Probing the composition of processes with first-order-ISOMIX logic

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1 Introduction and context

1.1 Motivation: composition of quantum supermaps

Quantum supermaps [1] are higher-order quantum transformations that map one or several quantum channels to another quantum channel. Quantum supermaps generalise parallel and serial composition of quantum channels while allowing for new, more general types of compositions. For instance, the order of composition can be coherently controlled by quantum systems, as in the quantum switch [2], leading to quantum supermaps with so-called indefinite causal order [3]. Such supermaps can provide advantages in various computational tasks [4, 5], motivating interest in understanding interesting classes of quantum supermaps [6] and exploring their information-processing capabilities.

In information-theoretic tasks, it is often crucial to be able to compose the fundamental objects: quantum circuits and computations are obtained by composing gates in parallel or sequence [7], while communication protocols often involve composing together several sub-protocols or tasks. It has been understood for some time, however, that more care must be taken when composing together higher-order quantum operations to ensure the causal compatibility of the objects being composed, and hence the consistency of the resulting objects [8]. Several recent works have made significant advances in understanding the composition of quantum supermaps. Working in the process matrix framework [3], which allows supermaps to be represented as positive semidefinite (PSD) matrices in the Choi picture, Refs. [9, 10] characterise the valid transformations between higher-order quantum operations – which are themselves higher-order operations.

A different, more abstract perspective that has been pursued is the study of causal categories [11, 12]. Simmons and Kissinger notably introduced the Causal Logic in [13] as a correct and complete logic to decide causal consistency in the framework of causal categories. In Causal Logic, a formula F is valid according to a proof-net criterion if and only if F is causally consistent. When applying the $Caus[-]$ construction to the category of completely positive (CP) maps between finite-dimensional C* algebras, the resulting causal category describes CP maps between sets of PSD matrices – i.e., the (iterated) Choi representation of higher-order quantum operations. Causal Logic thereby provides a means to determine the validity of a potential composition of quantum supermaps, although the proof-net criterion is coNP-hard to verify.

Here, we pursue a problem motivated by considering a slightly more restrictive form of composition. As a motivation, consider the setting in which one is given a set of quantum supermaps – each (by definition) mapping several quantum channels to a quantum channel – and one would like to understand how these can be composed together using some simple, or "free", compositional procedures. For instance, one may wish to know what supermaps can be obtained by composition together quantum switches, where the compositional procedure itself is causally well-defined. In such a resource-theoretic view [14], a simple choice for free operations might be to consider just "wirings" between input systems of one process and output systems of another.

The validity of such compositions can be studied using Causal Logic, but this is computationally difficult and such problems have additional structure. For this specific task, it is in fact sufficient to consider a fragment of Causal Logic, which we call First-Order ISOMIX (FO-ISOMIX) due to its resemblance to ISOMIX logic. While, just as for Causal Logic, it was shown in [13] that verifying the validity of a formula in FO-ISOMIX is in general a coNP-complete problem, the analogous problem for ISOMIX formulas can be solved efficiently. Here we study the relation between these two logics more carefully, and in particular a simple mapping from FO-ISOMIX formulas F to ISOMIX formulas $\mathcal{HO}(F)$, obtained by replacing first-order variables by higher-order variables. While it is easy to show that if $\mathcal{HO}(F)$ is valid, then so is F, the converse does not hold in general. We show, however, that if F is valid and $\mathcal{HO}(F)$ is not valid, then F must contain a "forbidden structure", which can be detected in linear time by parsing the formula F. This provides an efficient pre-check when aiming to determine the validity of F: if F has no forbidden structure, then the non-validity of $\mathcal{HO}(F)$ – which can be checked efficiently – implies the non-validity of F.

