) \to \{\bullet\}$.

6 Classical types and full-abstraction

SPL distinguishes between quantum and classical registers, taking measurement and classically controlled Pauli operators which interface between quantum and classical systems as primitive operations. However, in the preceding section, we have not described how classical systems can be formulated: neither within the framework of mixed-state quantum mechanics [25, Section 8.2] nor in terms of total affine coisotropic relations. In this section, we address this problem, adding a notion of classical system to the Hilbert space picture and the symplectic picture.

Fix an orthonormal basis $B = \{|\lambda_1\rangle, \ldots, |\lambda_n\rangle\}$. A state is classical with respect to B iff it is diagonal in B, i.e. $\rho = \sum_i p_i \, |\lambda_i\rangle\langle\lambda_i|$ for some probability distribution $(p_i)_i$. Measuring the quantum system $\mathcal{H}$ according to the basis B is said to *decohere* the system, producing a classical state with respect to B. In our operational semantics, this is exactly the probabilistic branching rule governed by the Born rule: measurement branches on an outcome j with probability $|\langle\lambda_j|\varphi\rangle|^2$, updating the residual state by normalisation. In mixed state quantum mechanics, the same branching behaviour is encoded by the completely positive trace-preserving map $\mathcal{E}_B : \mathcal{H} \otimes \mathcal{H}^* \to \mathcal{H} \otimes \mathcal{H}^*$:

$$\mathcal{E}_B(|\varphi\rangle\langle\varphi|) \coloneqq \sum_{j=1}^{n} |\lambda_j\rangle\langle\lambda_j| \, |\varphi\rangle\langle\varphi| \, |\lambda_j\rangle\langle\lambda_j| = \sum_{j=1}^{n} |\langle\lambda_j|\varphi\rangle|^2 \, |\lambda_j\rangle\langle\lambda_j| \, .$$

The diagonal states $|\lambda_j\rangle\langle\lambda_j| \in \mathcal{H} \otimes \mathcal{H}^*$ represent distinct measurement outcomes, and the coefficients $|\langle\lambda_j|\varphi\rangle|^2$ are the probabilities used by the branching rule in the operational semantics. However, this view does not yet distinguish between classical and quantum systems: not all mixed states $\rho \in \mathcal{H} \otimes \mathcal{H}^*$ are probabilistic mixtures of diagonal states, nor will all completely positive trace-preserving maps preserve this property of classicality.

In order to give denotational semantics to a quantum programming language, we seek to distinguish between quantum and classical systems so that the classicality is reflected at the level of *types*. The solution to this problem is to consider classical systems a *subsystems* of quantum systems. More precisely, as the subspace $\mathrm{Span}\{|\lambda_j\rangle\langle\lambda_j| : \forall\, |\lambda_j\rangle \in B\} \subseteq \mathcal{H} \otimes \mathcal{H}^*$. This is precisely the linear subspace that is fixed by the decoherence map $\mathcal{E}_B$, therefore we can think of a classical subsystem being specified not only by this subspace *but by a decoherence map itself.*

There is an abstract theory for quantum-classical systems, which can be formulated in any †-compact closed category. We will apply this notion both to stabiliser quantum processes (the operational semantics) and to their symplectic presentation (the denotational semantics). To this end, we first introduce the following notion following that of Selinger [36, Def. 3.6]:

▶ **Definition 27.** *An abstract* ***subsystem*** *of a quantum system X in $\mathrm{CP}(\mathcal{C})$ is a map $f : X \to X$ which is both self adjoint, $f^\dagger = f$, and idempotent, $f \circ f = f$.*

In the setting of finite-dimensional, mixed quantum theory, this produces what we expect:

▶ **Example 28** ([20, Thm. 2.5], [12, Prop. 3.5])**.** The abstract subsystems in CP(FHilb) are in bijection with tensor products of decoherence maps and identity maps.

The decoherence map in a basis corresponds to a fully classical subsystem of $\mathcal{H} \otimes \mathcal{H}^*$. On the other hand, the identity on $\mathcal{H}$ corresponds to the completely quantum system $\mathcal{H} \otimes \mathcal{H}^*$.

Tensor products of both of these classes of maps represent systems in which some subsystems are classical and others are quantum.

Just as before, we construct a category of processes between quantum and classical systems; which are captured by the following construction for $\mathcal{C} \coloneqq \mathsf{FHilb}$:

▶ **Definition 29** ([36, Def. 3.13])**.** *Given a $\dagger$-compact closed category $\mathcal{C}$, the category of abstract* ***quantum-classical operations*** $\mathrm{CPTP}_{\mathsf{qc}}(\mathcal{C})$ *has objects given by pairs (A, a) where A is an object of $\mathcal{C}$ and $a : A \to A$ is an abstract subsystem in $\mathrm{CP}(\mathcal{C})$. The morphisms $f : (A, a) \to (B, b)$ are given by morphisms $f : A \to B$ in $\mathrm{CPTP}(\mathcal{C})$ which are invariant under decoherence onto the input and output subsystems so that $b \circ f \circ a = f$.*

Applied to FHilb, this produces the quantum processes between quantum-classical systems:

▶ **Theorem 30** ([20, Thm. 2.5], [12, Prop. 3.5])**.** $\mathrm{CPTP}_{\mathsf{qc}}(\mathsf{FHilb})$ *is equivalent to the category* QChan *of quantum channels between quantum-classical systems (i.e. between C^*-algebras).*

Here, the decoherence map $\mathcal{E}_B$ corresponds to four different quantum channels:

- $\mathcal{E}_B : (\mathcal{H}, 1_{\mathcal{H}}) \to (\mathcal{H}, 1_{\mathcal{H}})$: which decoheres the quantum system in the basis B;
- $\mathcal{E}_B : (\mathcal{H}, \mathcal{E}_B) \to (\mathcal{H}, \mathcal{E}_B)$: which does nothing to the classical system;
- $\mathcal{E}_B : (\mathcal{H}, 1_{\mathcal{H}}) \to (\mathcal{H}, \mathcal{E}_B)$: which measures in the basis B;
- $\mathcal{E}_B : (\mathcal{H}, \mathcal{E}_B) \to (\mathcal{H}, 1_{\mathcal{H}})$: which prepares a pure state in the basis B.

