A unified framework for Bell inequalities from continuous-variable contextuality

C. E. Lopetegui-González,[1,*] G. Massé,[2] E. Oudot,[2] U. I. Meyer,[2] F. Centrone,[3,4] F. Grosshans,[2] P. E. Emeriau,[5] U. Chabaud,[6] and M. Walschaers[1,†]

[1] *Laboratoire Kastler Brossel, Sorbonne Université, CNRS, ENS-Université PSL, Collège de France, 4 place Jussieu, Paris F-75252, France*
[2] *Sorbonne Université, CNRS, LIP6*
[3] *ICFO–Institut de Ciencies Fotoniques, The Barcelona Institute of Science and Technology*
[4] *Universidad de Buenos Aires, Instituto de Física de Buenos Aires (IFIBA)*
[5] *Quandela, 7 rue Léonard de Vinci, 91300 Massy, France*
[6] *DIENS, École Normale Supérieure, PSL University, CNRS, INRIA, 45 rue d'Ulm, Paris, 75005, France*

(Dated: March 6, 2026)

Link to the pre-print: A unified framework for Bell inequalities from continuous-variable contextuality.

It was the original insights of Einstein, Podolsky and Rosen [1] that steered the debate around the non-locality of the quantum theory. Though this initial discussion was phrased in term of position and momentum of harmonic oscillators, most of the theoretical proposals and experimental tests of non-locality based on Bell inequalities [2, 3], have dealt with discrete variable systems. Developing continuous variable Bell inequalities and proposing robust experimental tests has thus a natural foundational appealing. Also, homodyne detection of the quadratures of an electromagnetic field offers exceptional efficiencies that could permit to realize loophole-free Bell tests on continuous variable optical systems. However, this remains a big challenge, despite extensive theoretical efforts [4–12].

Most of the proposals for performing Bell tests on CV systems rely on imposing a predefined binning strategy, that maps the continuum of possible outcomes to either +1 or 1, to then apply the Clauser–Horne–Shimony–Holt (CHSH) inequality [3]. In [10], using similar binning strategies, the Mermin–Klyshko inequality [13, 14] was used to probe non-locality for more than two modes, which allows to obtain larger normalized Bell inequality violations than what is possible with CHSH [15]. Recent efforts have been devoted to automatizing the binning procedure [12], leading to considerable improvements in the expected violation of the local bounds. Nevertheless, the procedure is constrained to states with a limited number of photons.

A genuinely CV Bell inequality has only been proposed in [9], which does not rely on any discretization of the continuum of measurement outcomes, but rather on a non-linear combination of their moments. The down-side of this approach is that at least ten modes are required to obtain a violation of the local bound, which limits its experimental pertinence.

All the above approaches start from a Bell inequality without any guarantee that it will be practical for the measurement scenario at hand. Here, we introduce a general framework which takes a radically different approach in that it starts from experimental measurement data and finds the optimal Bell inequality associated with them. Our starting point is a recently proven CV version [16] of the Fine–Abramsky–Brandenburger theorem [17, 18], which shows that CV Bell non-locality is a specific instance of contextuality [18]. This connection provides a well-defined quantifier of the amount of non-locality of a given experiment, via its contextual fraction [15]. For local measurements scenarios, like the one described in Fig.1, the contextual fraction explains how much of the observed statistics cannot be explained through a local hidden- variable model. It is equivalent to the maximal normalized Bell inequality violation possible with the available data.

Both the value of the contextual fraction and the optimal Bell inequality can be obtained through a linear optimization program. In the CV setting, it becomes an infinite linear program, which cannot be solved exactly. In [16], a convergent hierarchy of relaxations of this infinite linear program was provided. This hierarchy consists of positive semidefinite optimization programs (SDP), based on the matrices of moments of the empirical data. However, after an in-depth numerical exploration of this approach, we find it to be unfruitful for detecting Bell non-locality in CV quantum experiments.