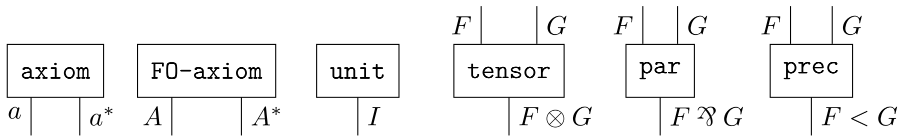
1.2 Causal Logic

A formula in Causal Logic [13] is defined according to the grammar

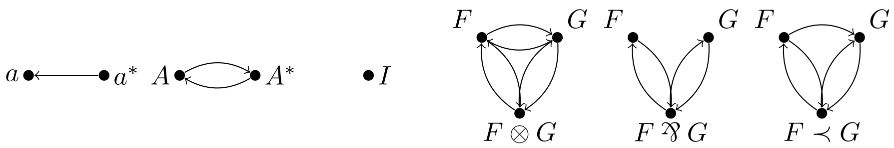
$$F, G ::= A, B, \dots \mid A^*, B^*, \dots \mid a, b, \dots \mid a^*, b^*, \dots \mid I \mid F \otimes G \mid F \parr G \mid F < G, \tag{1}$$

where variables $A, B, \dots$ and $a, b, \dots$ belong to two disjoint sets: regular variables (denoted in uppercase) and first-order (FO) variables (denoted in lowercase).

The validity of a Causal Logic formula is defined using a proof-net criterion. One recursively constructs syntactic string-diagram-like objects, called proof-structures, using the following "links":



Each wire is labelled with a formula, and the formulas labelling the input and output wires of each link must follow the rules depicted above. The wires whose bottom-ends are not attached to links are called the open wires. Given a proof-structure, one then considers maps s that take each link ℓ to an arc $s(\ell)$ between two input and/or output wires of this link. In this mapping, the allowed arcs depend on the corresponding type of this link, as depicted below.



Each map s that respects this constraint is called a switching function. Each switching function produces a graph whose nodes are the wires of the proof-structure and whose arcs are the $\{s(\ell)\}_\ell$. The correctness of a proof-structure is defined according to the cyclicity of these graphs: a proof-structure π is called a proof-net if for each switching function s of π, the resulting graph is acyclic. Finally, a formula is called valid (denoted $\vdash F$) when there exists a proof-net with exactly one open wire which is labelled with this formula.

1.3 ISOMIX and First-Order ISOMIX

The fragment of Causal Logic which we call FO-ISOMIX consists of the formulas defined over the grammar $F, G ::= a, b, \dots \mid a^*, b^*, \dots \mid I \mid F \otimes G \mid F \parr G$. A wiring between input and output Hilbert spaces of some quantum supermaps can be represented by an FO-ISOMIX formula U. Each FO variable of U is interpreted as a Hilbert space of a supermap, and for each pair of input and output Hilbert spaces wired together, their corresponding FO variables are connected with the $\parr$ connector. Valid compositions of supermaps of some given types correspond to a fixed formula R in FO-ISOMIX. A wiring, denoted by the formula U, then constitutes such a valid composition if and only if the formula $U \multimap R := U^* \parr R$ is valid [13].

Proof-net verification is the problem of checking if a given proof-structure is a proof-net. As mentioned earlier, proof-net verification for FO-ISOMIX is coNP-complete: it is easy to verify that some appropriate switching function s produces a cyclic graph, but finding such a switching function is typically difficult. On the other hand, for the fragment ISOMIX where formulas are defined over $F, G ::= A, B, \dots \mid A^*, B^*, \dots \mid I \mid F \otimes G \mid F \parr G$, validity corresponds to a simpler proof-net criterion, allowing one to verify correctness of a proof-net efficiently.

2 Results

2.1 Mapping FO-ISOMIX to ISOMIX

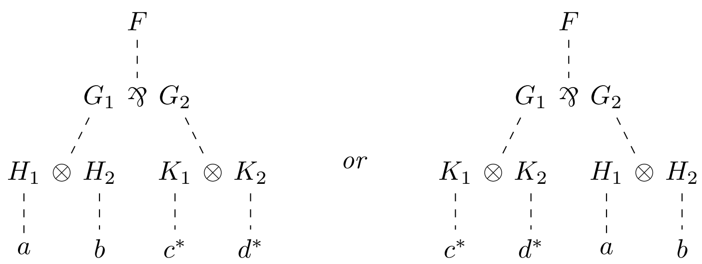
Because proof-net verification in ISOMIX is easier than in FO-ISOMIX, we consider a simple mapping of FO-ISOMIX formulas to ISOMIX formulas, allowing us in some cases to deduce the (in)validity of the FO-ISOMIX formula from the (in)validity of the corresponding ISOMIX formula. The mapping we consider takes an FO-ISOMIX formula F to an ISOMIX formula $\mathcal{HO}(F)$ by replacing first-order variables $a, b, \ldots$ by regular variables $\mathcal{HO}(a), \mathcal{HO}(b), \ldots$ such that