Note that measurement followed by state preparation decoheres the quantum system; whereas, state preparation followed by measurement acts trivially on the classical system.

Classical systems in the stabiliser theory

We now apply this construction in the novel setting of the stabiliser theory. We proceed to analyse how to add classical systems to the stabiliser theory; applying this construction to our two categories of interest Stab and ALR, starting with Stab:

▶ **Lemma 31.** *The stabiliser quantum channels are generated by the stabiliser operations between quantum systems in addition to the maps which measure and prepare in the Pauli Z-basis $\{|n\rangle : n \in \mathbb{Z}_p\}$ and Pauli X-basis $\{H|n\rangle : n \in \mathbb{Z}_p\}$. Moreover, adding state preparation and measurement is equivalent to adding state preparation and classically controlled Paulis.*

See proof on page 21.

On the other hand applying this construction to the symplectic picture is equivalent to the following category, given by adding a **classical modality**[3] $!(-)$ to ACR:

▶ **Definition 32.** *Consider the induced embedding $!(-) : \mathsf{AR} \xrightarrow{\mathsf{CP}[-]} \mathrm{CP}(\mathsf{ALR}) \cong \mathsf{ACR}$. The SMC $\mathsf{AR}_{\mathsf{qc}}$ of* ***quantum-classical affine relations*** *whose morphisms consist of the total affine relations $!R : (!\mathbb{Z}_p^n) \to (!\mathbb{Z}_p^m)$, total affine coisotropic relations $S : (\mathbb{Z}_p^n, \omega_n) \to (\mathbb{Z}_p^m, \omega_m)$, in addition to four more total affine relations*

$$\mu_Z \coloneqq \left\{\left(\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right], x\right)\right\} : (\mathbb{Z}_p^2, \omega_1) \to (!\mathbb{Z}_p), \quad \mu_Z^\dagger \coloneqq \left\{\left(x, \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right]\right)\right\} : (!\mathbb{Z}_p) \to (\mathbb{Z}_p^2, \omega_1),$$
$$\mu_X \coloneqq \left\{\left(\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right], z\right)\right\} : (\mathbb{Z}_p^2, \omega_1) \to (!\mathbb{Z}_p), \quad \mu_X^\dagger \coloneqq \left\{\left(z, \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right]\right)\right\} : (!\mathbb{Z}_p) \to (\mathbb{Z}_p^2, \omega_1),$$

corresponding to the measurement and state preparation maps in the Pauli-Z and X bases.

[3] $!(-)$ is not a modality in the sense of being induced by adjoint *functors* [4]; rather, it witnesses the *posetal* adjunction $\mu_P \dashv \mu_P^\dagger$ given by $1_{(\mathbb{Z}_p^2, \omega_1)} \subseteq \mu_P^\dagger \circ \mu_P$ and $\mu_P \circ \mu_P^\dagger = 1_{!(\mathbb{Z}_p)}$.

▶ **Proposition 33.** *There is an isomorphism* $\mathsf{AR}_{\mathsf{qc}} \cong \mathrm{CPTP}_{\mathsf{qc}}(\mathsf{ALR})$.

Proof. Define the Z-decohering relations,

$$E_Z \coloneqq \left\{ \left(\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right], \left[\begin{smallmatrix} x \\ z' \end{smallmatrix}\right] \right) \in \mathbb{Z}_p^2 \times \mathbb{Z}_p^2 \;\middle|\; x, z, z' \in \mathbb{Z}_p \right\} : (\mathbb{Z}_p^2, \omega_1) \to (\mathbb{Z}_p^2, \omega_1).$$

Along the equivalence $\mathsf{ACR} \cong \mathrm{CP}(\mathsf{ALR})$ of Proposition 24, using symplectic Gaussian elimination ([31, Corollary 7.1.5]), it follows that subsystems in ACR are symplectomorphic to a direct sum $E_Z^{\oplus n} \oplus 1_{(\mathbb{Z}_p^{2m}, \omega_m)}$ for some $n, m \in \mathbb{N}$.

For the split case in characteristic 2, define the X-decohering relation

$$E_X \coloneqq \left\{ \left(\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right], \left[\begin{smallmatrix} x' \\ z \end{smallmatrix}\right] \right) \in \mathbb{Z}_p^2 \times \mathbb{Z}_p^2 \;\middle|\; x, x', z \in \mathbb{Z}_p \right\} : (\mathbb{Z}_p^2, \omega_1) \to (\mathbb{Z}_p^2, \omega_1).$$

Then via a restricted form of symplectic Gaussian elimination, every subsystem is symplectomorphic via block diagonal operators to $E_Z^{\oplus n} \oplus E_X^{\oplus m} \oplus 1_{(\mathbb{Z}_p^{2k}, \omega_k)}$ for some $n, m, k \in \mathbb{N}$.

By construction, $\mathrm{CPTP}_{\mathsf{qc}}(\mathsf{ALR})$ is generated by $\mathrm{CPTP}(\mathsf{ALR})$ together with the state preparation and measurement morphisms for all subsystems in $\mathrm{CP}(\mathsf{ALR})$. Therefore, the above normal form implies that $\mathrm{CPTP}_{\mathsf{qc}}(\mathsf{ALR})$ is generated by $\mathrm{CPTP}(\mathsf{ALR})$ together with state preparation and measurement for E_Z and E_X.

Along the equivalence $\mathsf{TACR} \cong \mathrm{CPTP}(\mathsf{ALR})$ of Proposition 25, these morphisms are given by μ_P and $\mu_P^\dagger$ for $P \in \{X, Z\}$, as they satisfy $\mu_P^\dagger \circ \mu_P = E_P$ and $\mu_P \circ \mu_P^\dagger = 1$ because $\mathrm{CP}_{\mathsf{qc}}(\mathsf{ALR})$ is defined to be the †-idempotent completion of $\mathrm{CP}(\mathsf{ALR})$ (see [36, Def. 3.3, & Def. 3.6]), it follows that $\mathrm{CPTP}_{\mathsf{qc}}(\mathsf{ALR})$ is generated by total affine coisotropic relations, total affine relations in the image of $!(-)$, as well as μ_P and $\mu_P^\dagger$ for $P \in \{X, Z\}$. ◀

Interpreting the types of SPL in $\mathsf{AR}_{\mathsf{qc}}$ as $[\![\mathbf{init}]\!] \coloneqq !(\mathbb{Z}_p)$ and $[\![\mathbf{qinit}]\!] \coloneqq (\mathbb{Z}_p^2, \omega_1)$, from Lemma 31 and Proposition 33 it follows immediately that:

▶ **Theorem 34** (Full definability). *For all typed environments* Γ *and* Δ *and morphisms* $f : [\![\Gamma]\!] \to [\![\Delta]\!]$ *in* $\mathsf{AR}_{\mathsf{qc}}$*, there exists a well-formed judgement* $\Gamma \vdash c \triangleright \Delta$ *such that* $[\![c]\!] = f$.