For this reason, we focus on a different relaxation of the infinite linear program to compute the contextual fraction. The relaxation is constructed by performing a fine–grained binning of the probability distributions of the measurement outcomes. This results in a hierarchy of finite linear optimization programs, whose results converge to the exact one by increasing the number of bins. Notice that this implies that we also discretize the outcomes, but not as a predefined

*Electronic address: carlos-ernesto.lopetegui-gonzalez@lkb.upmc.fr
†Electronic address: mattia.walschaers@lkb.upmc.fr

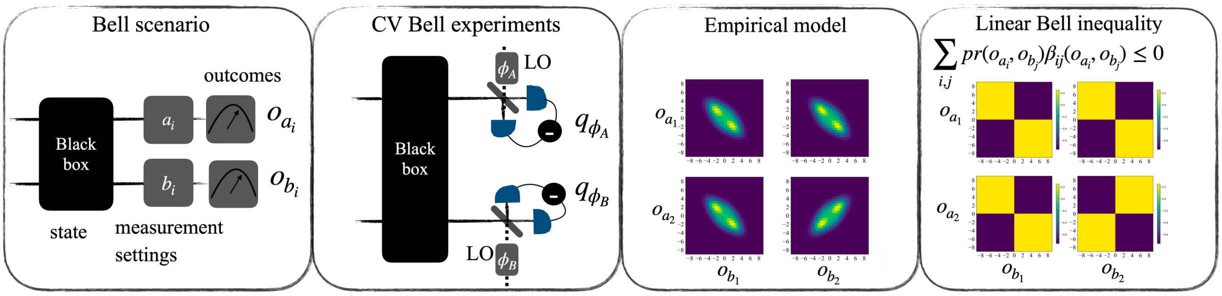

FIG. 1: From left to right: 1) description of a general Bell scenario for two parties. Each party performs a measurement among different available settings a_i (b_i), and obtain measurement outcomes $o_{a_i}(o_{b_i})$. 2) CV Bell experiment. Each party performs a homodyne detection measurement, setting different quadrature measurements by tuning the phases $\varphi_A(\varphi_B)$ of their local oscillators. 3) The data obtained from this kind of experiment usually comes in the form of histograms of commuting variables, or contexts. The set of all histograms provides an empirical model. 4) We test each empirical model for Bell non-locality (contextuality), and as a byproduct, obtain the optimal Bell inequality for the given empirical model. The plots represent the distribution of values for the *filters* β_{ij} that parametrize the Bell inequality. For the specific example shown, the optimal Bell inequality is equivalent to sign-binning the outcomes followed by CHSH.

physical post-processing to force a specific Bell inequality such as CHSH. Instead, binning is used as a convergent numerical relaxation of the underlying infinite linear program. In [16], it was proven that the contextual fraction decreases monotonously with such binning, which guarantees that a contextual fraction obtained for a finite number of bins provides a lower bound of the exact one. This approach allows us to develop a framework that, given access to experimental data, quantifies the amount of non-locality and provides the corresponding optimal Bell inequality.

Using our framework, we perform extensive numerical searches over potential CV experiments (testing both states and measurement settings). The implementation of the programs can be found as a package in [19]. As a result of our search we could confirm that binning strategies are in general optimal. In particular, we recover many existing results [6–8, 12], providing thus an argument for their optimality.

We also show several new examples of CV Bell violations. Finding states that display non-locality under homodyne detection proves to be a challenging task. Looking for such states involves a search over a huge parameter space. One would be tempted to think of this as an optimization problem: we could define a parameterized family of states and search for the one(s) providing the largest contextual fraction. However, the corresponding optimization landscape is mostly flat, i.e., for most states the contextual fraction is zero. This makes it impossible to find good directions to move along in the parameter space. In other words, we are looking for a needle in a haystack.