$$\mathcal{HO}(a^*) = \mathcal{HO}(a)^*, \qquad \mathcal{HO}(I) = I,$$
$$\mathcal{HO}(F \otimes G) = \mathcal{HO}(F) \otimes \mathcal{HO}(G), \qquad \mathcal{HO}(F \parr G) = \mathcal{HO}(F) \parr \mathcal{HO}(G).$$

A straightforward consequence of the proof-net criterion with this mapping is that given an FO-ISOMIX formula F, if $\vdash \mathcal{HO}(F)$ then $\vdash F$. The converse does not hold in general. Our main result is that formulas that do not satisfy the converse implication must contain a specific "forbidden" structure.

Definition 1. *We say that G and H are opposite subformulas of a formula $F = F_1 \circ F_2$, where $\circ \in \{\otimes, \parr\}$, if G is a subformula of F_1 and H is a subformula of F_2, or vice versa.*

Definition 2. *We say that a formula F contains a forbidden structure if there exist $G = G_1 \parr G_2$, $H = H_1 \otimes H_2$ and $K = K_1 \otimes K_2$, as well as FO variables a, b, c, d such that G is a subformula of F, H and K are opposite subformulas of G, and a, b, c^*, d^* are subformulas of H_1, H_2, K_1, K_2, respectively:*



Theorem 3. *Let F be a formula such that $\vdash F$ and $\nvdash \mathcal{HO}(F)$. Then, F contains a forbidden structure.*

To prove this result, first prove an intermediate result concerning the structure of cycles in the graphs obtained from switching functions. We show that, given a proof-structure that is not a proof-net (i.e., in which there exists a switching function producing a cyclic graph), one can construct a switching function whose graph contains a cycle of a specific form.

Given an FO-ISOMIX formula, we show that one can compute whether this formula contains a forbidden structure in linear time by parsing the formula. This result allows one to perform an efficient pre-check when one wishes to determine whether $\vdash F$: if F does not contain a forbidden structure, then $\nvdash \mathcal{HO}(F)$ implies $\nvdash F$, and one can determine whether $\nvdash \mathcal{HO}(F)$ efficiently.

Finally, coming back to our motivating problem of wiring supermaps, we restrict ourselves to the case of formulas of the form $U \multimap R$. We prove that a given formula R contains a forbidden structure if and only iff there exists a formula U of this form such that $\vdash U \multimap R$ and $\nvdash \mathcal{HO}(U \multimap R)$.

3 Conclusion and outlook

Our results help understand the relationship between FO-ISOMIX and ISOMIX, and allow one to detect certain formulas whose (in)validity can be efficiently determined, in contrast to general FO-ISOMIX formulas. This has immediate applications to our motivating question of wiring together quantum supermaps. In a follow-up work, we are studying in detail the possible valid wirings of bipartite quantum supermaps. This is an important first step towards understanding how small supermaps can be consistently combined into more complex ones.

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The Causal Logic of Simmons and Kissinger [1] provides a sound and complete logic to decide causal consistency in the framework of causal categories. Given a formula F in Causal Logic, the validity of F (denoted $\vdash F$) can be determined according to a proof-net criterion, and F is logically valid if and only if it is causally consistent. In general, however, determining the validity of a balanced formula, i.e. verifying the correctness of a given proof-net, is a co-NP-complete problem.

Here, we interest ourselves in a slightly more specific problem: determining the validity of compositions of higher-order processes (e.g., the composition of multiple quantum supermaps). While Causal Logic can be used to address this problem, it is sufficient to study some pertinent fragments of the logic for this purpose. In Causal Logic, formulas can contain two sorts of variables: first-order variables, and regular (or higher-order) variables, in addition to the unit atom I, and connectors $\otimes$, $\parr$ and $\prec$. ISOMIX is the fragment of Causal Logic containing only formulas with regular variables, I and the connectors $\otimes$ and $\parr$, while First-Order-ISOMIX (FO-ISOMIX) is defined analogously, but with only first-order variables. FO-ISOMIX turns out to be the relevant fragment for the compositional problem described above, but formulas' validity remains coNP-hard to decide. On the other hand, the validity of (balanced) ISOMIX formulas can be efficiently determined.