Now we relate the denotational semantics to the operational semantics, first observing:

▶ **Proposition 35.** *There is an equivalence* $\mathrm{CPTP}_{\mathsf{qc}}(\mathsf{Stab}) \simeq \mathsf{AR}_{\mathsf{qc}}$ *identifying stabiliser quantum channels between quantum-classical systems with quantum-classical relations.*

See proof on page 21.

▶ **Lemma 36.** *There is a mapping* $\langle - \rangle$ *of* SPL *terms into* $\mathrm{CPTP}_{\mathsf{qc}}(\mathsf{Stab})$ *such that* $c \approx_{\mathrm{obs}} d$ *if and only if* $\langle c \rangle = \langle d \rangle$.

See proof on page 21.

From Proposition 35 and Lemma 36 it follows that:

▶ **Theorem 37** (Full abstraction). *Well-formed judgements* c *and* d *are observationally equivalent if and only if their denotations are equal* $[\![c]\!] = [\![d]\!]$.

See proof on page 22.

References

1 Scott Aaronson and Daniel Gottesman. Improved simulation of stabilizer circuits. *Physical Review A*, 70(5), November 2004. arXiv:quant-ph/0406196, doi:10.1103/physreva.70.052328.

2 Rajeev Acharya, Dmitry A. Abanin, Laleh Aghababaie-Beni, Igor Aleiner, Trond I. Andersen, Markus Ansmann, Frank Arute, Kunal Arya, Abraham Asfaw, Nikita Astrakhantsev, Juan Atalaya, Ryan Babbush, Dave Bacon, Brian Ballard, Joseph C. Bardin, Johannes Bausch, Andreas Bengtsson, Alexander Bilmes, Sam Blackwell, Sergio Boixo, Gina Bortoli, Alexandre Bourassa, Jenna Bovaird, Leon Brill, Michael Broughton, David A. Browne, Brett Buchea, Bob B. Buckley, David A. Buell, Tim Burger, Brian Burkett, Nicholas Bushnell, Anthony Cabrera, Juan Campero, Hung-Shen Chang, Yu Chen, Zijun Chen, Ben Chiaro, Desmond Chik, Charina Chou, Jahan Claes, Agnetta Y. Cleland, Josh Cogan, Roberto Collins, Paul Conner, William Courtney, Alexander L. Crook, Ben Curtin, Sayan Das, Alex Davies, Laura De Lorenzo, Dripto M. Debroy, Sean Demura, Michel Devoret, Agustin Di Paolo, Paul Donohoe, Ilya Drozdov, Andrew Dunsworth, Clint Earle, Thomas Edlich, Alec Eickbusch, Aviv Moshe Elbag, Mahmoud Elzouka, Catherine Erickson, Lara Faoro, Edward Farhi, Vinicius S. Ferreira, Leslie Flores Burgos, Ebrahim Forati, Austin G. Fowler, Brooks Foxen, Suhas Ganjam, Gonzalo Garcia, Robert Gasca, Élie Genois, William Giang, Craig Gidney, Dar Gilboa, Raja Gosula, Alejandro Grajales Dau, Dietrich Graumann, Alex Greene, Jonathan A. Gross, Steve Habegger, John Hall, Michael C. Hamilton, Monica Hansen, Matthew P. Harrigan, Sean D. Harrington, Francisco J. H. Heras, Stephen Heslin, Paula Heu, Oscar Higgott, Gordon Hill, Jeremy Hilton, George Holland, Sabrina Hong, Hsin-Yuan Huang, Ashley Huff, William J. Huggins, Lev B. Ioffe, Sergei V. Isakov, Justin Iveland, Evan Jeffrey, Zhang Jiang, Cody Jones, Stephen Jordan, Chaitali Joshi, Pavol Juhas, Dvir Kafri, Hui Kang, Amir H. Karamlou, Kostyantyn Kechedzhi, Julian Kelly, Trupti Khaire, Tanuj Khattar, Mostafa Khezri, Seon Kim, Paul V. Klimov, Andrey R. Klots, Bryce Kobrin, Pushmeet Kohli, Alexander N. Korotkov, Fedor Kostritsa, Robin Kothari, Borislav Kozlovskii, John Mark Kreikebaum, Vladislav D. Kurilovich, Nathan Lacroix, David Landhuis, Tiano Lange-Dei, Brandon W. Langley, Pavel Laptev, Kim-Ming Lau, Loïck Le Guevel, Justin Ledford, Joonho Lee, Kenny Lee, Yuri D. Lensky, Shannon Leon, Brian J. Lester, Wing Yan Li, Yin Li, Alexander T. Lill, Wayne Liu, William P. Livingston, Aditya Locharla, Erik Lucero, Daniel Lundahl, Aaron Lunt, Sid Madhuk, Fionn D. Malone, Ashley Maloney, Salvatore Mandrà, James Manyika, Leigh S. Martin, Orion Martin, Steven Martin, Cameron Maxfield, Jarrod R. McClean, Matt McEwen, Seneca Meeks, Anthony Megrant, Xiao Mi, Kevin C. Miao, Amanda Mieszala, Reza Molavi, Sebastian Molina, Shirin Montazeri, Alexis Morvan, Ramis Movassagh, Wojciech Mruczkiewicz, Ofer Naaman, Matthew Neeley, Charles Neill, Ani Nersisyan, Hartmut Neven, Michael Newman, Jiun How Ng, Anthony Nguyen, Murray Nguyen, Chia-Hung Ni, Murphy Yuezhen Niu, Thomas E. O'Brien, William D. Oliver, Alex Opremcak, Kristoffer Ottosson, Andre Petukhov, Alex Pizzuto, John Platt, Rebecca Potter, Orion Pritchard, Leonid P. Pryadko, Chris Quintana, Ganesh Ramachandran, Matthew J. Reagor, John Redding, David M. Rhodes, Gabrielle Roberts, Eliott Rosenberg, Emma Rosenfeld, Pedram Roushan, Nicholas C. Rubin, Negar Saei, Daniel Sank, Kannan Sankaragomathi, Kevin J. Satzinger, Henry F. Schurkus, Christopher Schuster, Andrew W. Senior, Michael J. Shearn, Aaron Shorter, Noah Shutty, Vladimir Shvarts, Shraddha Singh, Volodymyr Sivak, Jindra Skruzny, Spencer Small, Vadim Smelyanskiy, W. Clarke Smith, Rolando D. Somma, Sofia Springer, George Sterling, Doug Strain, Jordan Suchard, Aaron Szasz, Alex Sztein, Douglas Thor, Alfredo Torres, M. Mert Torunbalci, Abeer Vaishnav, Justin Vargas, Sergey Vdovichev, Guifre Vidal, Benjamin Villalonga, Catherine Vollgraff Heidweiller, Steven Waltman, Shannon X. Wang, Brayden Ware, Kate Weber, Travis Weidel, Theodore White, Kristi Wong, Bryan W. K. Woo, Cheng Xing, Z. Jamie Yao, Ping Yeh, Bicheng Ying, Juhwan Yoo, Noureldin Yosri, Grayson Young, Adam Zalcman, Yaxing Zhang, Ningfeng Zhu, Nicholas Zobrist, and Google Quantum AI and Collaborators. Quantum error correction below the surface code threshold. *Nature*, 638(8052):920–926, 2025. Publisher: Nature Publishing Group. doi:10.1038/s41586-024-08449-y.

