Higher-order quantum computing with known input state

Vanessa Brzić[1] Satoshi Yoshida[2] Mio Murao[2] Marco Túlio Quintino[1]

[1] *Sorbonne Université, CNRS, LIP6, 75005 Paris, France*
[2] *Department of Physics, The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan*

Abstract.
In higher-order quantum computing (HOQC), one typically considers the universal transformation of unknown quantum operations, treated as blackboxes. It is also implicitly assumed that the resulting operation must act on arbitrary, and thus unknown, input states. In this work, we explore a variant of this framework in which the operation remains unknown, but the input state is fixed and known. We argue that this assumption is well-motivated in certain practical contexts, such as unitary programming, and show that classical knowledge of the input state can significantly enhance performance. We demonstrate that in the SAR protocol, this knowledge leads to an exponential advantage through a repeat-until-success strategy, highlighting the operational power of known-state higher-order transformations. Moreover, this assumption allows us to distinguish between protocols designed for pure, bipartite, and mixed states, and to identify the class of mixed states for which deterministic and exact implementation becomes possible. Finally, in the numerically tested regimes, general strategies offer no advantage over adaptive or sequential ones, suggesting that indefinite causal order does not help in the known-input setting.

1 Introduction

In higher-order quantum computing (HOQC) the operations themselves are treated as objects that can be transformed. Such transformations are carried out by what are known as higher-order quantum operations [1–3]. For example, in the case of unitary operations, a standard HOQC task is to transform an arbitrary unitary operation U into another unitary operation $f(U)$, where $f : \mathrm{SU}(d) \to \mathrm{SU}(d)$ is a function mapping unitary operators to unitary operators. The HOQC framework can be used, for instance, to design quantum circuits that invert an arbitrary unitary operation $f(U) = U^{-1}$, transpose it, $f(U) = U^T$, or take its complex conjugate, $f(U) = U^*$ [4–6]. Another relevant task that can be naturally analysed within the HOQO framework is unitary storage-and-retrieval (SAR) [7], also known as unitary learning [8], and closely related to unitary programming [9], which corresponds to the delayed input state variant of $f(U) = U$.

In this work, we consider higher-order quantum operations (HOQO) in a setting where the output operation is applied to a quantum state known to the user. This classical knowledge allows the HOQO to be specifically tailored to the given input state (see Fig. 1), leading to significant performance gains in certain tasks. Interestingly, we also identify tasks, such as probabilistic exact conjugation, for which knowledge of the input state offers no advantage.

Since the input state is now assumed to be known, the structure of the HOQO protocol depends on the nature of the state, whether it is pure known state implementation $U \mapsto f(U)\left|\psi\right\rangle\!\left\langle\psi\right| f(U)^\dagger$, mixed $U \mapsto f(U)\rho f(U)^\dagger$, or acting on part of a bipartite system $U \mapsto (\mathbb{1}_A \otimes f(U)_B)\left|\psi\right\rangle\!\left\langle\psi\right|_{AB}(\mathbb{1}_A \otimes f(U)_B^\dagger)$, with each case requiring a different characterization and leading to different optimal strategies. For bipartite protocols, we focus on the probabilistic case, which includes, as a special case, the single-subsystem pure-state $f(U)\left|\psi\right\rangle\!\left\langle\psi\right| f(U)^\dagger$ probabilistic implementation. Our results demonstrate that, the success probability increases with the amount of entanglement. For mixed states, we prove that a deterministic and exact implementation is possible using only a single call to the input unitary, for a certain class of states of the form $\rho = \eta\left|\psi\right\rangle\!\left\langle\psi\right| + \frac{1-\eta}{d}\mathbb{1}$, with the particular values of visibility η, as discussed in the Thm. 2.

2 HOQC with a single call of the input operation and known bipartite input state

Let us consider a class of protocols implementing a transformation on a part of a bipartite state, $U \mapsto f(U)_A|\psi\rangle\langle\psi|_{AB}f(U)_A^\dagger$, as depicted in Figure (1).