We study the relation between ISOMIX and FO-ISOMIX, and in particular a mapping from formulas F of the latter to the former, $\mathcal{HO}(F)$, obtained by replacing first-order variables by higher-order variables. While it is easy to show that if $\mathcal{HO}(F)$ is valid, then so is F, the converse does not hold in general. We show, however, that if F is valid and $\mathcal{HO}(F)$ is not valid, then F must contain a "forbidden structure", which can be detected in linear time by parsing the formula F. This provides an efficient pre-check when aiming to determine the validity of F: if F has no forbidden structure, then the non-validity of $\mathcal{HO}(F)$ – which can be checked efficiently – implies the non-validity of F.

These results provide a criteria for determining when, e.g., the validity of the composition of higher-order processes can efficiently determined. In work in progress, we apply this to the composition of quantum supermaps, enumerating all possible compositions of two bipartite supermaps, and determining the sets of valid compositions. This provides a first step in the study of how small supermaps can be consistently combined into more complex ones.

Note: the following is a draft version of the paper in progress, provided for the reviewers and accompanying our extended abstract. The proofs and technical content below is essentially complete, but further work is required on the presentation.

1 Introduction

In this article we present a connection between two logics: ISOMIX with regular variables and ISOMIX with first order variables, which we call First-Order ISOMIX (FO-ISOMIX). ISOMIX and FO-ISOMIX are two fragments of Causal Logic, which is defined in [1].

Validity in FO-ISOMIX is defined in [1] using a proof-net criterion. First some string-diagram objects are defined, called proof-structures. A proof-structure is a finite collection of wires labelled with formulas, and connected together with links. Each proof-structure corresponds to a given multiset of formulas, called its conclusions: those formulas labelling wires which are attached only to one link. Then, a subset of proof-structures is defined, the proof-structures of this subset are called proof-nets. A formula in FO-ISOMIX is said to be valid when there exists a proof-net whose only conclusion is this formula. Verifying whether a given proof-structure is a proof-net is a co-NP-complete problem for FO-ISOMIX. Validity in ISOMIX is defined using a similar proof-net criterion, but checking whether a given ISOMIX proof-structure is a proof-net takes polynomial time.

We study a simple transformation from FO-ISOMIX formulas to ISOMIX formulas, which consists merely in injectively replacing the first-order variables of an FO-ISOMIX formula by regular variables. We show that if the resulting ISOMIX formula is valid, then the initial FO-ISOMIX formula is valid. This allows to pre-check whether a given proof-structure of FO-ISOMIX is a proof-net: if the ISOMIX proof-structure produced by this transformation is a proof-net, so is the initial FO-ISOMIX proof-structure. However this pre-check may fail (the ISOMIX proof-structure isn't a proof-net, but the FO-ISOMIX proof-structure is a proof-net). We show that in this case the initial FO-ISOMIX formula must present a specific syntactic feature, which we call a forbidden structure.

Finally, we consider the FO-ISOMIX formulas U which are of a specific form, which we call tensor of FO maps (as, in our goal application of FO-ISOMIX, the formulas of this form represent tensor products of quantum channels). In a later project, we will consider the following task: given an FO-ISOMIX formula R where each variable appears only in one literal, enumerate all tensor of FO maps U which have the same literals as R and such that $U \multimap R$ is valid. We show here that for this task, it is sufficient to use the transformation of $U \multimap R$ in an ISOMIX formula if and only if R doesn't contain a forbidden structure.

1.1 Motivation

We are interested in the task of finding all the valid "wirings" connecting two process matrices together in a deterministic way. By "wiring", we mean applying quantum channels between pairs of output and input Hilbert spaces of the process matrices.

This task can be tackled using FO-ISOMIX logic. A given formula R in FO-ISOMIX represents a valid connection between process matrices. Each wiring can be represented by a formula U in FO-ISOMIX, such that $U \multimap R$ is a valid FO-ISOMIX formula iff the wiring is a valid composition.