3 Rajeev Acharya, Igor Aleiner, Richard Allen, Trond I. Andersen, Markus Ansmann, Frank Arute, Kunal Arya, Abraham Asfaw, Juan Atalaya, Ryan Babbush, Dave Bacon, Joseph C. Bardin, Joao Basso, Andreas Bengtsson, Sergio Boixo, Gina Bortoli, Alexandre Bourassa,

Jenna Bovaird, Leon Brill, Michael Broughton, Bob B. Buckley, David A. Buell, Tim Burger, Brian Burkett, Nicholas Bushnell, Yu Chen, Zijun Chen, Ben Chiaro, Josh Cogan, Roberto Collins, Paul Conner, William Courtney, Alexander L. Crook, Ben Curtin, Dripto M. Debroy, Alexander Del Toro Barba, Sean Demura, Andrew Dunsworth, Daniel Eppens, Catherine Erickson, Lara Faoro, Edward Farhi, Reza Fatemi, Leslie Flores Burgos, Ebrahim Forati, Austin G. Fowler, Brooks Foxen, William Giang, Craig Gidney, Dar Gilboa, Marissa Giustina, Alejandro Grajales Dau, Jonathan A. Gross, Steve Habegger, Michael C. Hamilton, Matthew P. Harrigan, Sean D. Harrington, Oscar Higgott, Jeremy Hilton, Markus Hoffmann, Sabrina Hong, Trent Huang, Ashley Huff, William J. Huggins, Lev B. Ioffe, Sergei V. Isakov, Justin Iveland, Evan Jeffrey, Zhang Jiang, Cody Jones, Pavol Juhas, Dvir Kafri, Kostyantyn Kechedzhi, Julian Kelly, Tanuj Khattar, Mostafa Khezri, Mária Kieferová, Seon Kim, Alexei Kitaev, Paul V. Klimov, Andrey R. Klots, Alexander N. Korotkov, Fedor Kostritsa, John Mark Kreikebaum, David Landhuis, Pavel Laptev, Kim-Ming Lau, Lily Laws, Joonho Lee, Kenny Lee, Brian J. Lester, Alexander Lill, Wayne Liu, Aditya Locharla, Erik Lucero, Fionn D. Malone, Jeffrey Marshall, Orion Martin, Jarrod R. McClean, Trevor McCourt, Matt McEwen, Anthony Megrant, Bernardo Meurer Costa, Xiao Mi, Kevin C. Miao, Masoud Mohseni, Shirin Montazeri, Alexis Morvan, Emily Mount, Wojciech Mruczkiewicz, Ofer Naaman, Matthew Neeley, Charles Neill, Ani Nersisyan, Hartmut Neven, Michael Newman, Jiun How Ng, Anthony Nguyen, Murray Nguyen, Murphy Yuezhen Niu, Thomas E. O'Brien, Alex Opremcak, John Platt, Andre Petukhov, Rebecca Potter, Leonid P. Pryadko, Chris Quintana, Pedram Roushan, Nicholas C. Rubin, Negar Saei, Daniel Sank, Kannan Sankaragomathi, Kevin J. Satzinger, Henry F. Schurkus, Christopher Schuster, Michael J. Shearn, Aaron Shorter, Vladimir Shvarts, Jindra Skruzny, Vadim Smelyanskiy, W. Clarke Smith, George Sterling, Doug Strain, Marco Szalay, Alfredo Torres, Guifre Vidal, Benjamin Villalonga, Catherine Vollgraff Heidweiller, Theodore White, Cheng Xing, Z. Jamie Yao, Ping Yeh, Juhwan Yoo, Grayson Young, Adam Zalcman, Yaxing Zhang, Ningfeng Zhu, and Google Quantum AI. Suppressing quantum errors by scaling a surface code logical qubit. *Nature*, 614(7949):676–681, 2023. Publisher: Nature Publishing Group. doi:10.1038/s41586-022-05434-1.

4 P. N. Benton. *A mixed linear and non-linear logic: Proofs, terms and models: Extended abstract*, page 121–135. Springer Berlin Heidelberg, 1995. URL: https://ncatlab.org/nlab/files/BentonLinearLogic.pdf, doi:10.1007/bfb0022251.