Theorem 1 *Given a single call of the d-dimensional unitary operation U_d, in the case of a supermap realized on a subsystem of a known* bipartite *pure input state $|\psi\rangle_{AB}$,*

- *the transposition operation U_d^T (and equivalently, the storage and retrieval protocol) can be implemented with the maximal success probability,*

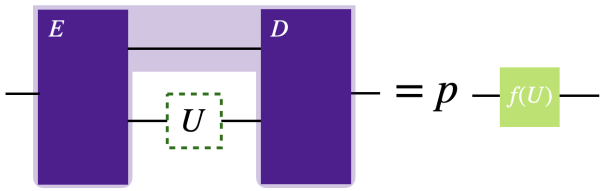
$$p^{\psi_{AB}}_{max,trans} = \frac{1}{d||\,\mathrm{Tr}_A(|\psi\rangle\langle\psi|_{AB})||_{op}} \tag{1}$$

- *the conjugation operation $\bar{U}_d$ can be implemented with maximal success probability*

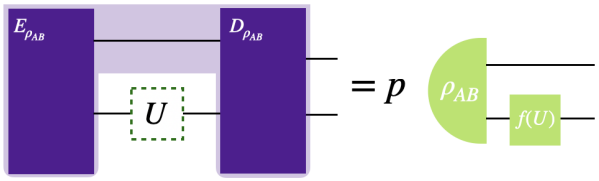
$$p^{\psi_{AB}}_{max,conjg} = \begin{cases} 1, & \text{if } d = 2 \\ 0, & \text{if } d > 2 \end{cases} \tag{2}$$

- *the inversion operation U_d^{-1} can be implemented with maximal success probability*

$$p^{\psi_{AB}}_{max,inv} = \begin{cases} \frac{1}{2||\,\mathrm{Tr}_A(|\psi\rangle_{AB}\langle\psi|)||_{op}}, & \text{if } d = 2 \\ 0, & \text{if } d > 2 \end{cases} \tag{3}$$



(a) Standard HOQC case where the input operation and the input state are unknown.



(b) The scenario considered in our work assumes classical knowledge of the input state, enabling the design of the encoder and decoder to implement a transformation of interest on the known program state.

Figure 1: A protocol implementing unitary transformation via action of a supermap. Elements shown in purple are optimized, dashed elements represent unknown and universal components, and green indicates that their actions are fixed as a consequence of the other elements.

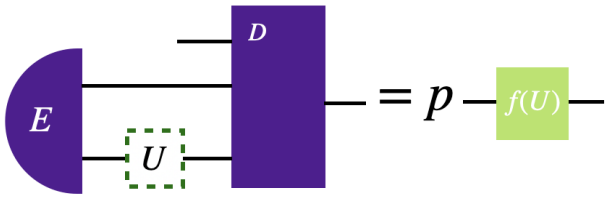
From these results, we see that greater entanglement with an external reference leads to a higher attainable success probability. In the limiting cases of no entanglement, we obtain the result for a supermap acting on a single pure unentangled state, $p(|\psi\rangle\langle\psi|_A \otimes \mathbb{1}_B) = \frac{1}{d}$. On the other hand, for a maximally entangled state, we obtain $p(|\phi^+\rangle\langle\phi^+|_{AB}) = 1$.

3 HOQC with a single call of the input operation and known mixed input state

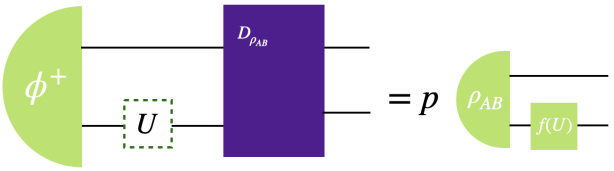
In the technical manuscript, make use of the optimal average fidelity of pure state presented above to prove the following theorem.

Theorem 2 *Let ρ be arbitrary, known state of the form $\rho = \eta|\psi\rangle\langle\psi| + \frac{1-\eta}{d}\mathbb{1}$. There exists a protocol such that we have deterministic and exact implementation of $f(U)\rho f(U)^\dagger$ with the visibility $\eta = \frac{d\langle\check{F}\rangle - 1}{d-1}$, where $\langle\check{F}\rangle$ is optimal average fidelity of the known pure state protocol. Then the smallest value of visibilities for which the perfect implementation is still possible are:*

- *transposition* $\eta = \frac{d}{d^2-1}$
- *conjugation* $\eta = \begin{cases} 1, & \text{if } d = 2 \\ \frac{1}{d+1} & \text{if } d > 2 \end{cases}$
- *inversion* $\eta = \frac{2}{d^2-1}$



(a) The standard HOQC case where the input operation and the input state are unknown and where the input state is delayed.



(b) The scenario considered in our work assumes that the known input state is delayed, arriving after the storage phase. This enables the decoder to be designed using classical knowledge of the state, in general enhancing the probability of successful retrieval.

Figure 2: A protocol implementing unitary transformation via action of a supermap with delayed input state.