2 Notations

Notation 1. *A multiset on a set X is a function $X \to \mathbb{N}$. Given a function $f : X \to Y$, we denote the multiset $m : Y \to \mathbb{N}, y \mapsto \#f^{-1}(y)$ (the number of preimages of each element of Y by f) as $[f(x), x \in X]$. We denote the empty multiset (the zero function) as $[]$. Given two multisets m, m' on a set X, we denote $m + m' : y \mapsto m(y) + m'(y)$ (the usual sum of functions $X \to \mathbb{N}$) and we denote $m \cap m' : y \mapsto \min(m(y), m'(y))$. Given a multiset m on a set X and $x \in X$, we denote $x \in m$ for $m(x) > 0$ and $x \notin m$ for $m(x) = 0$. Given a multiset m on a set X and a function $f : X \to Y$ where the elements of Y can be summed together, we write $\sum_{x \in m} f(x)$ for $\sum_{x \in X} m(x)f(x)$ (i.e. the elements of m are summed with multiplicity).*

3 ISOMIX and FO-ISOMIX

We introduce Causal Logic and its two fragments ISOMIX and FO-ISOMIX.

3.1 Formulas

Definition 2 (Formulas)**.** *Let $\mathcal{V}$ and $\mathcal{V}_{FO}$ be two disjoint infinite sets of same cardinality. $\mathcal{V}$ is called the set of regular (or higher-order) variables and $\mathcal{V}_{FO}$ is called the set of first-order variables. A formula in Causal Logic is an implicitely parenthesized term constructed from the grammar:*

$$F, G ::= \mathcal{V}|\mathcal{V}^*|\mathcal{V}_{FO}|{\mathcal{V}_{FO}}^*|I|F \otimes G|F \parr G|F < G\ . \tag{1}$$

An atomic formula is a formula of one of the forms $\mathcal{V}|\mathcal{V}^*|\mathcal{V}_{FO}|{\mathcal{V}_{FO}}^*|I$.
ISOMIX is the fragment of Causal Logic which consists of the formulas over the grammar:

$$F, G ::= \mathcal{V}|\mathcal{V}^*|I|F \otimes G|F \parr G\ . \tag{2}$$

First-Order ISOMIX (FO-ISOMIX) is the fragment of Causal Logic which consists of the formulas over the grammar:

$$F, G ::= \mathcal{V}_{FO}|\mathcal{V}^*_{FO}|I|F \otimes G|F \parr G\ . \tag{3}$$

Definition 3 (Subformulas)**.** *Given a formula F, the set of direct subformulas, the set of proper subformulas and the set of subformulas of F, denoted D_F, P_F and S_F respectively here, are defined recursively as:*

$$\forall a \in \mathcal{V}_{FO}, D_a = D_{a^*} = \emptyset\ , \tag{4}$$
$$\forall A \in \mathcal{V}, D_A = D_{A^*} = \emptyset\ , \tag{5}$$
$$D_I = \emptyset\ , \tag{6}$$
$$D_{F<G} = D_{F\otimes G} = D_{F\parr G} = \{F, G\}\ , \tag{7}$$
$$\forall a \in \mathcal{V}_{FO}, P_a = P_{a^*} = \emptyset\ , \tag{8}$$
$$\forall A \in \mathcal{V}, P_A = P_{A^*} = \emptyset\ , \tag{9}$$
$$P_I = \emptyset\ , \tag{10}$$
$$P_{F<G} = P_{F\otimes G} = P_{F\parr G} = \{F, G\} \cup P_F \cup P_G \tag{11}$$
$$S_F = \{F\} \cup P_F\ . \tag{12}$$

3.2 Proof-nets

Definition 4 (Proof-structure)**.** *A proof-structure π in Causal Logic consists of the following data:*

- *a finite set W of wires and a finite set L of links,*
- *for each wire $x \in W$, a label $f(x)$ which is a formula,*
- *for each link $\ell \in L$, a type $t(\ell) \in \{\texttt{axiom}, \texttt{FO-axiom}, \texttt{unit}, \texttt{tensor}, \texttt{par}, \texttt{prec}\}$, a set of premisses $p(\ell) \subset W$ which is totally ordered from the leftmost to the rightmost premisse and a set of conclusions $c(\ell) \subset W$ which is totally ordered from the leftmost to the rightmost conclusion,*

such that each wire is the conclusion of exactly one link and the premisse of at most one link, and for each link $\ell \in L$:

- *if $t(\ell) = \texttt{axiom}$ then ℓ has no premisse and has exactly two conclusions x (left) and y (right) and there exists $A \in \mathcal{V}$ such that $f(x) = A$ and $f(y) = A^*$,*
- *if $t(\ell) = \texttt{FO-axiom}$ then ℓ has no premisse and has exactly two conclusions x (left) and y (right), and there exists $a \in \mathcal{V}_{FO}$ such that $f(x) = a$ and $f(y) = a^*$,*
- *if $t(\ell) = \texttt{unit}$ then ℓ has no premisse and has exactly one conclusion x and $f(x) = I$,*
- *if $t(\ell) = \texttt{tensor}$ then ℓ has exactly two premisses x (left) and y (right) and exactly one conclusion z and there exist two formulas F, G such that $f(x) = F$, $f(y) = G$ and $f(z) = F \otimes G$,*

- *if* $t(\ell) = \texttt{par}$ *then* ℓ *has exactly two premisses* x *(left) and* y *(right) and exactly one conclusion* z *and there exist two formulas* F, G *such that* $f(x) = F$*,* $f(y) = G$ *and* $f(z) = F \parr G$*,*

- *if* $t(\ell) = \texttt{prec}$ *then* ℓ *has exactly two premisses* x *(left) and* y *(right) and exactly one conclusion* z *and there exist two formulas* F, G *such that* $f(x) = F$*,* $f(y) = G$ *and* $f(z) = F < G$*.*

A wire $x \in W$ *is called open when is isn't a premisse of any link* $\ell \in L$*. We denote* Op *the set of open wires.*

The multiset of conclusion formulas is the multiset $[f_\pi(x), x \in \mathrm{Op}]$*. Given* Γ *a multiset of formulas, a proof-structure of* Γ *is a proof-structure whose multiset of conclusion formulas is* Γ*.*

A proof-structure π *is called an ISOMIX (respectively FO-ISOMIX) proof-structure when for each wire* $x \in W$*, the label* $f(x)$ *is an ISOMIX (respectively FO-ISOMIX) formula.*

Definition 5 (Switching graph)**.** *Let* π *be a proof-structure. A switching graph* $\mathcal{G}$ *of a proof-structure* π *is a directed graph whose set of nodes is the set of wires* W*, and whose set of arcs is the image of a partial function* $s : L \to W \times W$*, called a switching function of* π*, such that for each link* $\ell \in L$*:*

- *if* $t(\ell) = \texttt{axiom}$*, then, denoting* x *and* y *the left and right conclusions of* ℓ *respectively (i.e. such that* $f(x) = A$ *and* $f(y) = A^*$ *for some* $A \in \mathcal{V}$*),* $s(\ell) \in \{x \to y, y \to x\}$*,*

- *if* $t(\ell) = \texttt{FO-axiom}$*, then, denoting* x *and* y *the left and right conclusions of* ℓ *respectively (i.e. such that* $f(x) = a$ *and* $f(y) = a^*$ *for some* $a \in \mathcal{V}_{FO}$*),* $s(\ell) = y \to x$*,*

- *if* $t(\ell) = \texttt{unit}$*, then* s *doesn't map* ℓ *to any arc,*

- *if* $t(\ell) = \texttt{tensor}$*, then, denoting* x *and* y *the left and right premisses of* ℓ *respectively and* z *the conclusion of* ℓ *(i.e. such that* $f(x) = F$*,* $f(y) = G$ *and* $f(z) = F \otimes G$ *for some formulas* F, G*),* $s(\ell) \in \{x \to z, z \to x, y \to z, z \to y, x \to y, y \to x\}$*,*

- *if* $t(\ell) = \texttt{par}$*, then, denoting* x *and* y *the left and right premisses of* ℓ *respectively and* z *the conclusion of* ℓ *(i.e. such that* $f(x) = F$*,* $f(y) = G$ *and* $f(z) = F \parr G$ *for some formulas* F, G*),* $s(\ell) \in \{x \to z, z \to x, y \to z, z \to y\}$*,*