5 Michael E. Beverland, Prakash Murali, Matthias Troyer, Krysta M. Svore, Torsten Hoefler, Vadym Kliuchnikov, Guang Hao Low, Mathias Soeken, Aarthi Sundaram, and Alexander Vaschillo. Assessing requirements to scale to practical quantum advantage, 2022. arXiv:2211.07629.

6 S. B. Bravyi and A. Yu. Kitaev. Quantum codes on a lattice with boundary, 1998. arXiv:quant-ph/9811052.

7 Sergey Bravyi, Andrew W. Cross, Jay M. Gambetta, Dmitri Maslov, Patrick Rall, and Theodore J. Yoder. High-threshold and low-overhead fault-tolerant quantum memory. *Nature*, 627(8005):778–782, 2024. Publisher: Nature Publishing Group. doi:10.1038/s41586-024-07107-7.

8 Paul Busch, Pekka Lahti, Juha-Pekka Pellonpää, and Kari Ylinen. *Quantum Measurement*. Springer International Publishing, 2016. doi:10.1007/978-3-319-43389-9.

9 A. R. Calderbank and Peter W. Shor. Good quantum error-correcting codes exist. *Physical Review A*, 54(2):1098–1105, August 1996. arXiv:quant-ph/9512032, doi:10.1103/physreva.54.1098.

10 Earl T. Campbell, Barbara M. Terhal, and Christophe Vuillot. Roads towards fault-tolerant universal quantum computation. *Nature*, 549(7671):172–179, 2017. Publisher: Nature Publishing Group. arXiv:1612.07330, doi:10.1038/nature23460.

11 Kean Chen, Yuhao Liu, Wang Fang, Jennifer Paykin, Xin-Chuan Wu, Albert Schmitz, Steve Zdancewic, and Gushu Li. *Verifying Fault-Tolerance of Quantum Error Correction Codes*, page 3–27. Springer Nature Switzerland, 2025. doi:10.1007/978-3-031-98685-7_1.

12 Bob Coecke, Chris Heunen, and Aleks Kissinger. Categories of quantum and classical channels. *Quantum Information Processing*, 15(12):5179–5209, October 2014. arXiv:1305.3821, doi:10.1007/s11128-014-0837-4.

13 Cole Comfort. Stabiliser programming language (SPL). https://github.com/ColeComfort/SPL, 2025. GitHub repository.

14 Cole Comfort and Aleks Kissinger. A graphical calculus for Lagrangian relations. In *Electronic Proceedings in Theoretical Computer Science*, volume 372, pages 338–351. EPTCS, 2022. doi:10.4204/EPTCS.372.24.

15 Eric Dennis, Alexei Kitaev, Andrew Landahl, and John Preskill. Topological quantum memory. *Journal of Mathematical Physics*, 43(9):4452–4505, September 2002. arXiv:quant-ph/0110143, doi:10.1063/1.1499754.

16 Wang Fang and Mingsheng Ying. Symbolic execution for quantum error correction programs. *Proceedings of the ACM on Programming Languages*, 8:189:1040–189:1065, 2024. doi:10.1145/3656419.

17 Daniel Gottesman. An introduction to quantum error correction and fault-tolerant quantum computation, 2009. arXiv:0904.2557.

18 David Gross. Hudson's theorem for finite-dimensional quantum systems. *Journal of Mathematical Physics*, 47(12), December 2006. arXiv:quant-ph/0602001, doi:10.1063/1.2393152.

19 Victor Guillemin and Shlomo Sternberg. Some problems in integral geometry and some related problems in micro-local analysis. *American Journal of Mathematics*, 101(4):915, August 1979. URL: http://dx.doi.org/10.2307/2373923, doi:10.2307/2373923.

20 Chris Heunen, Aleks Kissinger, and Peter Selinger. Completely positive projections and biproducts. *Electronic Proceedings in Theoretical Computer Science*, 171:71–83, 2014. doi:10.4204/EPTCS.171.7.

21 Qifan Huang, Li Zhou, Wang Fang, Mengyu Zhao, and Mingsheng Ying. Efficient formal verification of quantum error correcting programs. *Proceedings of the ACM on Programming Languages*, 9:1068–1093, 2025. doi:10.1145/3729293.

22 A.Yu. Kitaev. Fault-tolerant quantum computation by anyons. *Annals of Physics*, 303(1):2–30, January 2003. arXiv:quant-ph/9707021, doi:10.1016/s0003-4916(02)00018-0.

23 Ian Mackie. Lilac: a functional programming language based on linear logic. *Journal of Functional Programming*, 4(4):395–433, 1994. doi:10.1017/S0956796800001131.

24 Yurii A. Neretin. *Lectures on Gaussian Integral Operators and Classical Groups*. EMS Press, February 2011. doi:10.4171/080.

25 Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information: 10th Anniversary Edition*. Cambridge University Press, June 2012. doi:10.1017/cbo9780511976667.

26 Alexandru Paler, Ilia Polian, Kae Nemoto, and Simon J Devitt. Fault-tolerant, high-level quantum circuits: form, compilation and description. *Quantum Science and Technology*, 2(2):025003, April 2017. doi:10.1088/2058-9565/aa66eb.

27 Jennifer Paykin and Sam Winnick. Qudit quantum programming with projective cliffords. *Proc. ACM Program. Lang.*, 10(POPL):89–116, January 2026. doi:10.1145/3776646.

28 Robert Rand, Aarthi Sundaram, Kartik Singhal, and Brad Lackey. Gottesman types for quantum programs. In *Proceedings of the 17th International Conference on Quantum Physics and Logic (QPL 2020)*, volume 340, page 279–290. Open Publishing Association, September 2021. doi:10.4204/eptcs.340.14.

29 Benjamin Rodatz, Boldizsár Poór, and Aleks Kissinger. Fault tolerance by construction. arXiv:2506.17181.

30 Benjamin Rodatz, Boldizsár Poór, and Aleks Kissinger. Floquetifying stabiliser codes with distance-preserving rewrites. arXiv:2410.17240.

31 Gerd Rudolph and Matthias Schmidt. Linear symplectic algebra. In *Differential Geometry and Mathematical Physics: Part I. Manifolds, Lie Groups and Hamiltonian Sys-*

tems, Theoretical and Mathematical Physics, pages 315–352. Springer Netherlands, 2013. doi:10.1007/978-94-007-5345-7_7.