4 Exponential advantage in known-state SAR protocol with multiple calls

The assumption of a known input state is particularly well motivated in the operational setting of unitary SAR. The storage-and-retrieval protocol (SAR) [7], also known as unitary learning [8], consists of a *storage phase*, in which access to the unknown operation is available and its action is encoded into a quantum memory, and a *retrieval phase*, in which the target state ρ becomes available but access to the operation is no longer possible. A retrieving channel acts on the stored memory and ρ with the goal of achieving $U\rho U^\dagger$. By the no-programming theorem [9], deterministic exact universal SAR is impossible for $\forall U \in \mathrm{SU}(d)$; thus one considers approximation or probabilistic exact retrieval.

The known-input variant is operationally natural: at retrieval time, the user typically knows the state on which they wish to retrieve the stored transformation. In our setting, the storage is universal and independent of the input, while the retrieval may exploit classical information available in the retrieval phase (a perspective comparable to remote state preparation versus teleportation). As shown in the main manuscript, known-state SAR remains equivalent to implementing transposition [4]. The SAR results extend from unitaries to general CPTP maps, as we further discuss in the main manuscript.

Theorem 3 *Given $k > 1$ call of the $d-$dimensional unitary operation U_d, in the case of a supermap realized on a known pure input state $|\psi\rangle\langle\psi|$, the transposition operation U_d^T, and equivalently storage and retrieval, can be implemented with success probability*

$$p_{SAR} \geq 1 - \left(1 - \frac{1}{d}\right)^k. \tag{4}$$

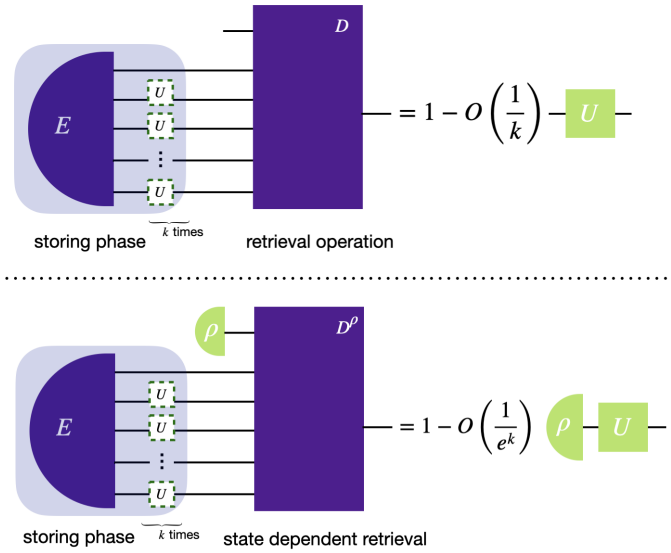


Figure 3: Top figure: SAR task with an unknown input state and k calls to the unknown unitary. The optimal success probability asymptotically in the number of calls k reaches $p = 1$ with $O(\frac{1}{k})$. Bottom figure: SAR task with a known input state. The success probability attainable via a repeat-until-success strategy asymptotically in k reaches $p = 1$ with $O\left(\frac{1}{e^k}\right)$.

From here, we see that the probability of success approaches one exponentially on the number of calls k. We remark that the optimal performance of unitary SAR for the unknown state case is $p = 1 - \left(1 - \frac{k}{k-1+d^2}\right)$, hence we have an exponential improvement.

5 HOQC with multiple calls

When multiple calls are available, we consider three classes of higher-order processes: parallel strategies, sequential strategies, and general strategies—where the input operations can be called without respecting a definite causal order [10–12], as depicted in Fig. 4. As shown in the main manuscript, the optimal performance of unitary transposition, inversion, and complex conjugation can be formulated as semidefinite programs and solved numerically. Here we summarise qualitative conclusions supported by the numerical optimisations (details and full data in the main manuscript).

For $k = 3$ and $d = 2$, both unitary transposition and inversion on an arbitrary known state can be implemented deterministically and exactly, even in the pure input state setting. This parallels the universal input state scenario, where $k = 4$ queries were required at $d = 2$ [13].

We numerically investigated the advantages of nonparallel strategies and found that, similarly to [14], sequential (adaptive) strategies outperform parallel ones, showing that causal order plays a nontrivial role. However, within the tested regime, general strategies appear to provide no advantage over sequential ones, in contrast to the unknown-input case, where indefinite causal order can outperform sequential protocols [4, 5].