- *if* $t(\ell) = \texttt{prec}$*, then, denoting* x *and* y *the left and right premisses of* ℓ *respectively and* z *the conclusion of* ℓ *(i.e. such that* $f(x) = F$*,* $f(y) = G$ *and* $f(z) = F < G$ *for some formulas* F, G*),* $s(\ell) \in \{x \to z, z \to x, y \to z, z \to y, x \to y\}$*.*

We denote $\textsc{Graphs}(\pi)$ *the set of switching graphs of* π*.*

Definition 6 (Clarification: cycle)**.** *A cycle of a directed graph* $\mathcal{G} = (V, E)$ *is a sequence* $C = (e_1, \ldots, e_n)$ *of arcs for some* $n \geq 1$ *such that, denoting* $e_i = x_i \to y_i$ *for each* $i \in \{1, \ldots, n\}$*,*

- $y_i = x_{i+1}$ *for each* $i \in \{1, \ldots, n-1\}$ *and* $y_n = x_1$*, and*

- $x_i \neq x_{i'}$ *for each* $i, i' \in \{1, \ldots, n\}$ *such that* $i \neq i'$*.*

We call $(x_1, \ldots, x_n)$ *the vertex sequence of* C*. Usually, we identify cycles up to a circular permutation and we identify* C *with its set of arcs, so we write* $e \in C$ *when* $e = e_i$ *for some* $i \in \{1, \ldots, n\}$*. For each* $i \in \mathbb{Z}$*, we use the notation* $e_i = e_{i+kn}$ *for the appropriate* k *such that* $i + kn \in \{1, \ldots, n\}$*.*

Definition 7 (Proof-net)**.** *A proof-net is a proof-structure such that all its switching graphs are acyclic. A multiset of formulas* Γ *is called valid when there exists a proof-net of* Γ *i.e. a proof-structure of* Γ *which is a proof-net, and in that case we write* $\vdash \Gamma$*. A formula* F *is called valid when* $[F]$ *is valid, and in that case we write* $\vdash F$*.*

4 Technical results

4.1 Size of a formula

Definition 8. *We define the size of a formula $s(F)$ by induction on F:*

$$\forall A \in \mathcal{V}, s(A) = s(A^*) = 1 \ ,$$
$$\forall a \in \mathcal{V}_{FO}, s(a) = s(a^*) = 1 \ ,$$
$$s(I) = 1 \ ,$$
$$s(F \otimes G) = s(F \parr G) = s(F < G) = 1 + s(F) + s(G) \ .$$

Fact 9. *Let π be a proof-structure, $\ell \in L_\pi$, $x \in p_\pi(\ell)$ and $y \in c_\pi(\ell)$. Then, $s(f_\pi(x)) < s(f_\pi(y))$.*

Proof. If $t_\pi(\ell) \in \{\texttt{axiom}, \texttt{FO-axiom}, \texttt{unit}\}$, then from Definition 4 $p_\pi(\ell) = \emptyset$ which contradicts $x \in p_\pi(\ell)$.

Otherwise if $t(\ell) \in \{\texttt{tensor}, \texttt{par}, \texttt{prec}\}$, then from Definition 4 there exist two formulas F and G such that $f_\pi(x) \in \{F, G\}$ and $f_\pi(y) \in \{F \otimes G, F \parr G, F < G\}$. Since $f_\pi(x) \in \{F, G\}$ and $s(F), s(G) > 0$, from Definition 8 $s(f_\pi(x)) < 1 + s(F) + s(G) = s(F \otimes G) = s(F \parr G) = s(F < G) = s(f_\pi(y))$. ☐

Fact 10. *Let π be a proof-structure, let $\ell \in L_\pi$ and let $x, y \in c_\pi(\ell)$. Then, $s(f_\pi(x)) = s(f_\pi(y))$.*

Proof. If $t_\pi(\ell) = \texttt{axiom}$, then there exists $A \in \mathcal{V}$ such that $c_\pi(\ell) = \{A, A^*\}$, and since $s(A) = s(A^*) = 1$ the conclusion holds. Otherwise if $t_\pi(\ell) = \texttt{FO-axiom}$, then there exists $a \in \mathcal{V}_{FO}$ such that $c_\pi(\ell) = \{a, a^*\}$, and since $s(a) = s(a^*) = 1$ the conclusion holds. Otherwise if $t_\pi(\ell) \in \{\texttt{unit}, \texttt{tensor}, \texttt{par}, \texttt{prec}\}$ then $\#c_\pi(\ell) = 1$ so the conclusion holds. ☐

Proposition 11. *Let F be a formula and let G be a subformula of F. Then, $s(F) \geq s(G)$.*

Proof. We prove this statement by induction on F.