32 C. Ryan-Anderson, J. G. Bohnet, K. Lee, D. Gresh, A. Hankin, J. P. Gaebler, D. Francois, A. Chernoguzov, D. Lucchetti, N. C. Brown, T. M. Gatterman, S. K. Halit, K. Gilmore, J. A. Gerber, B. Neyenhuis, D. Hayes, and R. P. Stutz. Realization of real-time fault-tolerant quantum error correction. *Physical Review X*, 11(4):041058, 2021. Publisher: American Physical Society. doi:10.1103/PhysRevX.11.041058.

33 Maximilian Rüsch, Benjamin Rodatz, and Aleks Kissinger. Completeness for fault equivalence of clifford ZX diagrams. arXiv:2510.08477.

34 Peter Selinger. Towards a quantum programming language. *Mathematical. Structures in Comp. Sci.*, 14(4):527–586, August 2004. URL: https://www.mathstat.dal.ca/~selinger/papers/qpl.pdf, doi:10.1017/S0960129504004256.

35 Peter Selinger. Dagger compact closed categories and completely positive maps. *Electronic Notes in Theoretical Computer Science*, 170:139–163, March 2007. doi:10.1016/j.entcs.2006.12.018.

36 Peter Selinger. Idempotents in dagger categories. In *Proceedings of the 4th International Workshop on Quantum Programming Languages (QPL 2006)*, volume 210, pages 107–122, 2008. doi:10.1016/j.entcs.2008.04.021.

37 A. M. Steane. Error correcting codes in quantum theory. *Physical Review Letters*, 77(5):793–797, July 1996. doi:10.1103/physrevlett.77.793.

38 Aarthi Sundaram, Robert Rand, Kartik Singhal, and Brad Lackey. Hoare meets Heisenberg: A lightweight logic for quantum programs, 2025. arXiv:2101.08939.

39 Anbang Wu, Gushu Li, Hezi Zhang, Gian Giacomo Guerreschi, Yuan Xie, and Yufei Ding. QECV: Quantum error correction verification, 2021. arXiv:2111.13728.

40 Mingsheng Ying. *Foundations of Quantum Programming*. Elsevier Inc., 2016. doi:10.1016/C2014-0-02660-3.

41 Uma Zalakain and Ornela Dardha. π with leftovers: A mechanisation in agda. In *Formal Techniques for Distributed Objects, Components, and Systems*, page 157–174. Springer International Publishing, 2021. doi:10.1007/978-3-030-78089-0_9.

A Operational semantics

For completeness, we begin by presenting a conventional operational semantics for SPL. In any quantum programming language, measurements are inherently probabilistic, hence properly describing their operational behaviour requires care. Measurements in SPL are *destructive* measurements in the computational basis: they consume a qupit to produce a pit. Furthermore, they are governed by the *Born* rule: given a (normalised) quantum state $|\psi\rangle$, a measurement of the quantum register $\underline{x}$ yields the outcome $k \in \mathbb{Z}_p$ with probability $\mathrm{Prob}[k \mid |\psi\rangle] \coloneqq \| \langle k|_{\underline{x}} |\psi\rangle \|^2$, and the (renormalised) state of the system after the measurement is $\langle k|_{\underline{x}} |\psi\rangle \,/\, \| \langle k|_{\underline{x}} |\psi\rangle \|$.

▶ **Definition 38.** *Given an environment* Γ*, a* ***state*** *is a pair* $(\sigma, |\psi\rangle)$ *consisting of a classical store* $\sigma : \Gamma^{-1}(\mathtt{pit}) \to \mathbb{Z}_p$ *and a* normalised *vector* $|\psi\rangle \in \bigotimes_{\Gamma^{-1}(\mathtt{qpit})} \mathcal{H}_p$*. We identify states* $(\sigma, |\psi\rangle)$ *and* $(\sigma, |\psi'\rangle)$ *when there is a complex phase* $\theta \in [0, 2\pi)$ *such that* $|\psi\rangle = \exp(i\theta)\,|\psi'\rangle$*.*

▶ **Definition 39.** *A* ***configuration*** *is a pair* $\left[\, \Gamma \vdash t \triangleright \Delta \;\middle|\; \sigma, |\psi\rangle \,\right]$*, or* $\left[\, t \;\middle|\; \sigma, |\psi\rangle \,\right]$ *for short, consisting of a well-formed judgement* $\Gamma \vdash t \triangleright \Delta$ *and a state* $(\sigma, |\psi\rangle)$ *of* Γ*.*

The operational semantics of SPL is given by the big-step reduction $\left[\, t \;\middle|\; \sigma, \psi \,\right] \Downarrow \mu$ defined in figure 6, where μ is a probability distribution on states for the output environment. In order to clarify presentation, we write such a probability distribution as $\boxplus_i (p_i, \sigma_i, |\psi_i\rangle)$ where $p_i \in [0, 1]$ is the probability of the corresponding state, but emphasise that this is purely for

$$\frac{\left[\, c \;\middle|\; \sigma, |\psi\rangle \,\right] \Downarrow \boxplus_i (p_i, \sigma_i, |\psi_i\rangle) \quad \left[\, d \;\middle|\; \sigma_i, |\psi_i\rangle \,\right] \Downarrow \boxplus_j (p'_{ij}, \sigma'_{ij}, |\psi'_{ij}\rangle)}{\left[\, c \fatsemi d \;\middle|\; \sigma, |\psi\rangle \,\right] \Downarrow \boxplus_j \left(\sum_i p_i p'_{ij}, \sigma'_{ij}, |\psi'_{ij}\rangle \right)}$$

$$\left[\, \mathbf{init}\ \underline{\mathtt{x}} \;\middle|\; \sigma, |\psi\rangle \,\right] \Downarrow (1, \sigma[\underline{\mathtt{x}} := 0], |\psi\rangle) \qquad \left[\, \underline{\mathbf{y}} \mathrel{+}= A * \underline{\mathbf{x}} \;\middle|\; \sigma, |\psi\rangle \,\right] \Downarrow (1, \sigma[\underline{\mathbf{y}} \mathrel{+}= A * \underline{\mathbf{x}}], |\psi\rangle)$$