That being said, it is still possible to find interesting states. Notably, bosonic codes, *i.e.,* CV states used to encode qubits, offer an interesting proxy to search for Bell inequality violations. Among those codes, GKP states [20], are particularly well suited to find Bell inequalities based on homodyne detection, given that logical Pauli measurements are performed by homodyne detection. The Pauli Z measurement is performed through measuring the amplitude quadrature q, while the X is performed by measuring the phase quadrature p.

Inspired by these encodings we proposed several examples of entangled GKP states that produce a large Bell violation. Moreover we show that even when the code-words are of very low quality we can still preserve a large contextual fraction. This is the case for example if we use a squeezed vacuum state as the encoding for a logical $|0\rangle$ state, and a squeezed cat state as encoding for the logical $|1\rangle$. Preparing the state

$$|\psi_{qubit}\rangle = \frac{\big(|11\rangle - |00\rangle - (1+\sqrt{2})(|01\rangle + |10\rangle)\big)}{2\sqrt{2+\sqrt{2}}}. \tag{1}$$

from these codeword, and measuring the position and momentum quadratures on the two modes leads to an empirical model that displays $\sim 23\%$ contextuality. A non-zero contextual fraction is preserved even after the state goes through a loss channel, up to around 15% losses.

For the first time, we report an example of a two-mode non-local CV experiment with homodyne measurements, whose non-locality cannot be detected using the CHSH inequality, but requires a Bell inequality which is equivalent to the CGLMP Bell inequality for qutrits [21]. For this we relied on qutrit GKP code-words. For details about the state we refer to the main manuscript. In Fig.2, we show the empirical statistics corresponding to the state considered, as well as the filter functions that parametrize the optimal Bell inequality obtained. These filters display three different values, at difference of all previously considered binning strategies that could always be mapped to CHSH (see for example the filters shown in Fig.1, that correspond to performing sign binning of the homodyne outcomes).

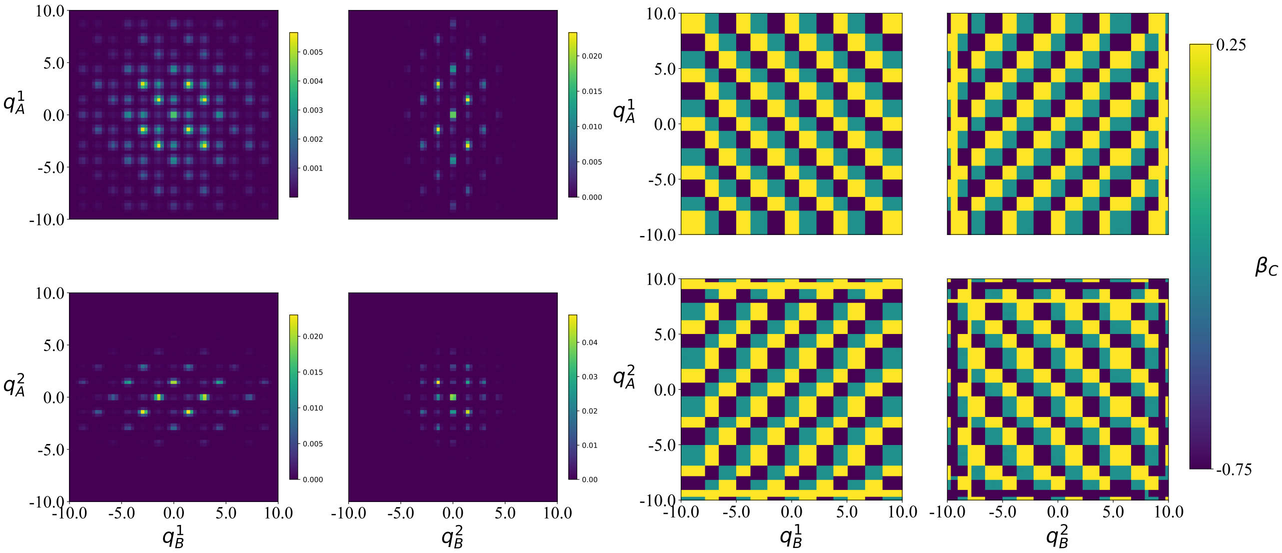

FIG. 2: To the left: joint histograms corresponding to the measurement scenario given by the homodyne settings $\varphi_{A,B} \in \{0, \pi/2\}$, on an entangled GKP qutrit state. To the right: Filter functions β_C that parameterize the optimal Bell inequality for this empirical model. A contextual fraction of 25.4% was obtained for this empirical model.