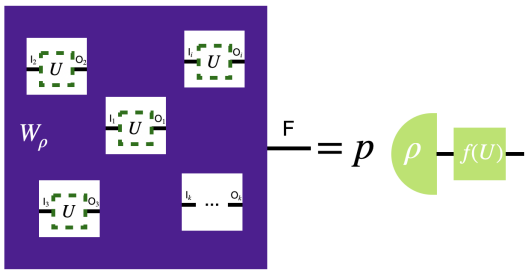


Figure 4: Known state supermap protocol realising $f(U)\rho f(U)^\dagger$ using k calls to the unknown unitary in a general strategy.

For tested regimes, numerical results indicate that the repeat-until-success strategy with $p_{\text{RUS}} = 1 - \left(1 - \frac{1}{d}\right)^k$ is optimal for both transposition and SAR in the pure known-input setting.

Furthermore, general strategies appear to offer no advantage over adaptive/sequential ones for the considered functions. By contrast, for unknown input states implementing inversion or transposition, general strategies with indefinite causal order can outperform sequential ones [4, 5].

6 Conclusions

- For protocols involving a single call to the input unitary, the implementations of transposition, SAR, and inversion exhibit a quadratic improvement—both in probabilistic exact and deterministic approximate realizations—compared to protocols designed for universal input states.
- In the case of protocols implementing probabilistic exact unitary conjugation, knowledge of the input state offers no advantage.
- For protocols implementing a desired transformation on part of a known bipartite state, the success probability increases with the amount of entanglement.
- For mixed states of the form $\rho = \eta|\psi\rangle\langle\psi| + \frac{1-\eta}{d}\mathbb{1}$, it is possible to construct deterministic and exact protocol for specific values of the visibility η, which is determined by the optimal average fidelity achieved in the corresponding pure-state protocol.
- In the known-state SAR protocol with multiple uses of the input unitary, an exponential advantage is observed over the corresponding unknown-state SAR protocol.
- Numerical results show that for $k = 3$ and $d = 2$, unitary transposition and inversion on an arbitrary known state can be performed deterministically and exactly, even in the pure-state case.
- In the numerically tested regimes, general strategies and hence indefinite causal order strategies appear to offer no advantage.

References

[1] P. Taranto, S. Milz, M. Murao, M. T. Quintino, and K. Modi, arXiv preprint (2025), arXiv:2503.09693 [quant-ph] .

[2] G. Chiribella, G. M. D'Ariano, and P. Perinotti, EPL (Europhysics Letters) **83**, 30004 (2008), arXiv:0804.0180 [quant-ph] .

[3] G. Chiribella, G. M. D'Ariano, and P. Perinotti, Phys. Rev. A **80**, 022339 (2009), arXiv:0904.4483 [quant-ph] .

[4] M. T. Quintino, Q. Dong, A. Shimbo, A. Soeda, and M. Murao, Phys. Rev. A **100**, 062339 (2019), arXiv:1909.01366 [quant-ph] .

[5] M. T. Quintino and D. Ebler, Quantum **6**, 679 (2022), arXiv:2109.08202 [quant-ph] .

[6] D. Ebler, M. Horodecki, M. Marciniak, T. Młynik, M. T. Quintino, and M. Studziński, IEEE Transactions on Information Theory **69**, 5069–5082 (2023), arXiv:2206.00107 [quant-ph] .

[7] M. Sedlák, A. Bisio, and M. Ziman, Phys. Rev. Lett. **122**, 170502 (2019), arXiv:1809.04552 [quant-ph] .

[8] A. Bisio, G. Chiribella, G. M. D'Ariano, S. Facchini, and P. Perinotti, Phys. Rev. A **81**, 032324 (2010).

[9] M. A. Nielsen and I. L. Chuang, Phys. Rev. Lett. **79**, 321–324 (1997), arXiv:quant-ph/9703032 [quant-ph] .

[10] M. Araújo, C. Branciard, F. Costa, A. Feix, C. Giarmatzi, and Č. Brukner, New Journal of Physics **17**, 102001 (2015), arXiv:1506.03776 [quant-ph] .

[11] G. Chiribella, G. M. D'Ariano, P. Perinotti, and B. Valiron, Phys. Rev. A **88**, 022318 (2013), arXiv:0912.0195 [quant-ph] .

[12] O. Oreshkov, F. Costa, and C. Brukner, Nature communications **3**, 1092 (2012), arXiv:1105.4464 [quant-ph] .

[13] S. Yoshida, A. Soeda, and M. Murao, Phys. Rev. Lett. **131**, 120602 (2023), arXiv:2209.02907 [quant-ph] .

[14] J. Bavaresco, M. Murao, and M. T. Quintino, Phys. Rev. Lett. **127**, 200504 (2021), arXiv:2011.08300 [quant-ph] .