$$\left[\, \mathbf{disc}\ \underline{\mathtt{x}} \;\middle|\; \sigma, |\psi\rangle \,\right] \Downarrow (1, \sigma \setminus \underline{\mathtt{x}}, |\psi\rangle) \qquad \left[\, \underline{\mathbf{x}} \mathrel{*}= U \;\middle|\; \sigma, |\psi\rangle \,\right] \Downarrow (1, \sigma, U_{\underline{\mathbf{x}}} |\psi\rangle)$$

$$\left[\, \mathbf{qinit}_Z\ \underline{\mathtt{x}} \;\middle|\; \sigma, |\psi\rangle \,\right] \Downarrow (1, \sigma, |\psi\rangle \otimes |0\rangle_{\underline{\mathtt{x}}}) \qquad \left[\, \mathbf{qinit}_X\ \underline{\mathtt{x}} \;\middle|\; \sigma, |\psi\rangle \,\right] \Downarrow (1, \sigma, |\psi\rangle \otimes H\,|0\rangle_{\underline{\mathtt{x}}})$$

$$\left[\, \mathbf{ctrl}_P\ \underline{\mathtt{x}}\ \underline{\mathtt{y}} \;\middle|\; \sigma, |\psi\rangle \,\right] \Downarrow (1, \sigma, P_{\underline{\mathtt{y}}}^{\sigma(\underline{\mathtt{x}})} |\psi\rangle) \qquad \left[\, \mathbf{skip} \;\middle|\; \sigma, |\psi\rangle \,\right] \Downarrow (1, \sigma, |\psi\rangle)$$

$$\left[\, \mathbf{meas}_Z\ \underline{\mathtt{x}} \;\middle|\; \sigma, |\psi\rangle \,\right] \Downarrow \boxplus_{k \in \mathbb{Z}_p} (\mathrm{Prob}[k \mid |\psi\rangle], \sigma[\underline{\mathtt{x}} := k], \langle k|_{\underline{\mathtt{x}}} |\psi\rangle \;/\; \| \langle k|_{\underline{\mathtt{x}}} |\psi\rangle \|)$$

$$\left[\, \mathbf{meas}_X\ \underline{\mathtt{x}} \;\middle|\; \sigma, |\psi\rangle \,\right] \Downarrow \boxplus_{k \in \mathbb{Z}_p} (\mathrm{Prob}[k \mid H\,|\psi\rangle], \sigma[\underline{\mathtt{x}} := k], \langle k|_{\underline{\mathtt{x}}} H\,|\psi\rangle \;/\; \| \langle k|_{\underline{\mathtt{x}}} H\,|\psi\rangle \|)$$

Figure 6 Operational semantics of SPL. The symbols $U_{\underline{\mathbf{x}}}$ and $P_{\underline{\mathtt{y}}}$ represent the application of the unitaries U and P at the qupits indexed by the registers $\underline{\mathbf{x}}$ and $\underline{\mathtt{y}}$ respectively. $P^{\sigma(\underline{\mathtt{x}})}$ represents the repeated application of the unitary P parametrised by the classical store value $\sigma(\underline{\mathtt{x}}) \in \mathbb{Z}_p$. The only reduction rule that introduces probabilistic branching is the rule for **meas**.

notational convenience and that μ should be thought of as a function from states into the interval $[0, 1]$ (i.e. a genuine distribution).

This operational semantics is *intensional*, in the sense that it distinguishes computational behaviours that are physically indistinguishable. This is because different distributions over the quantum states $\boxplus_i (p_i, \sigma_i, |\psi_i\rangle)$ and $\boxplus_j (p'_j, \sigma'_j, |\psi'_j\rangle)$ can represent the same physical state. Nevertheless, by postcomposing with terms which only output classical data, we can rely on the standard definition of observational equivalence for probabilistic transition systems:

▶ **Definition 40.** *Let* Θ *be a* classical *environment containing no quantum registers, i.e.* $\Theta^{-1}(\mathtt{qpit}) = \varnothing$. *Well-formed* SPL *terms* $\Sigma \vdash c \triangleright \Delta$ *and* $\Sigma \vdash d \triangleright \Delta$ *are* ***observationally equivalent*** *if for any well-formed terms* $\Gamma \vdash a \triangleright \Sigma$ *and* $\Delta \vdash b \triangleright \Theta$, $\left[\, a \fatsemi c \fatsemi b \;\middle|\; \sigma, |\psi\rangle \,\right] \Downarrow \mu$ *implies* $\left[\, a \fatsemi d \fatsemi b \;\middle|\; \sigma, |\psi\rangle \,\right] \Downarrow \mu$, *and vice-versa.*

We emphasise that in this definition μ is a probability distribution $\boxplus_i (p_i, \sigma_i, |\,\rangle)$ on the output classical values, i.e. a classical probability distribution, since there is no output quantum state after sequencing with Θ. This corresponds to the standard physical identification of quantum systems: two quantum states are identified if and only if they produce the same statistics for all possible measurements.

As we shall see, this *extensional* operational semantics (up to observational equivalence) *does* capture the correct behaviour of quantum systems under measurement. This is because the set of measurements in the stabiliser fragment is sufficiently strong to distinguish arbitrary quantum states, not just stabiliser ones.[4] This is made formal by lemma 36.

▶ Remark 41. *Density operators* provide the standard device in quantum physics for reasoning about probability distributions over quantum states. This amounts to quotienting our semantics by the equivalence relation induced by observational equivalence. We have chosen to give our operational semantics in a language which should be more familiar to the reader familiar with probabilistic programming, whilst avoiding the definition of density operators and operations on them,

[4] In the language of quantum information theory, stabiliser measurements are *informationally complete*.

B Proofs for Section 6: Measurement and classical types

▶ **Lemma 31.** *The stabiliser quantum channels are generated by the stabiliser operations between quantum systems in addition to the maps which measure and prepare in the Pauli Z-basis $\{|n\rangle : n \in \mathbb{Z}_p\}$ and Pauli X-basis $\{H|n\rangle : n \in \mathbb{Z}_p\}$. Moreover, adding state preparation and measurement is equivalent to adding state preparation and classically controlled Paulis.*

Proof of Lemma 31. We prove both claims for Pauli Z-measurements and classically controlled X-gates in the Pauli Z basis, from which the result follows.