We also explore three-mode scenarios, inspired by the results in [15], where the emergence of maximal contextuality for GHZ states [22] was discussed. Considering a GHZ-like GKP encoded state we obtain contextual fractions of around 60.5%, well beyond the Tsirelson bound for CHSH [23].

Moreover, our method has the strength of being agnostic to the dimension or general details of the underlying physical system. This makes it an ideal tool for the study of hybrid DV-CV systems [24, 25], with the added value that the DV system can be treated exactly, while the CV part can be treated with a higher accuracy. We identify several simple hybrid quantum states for which a large contextual fraction can be obtained. On the technical level, the methods described in this work are actually much more scalable with respect to the number of subsystems for hybrid or DV systems.

Beyond the technical interest that the results obtained in this work may bear, they offer several new insights into the non-locality of homodyne-detection-based experiments. In particular, our results question the existence and relevance of a genuinely continuous type of non-locality. Throughout all the examples that we have analyzed, a common feature emerges for all the optimal Bell inequalities: they are parametrized by discrete functions, *i.e.,* functions that evaluate to a finite number of values (2 or 3 in the cases we found). All the behaviors obtained can thus be simulated by experiments on discrete-variable systems. This hints at a discrete nature of homodyne-based non-locality. Given the extensive numerical search that we have performed, we conjecture that this may indeed be a general fact. However, as we have no formal proof, this clearly provides an interesting open question for further research.

Given that in all the examples considered the contextual fraction never increases by adding more than two homodyne settings per mode, we can furthermore conjecture that no advantage can be obtained from considering larger numbers of settings. We have not been able to prove (or disprove) this claim, and we also leave it as an open question.

Overall, the search for states and homodyne measurement settings that provide non-local behavior has proven intensive, and in many ways feels like looking for a needle in a haystack. This implies that some states may have potentially been overlooked in our search. Our results provide a powerful new tool to be used in this kind of systematic searches, but it also highlights that there is still a significant lack of understanding of CV non-locality.

[1] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935), URL `https://link.aps.org/doi/10.1103/PhysRev.47.777`.

[2] J. S. Bell, Physics Physique Fizika **1**, 195 (1964), URL `https://link.aps.org/doi/10.1103/PhysicsPhysiqueFizika.1.195`.

[3] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. **23**, 880 (1969), URL `https://link.aps.org/doi/10.1103/PhysRevLett.23.880`.

[4] A. Gilchrist, P. Deuar, and M. D. Reid, Phys. Rev. Lett. **80**, 3169 (1998), URL https://link.aps.org/doi/10.1103/PhysRevLett.80.3169.

[5] W. J. Munro, Phys. Rev. A **59**, 4197 (1999), URL https://link.aps.org/doi/10.1103/PhysRevA.59.4197.

[6] J. Wenger, M. Hafezi, F. Grosshans, R. Tualle-Brouri, and P. Grangier, Phys. Rev. A **67**, 012105 (2003), URL https://link.aps.org/doi/10.1103/PhysRevA.67.012105.

[7] R. García-Patrón, J. Fiurášek, N. J. Cerf, J. Wenger, R. Tualle-Brouri, and P. Grangier, Phys. Rev. Lett. **93**, 130409 (2004), URL https://link.aps.org/doi/10.1103/PhysRevLett.93.130409.

[8] R. García-Patrón, J. Fiurášek, and N. J. Cerf, Phys. Rev. A **71**, 022105 (2005), URL https://link.aps.org/doi/10.1103/PhysRevA.71.022105.