We prove the first claim. First observe that $\mathrm{CPTP}_{\mathsf{qc}}(\mathsf{Stab})$ is generated by $\mathrm{CPTP}(\mathsf{Stab})$ together with the state preparation and measurement morphisms for all subsystems in $\mathrm{CP}(\mathsf{Stab})$. Moreover because stabiliser codes are uniform mixtures of pure stabiliser states, and all pure stabiliser states are unitarily isomorphic via Clifford operators to $|0\rangle$ all subsystems in $\mathrm{CP}(\mathsf{Stab})$ are unitarily isomorphic, via conjugation with Clifford operators, to those of the form $(\mathcal{H}_p^{\otimes(n+m)}, \mathcal{E}_Z^{\otimes n} \otimes 1_{\mathcal{H}_p}^{\otimes m})$. Therefore, $\mathrm{CPTP}_{\mathsf{qc}}(\mathsf{Stab})$ is generated by the stabiliser quantum operations between quantum systems in addition to the Pauli-Z state preparation and measurement operations.

For the second claim, note that applying a classically controlled Pauli-X operator in the Pauli-Z basis to the state $|0\rangle$ and then discarding the classical pit is the same as state preparation in the Pauli Z-basis. Conversely, a classically controlled Pauli-X gate in the Z-basis can be performed by preparing the first qupit in the Pauli-Z basis, applying a quantum controlled X gate controlled by this prepared qupit and then measuring the same qupit in the Pauli Z-basis. ◀

▶ **Theorem 34** (Full definability)**.** *For all typed environments Γ and Δ and morphisms $f : [\![\Gamma]\!] \to [\![\Delta]\!]$ in $\mathsf{AR}_{\mathsf{qc}}$, there exists a well-formed judgement $\Gamma \vdash c \triangleright \Delta$ such that $[\![c]\!] = f$.*

▶ **Proposition 35.** *There is an equivalence $\mathrm{CPTP}_{\mathsf{qc}}(\mathsf{Stab}) \simeq \mathsf{AR}_{\mathsf{qc}}$ identifying stabiliser quantum channels between quantum-classical systems with quantum-classical relations.*

Proof of Proposition 35. By Theorem 26, $\mathrm{CPTP}(\mathsf{Stab}) \simeq \mathsf{TACR}$ modulo scalars and by Proposition 25, $\mathsf{TACR} \cong \mathrm{CPTP}(\mathsf{ALR})$; therefore by transitivity, $\mathrm{CPTP}(\mathsf{Stab}) \simeq \mathrm{CPTP}(\mathsf{ALR})$ modulo scalars. Because quotienting by scalars commutes with $\mathrm{CPTP}_{\mathsf{qc}}$, it follows that $\mathrm{CPTP}_{\mathsf{qc}}(\mathsf{Stab}) \simeq \mathrm{CPTP}_{\mathsf{qc}}(\mathsf{ALR})$ modulo scalars. Moreover, by Proposition 33 $\mathrm{CPTP}_{\mathsf{qc}}(\mathsf{ALR}) \simeq \mathsf{AR}_{\mathsf{qc}}$. By transitivity, it follows that $\mathrm{CPTP}_{\mathsf{qc}}(\mathsf{Stab}) \simeq \mathsf{AR}_{\mathsf{qc}}$ modulo scalars. By the same argument as in Theorem 26, it follows that this induces an honest equivalence $\mathrm{CPTP}_{\mathsf{qc}}(\mathsf{Stab}) \simeq \mathsf{AR}_{\mathsf{qc}}$ without quotienting by scalars. ◀

▶ **Lemma 36.** *There is a mapping $\langle - \rangle$ of* SPL *terms into $\mathrm{CPTP}_{\mathsf{qc}}(\mathsf{Stab})$ such that $c \approx_{\mathrm{obs}} d$ if and only if $\langle c \rangle = \langle d \rangle$.*

Proof of Lemma 36. Given a well-formed judgement $\Gamma \vdash c \triangleright \Delta$, for each state $(\sigma, |\psi\rangle)$ for Γ, there must be some probability distribution μ on the states for Δ such that $\left[\, c \;\middle|\; \sigma, |\psi\rangle \,\right] \Downarrow \mu$, and by construction of $\Downarrow$ this μ is unique.

Now, both the state $(\sigma, |\psi\rangle)$ and the probability distribution μ yield density operators: we encode classical data as a diagonal density operator then form the tensor product with $|\psi\rangle\langle\psi|$. μ then corresponds to a statistical ensemble of pure states, which canonically corresponds to a density operator [25, Section 2.4].[5] This yields a mapping from (pure) input density operators to output density operators.

[5] Recall that the mapping from statistical ensemble μ is surjective onto density operators but not injective.

It is straightforward to verify that each of the generating terms corresponds in this way to a map in $\mathrm{CPTP}_{\mathsf{qc}}(\mathsf{Stab})$, and that that the sequence rule corresponds to composition in $\mathrm{CPTP}_{\mathsf{qc}}(\mathsf{Stab})$. This gives the mapping $\langle - \rangle$

Finally, observational equivalence yields exactly the condition that all measurements in $\mathrm{CPTP}_{\mathsf{qc}}(\mathsf{Stab})$ have identical outcome probabilities, which uniquely characterises the output density operator. Hence, if $c \approx_{\mathrm{obs}} d$, the corresponding $\langle c \rangle, \langle d \rangle$ map every input density operator to the same output density operator, i.e. they are equal. ◀

▶ **Theorem 37** (Full abstraction)**.** *Well-formed judgements c and d are observationally equivalent if and only if their denotations are equal $[\![c]\!] = [\![d]\!]$.*

Proof of Theorem 37. By Lemma 36, we have $c \approx_{\mathrm{obs}} d$ if and only if $\langle c \rangle = \langle d \rangle$ in $\mathrm{CPTP}_{\mathsf{qc}}(\mathsf{Stab})$. Transporting along the equivalence $\mathrm{CPTP}_{\mathsf{qc}}(\mathsf{Stab}) \simeq \mathsf{AR}_{\mathsf{qc}}$ of Proposition 35, we have $\langle c \rangle = \langle d \rangle$ if and only if $[\![c]\!] = [\![d]\!]$. Combining these equivalences gives the result. ◀