[9] E. G. Cavalcanti, C. J. Foster, M. D. Reid, and P. D. Drummond, Phys. Rev. Lett. **99**, 210405 (2007), URL https://link.aps.org/doi/10.1103/PhysRevLett.99.210405.

[10] A. Acín, N. J. Cerf, A. Ferraro, and J. Niset, Phys. Rev. A **79**, 012112 (2009), URL https://link.aps.org/doi/10.1103/PhysRevA.79.012112.

[11] J. Etesse, R. Blandino, B. Kanseri, and R. Tualle-Brouri, New Journal of Physics **16**, 053001 (2014), URL https://doi.org/10.1088/1367-2630/16/5/053001.

[12] E. Oudot, G. Massé, X. Valcarce, and A. Acín, *Realistic bell tests with homodyne measurements* (2024), 2402.01530, URL https://arxiv.org/abs/2402.01530.

[13] N. D. Mermin, Phys. Rev. Lett. **65**, 1838 (1990), URL https://link.aps.org/doi/10.1103/PhysRevLett.65.1838.

[14] D. Klyshko, Physics Letters A **218**, 119 (1996), ISSN 0375-9601, URL https://www.sciencedirect.com/science/article/pii/0375960196004446.

[15] S. Abramsky, R. S. Barbosa, and S. Mansfield, Phys. Rev. Lett. **119**, 050504 (2017), URL https://link.aps.org/doi/10.1103/PhysRevLett.119.050504.

[16] R. S. Barbosa, T. Douce, P.-E. Emeriau, E. Kashefi, and S. Mansfield, Commun. Math. Phys. **391**, 1047 (2022), ISSN 0010-3616, 1432-0916, URL http://arxiv.org/abs/1905.08267.

[17] A. Fine, Phys. Rev. Lett. **48**, 291 (1982), URL https://link.aps.org/doi/10.1103/PhysRevLett.48.291.

[18] S. Abramsky and A. Brandenburger, New Journal of Physics **13**, 113036 (2011), URL https://doi.org/10.1088/1367-2630/13/11/113036.

[19] C. E. Lopetegui-González, *CV_contextuality* (2026), repository of the code used in this article, archived by Software Heritage as swh:1:dir:5351f8e69770cdb28cc2b81bd7b4bcbd965b349d, URL https://archive.softwareheritage.org/swh:1:dir:5351f8e69770cdb28cc2b81bd7b4bcbd965b349d.

[20] D. Gottesman, A. Kitaev, and J. Preskill, Phys. Rev. A **64**, 012310 (2001), URL https://link.aps.org/doi/10.1103/PhysRevA.64.012310.

[21] D. Collins, N. Gisin, N. Linden, S. Massar, and S. Popescu, Phys. Rev. Lett. **88**, 040404 (2002), URL https://link.aps.org/doi/10.1103/PhysRevLett.88.040404.

[22] D. M. Greenberger, M. A. Horne, A. Shimony, and A. Zeilinger, American Journal of Physics **58**, 1131 (1990), ISSN 0002-9505, URL https://doi.org/10.1119/1.16243.

[23] B. S. Tsirel'son, Journal of Soviet Mathematics **36**, 557 (1987), ISSN 1573-8795, URL https://doi.org/10.1007/BF01663472.

[24] A. Cavaillès, H. Le Jeannic, J. Raskop, G. Guccione, D. Markham, E. Diamanti, M. D. Shaw, V. B. Verma, S. W. Nam, and J. Laurat, Phys. Rev. Lett. **121**, 170403 (2018), URL https://link.aps.org/doi/10.1103/PhysRevLett.121.170403.

[25] M. Moradi, J. Camilo López Carreño, A. Buraczewski, T. McDermott, B. Elisabeth Asenbeck, J. Laurat, and M. Stobińska, New Journal of Physics **26**, 033019 (2024), URL https://doi.org/10.1088/1367-2630/ad2d40.