If $F = G$, then $s(F) = s(G)$.

Otherwise G is a proper subformula of F. If F is an atomic formula, then this contradicts G is a proper subformula of F. Otherwise if F is of one of the forms $X \otimes Y$, $X \parr Y$ or $X < Y$, then since G is a proper subformula of F, G is either a subformula of X or of Y, so there exists $Z \in \{X, Y\}$ such that G is a subformula of Z. By the induction hypothesis $s(G) \leq s(Z) \leq \max(s(X), s(Y))$. Since $s(X), s(Y) > 0$, from Definition 8 $s(G) < 1 + s(X) + s(Y) = s(X \otimes Y) = s(X \parr Y) = s(X < Y) = s(F)$, so $s(F) \geq s(G)$. ☐

4.2 Facts on proof-structures

Fact 12. *Let π be a proof-structure and let $\ell \in W_\pi$.*

- *$\#c_\pi(\ell) \geq 1$.*
- *If $\#p_\pi(\ell) > 0$, then $\#c_\pi(\ell) = 1$.*

Proof. If $t_\pi(\ell) \in \{\texttt{axiom}, \texttt{FO-axiom}\}$, then from Definition 4 ℓ has exactly two conclusions. Otherwise if $t_\pi(\ell) \in \{\texttt{unit}, \texttt{tensor}, \texttt{prec}, \texttt{par}\}$, then from Definition 4 ℓ has exactly one conclusion. In both cases there exists $x \in c_\pi(\ell)$.

If $\#p_\pi(\ell) > 0$, then from Definition 4 $t_\pi(\ell) \in \{\texttt{tensor}, \texttt{par}, \texttt{prec}\}$ and $\#c_\pi(\ell) = 1$. ☐

Fact 13. *Let π be a proof-structure and let $\ell \in L_\pi$. If there exists $x \in c_\pi(\ell) \cap \mathrm{Op}_\pi$ such that $s(f_\pi(x)) > 1$, then $c_\pi(\ell) \subset \mathrm{Op}_\pi$.*

Proof. Since $s(f_\pi(x)) > 1$, from Definition 8 f_π is of one of the forms $F \otimes G, F \parr G, F < G$. Hence from Definition 4, $t_\pi(\ell) \in \{\texttt{tensor}, \texttt{par}, \texttt{prec}\}$ and hence $\#c_\pi(\ell) = 1$. So, for each $y \in c_\pi(\ell)$, $y = x$, hence by assumption on x, $y \in \mathrm{Op}_\pi$. ☐

Fact 14. *Let π be a proof-structure and Γ be the multiset of conclusions of π. The following statements are equivalent:*

- $L_\pi = \emptyset$,
- $W_\pi = \emptyset$.

Proof. If $L_\pi \neq \emptyset$, then there exists $\ell \in L_\pi$. By Fact 12 $\#c_\pi(\ell) \geq 1$, hence $W_\pi \neq \emptyset$. If $W_\pi \neq \emptyset$ then there exists $x \in W_\pi$, so from Definition 4 there exists $\ell \in L_\pi$ such that $x \in c_\pi(\ell)$, so $L_\pi \neq \emptyset$. $\square$

Proposition 15. *Let π be a proof-structure such that $\#L_\pi > 0$. Then, there exists a link $\ell \in L_\pi$ such that $c_\pi(\ell) \subset \mathrm{Op}_\pi$.*

Proof. Since $L_\pi \neq \emptyset$, by Fact 14 $W_\pi \neq \emptyset$ and from Definition 4 W_π is finite, so there exists $x \in W_\pi$ such that $s(f_\pi(x))$ is maximum over W_π. From Definition 4 there exists ℓ such that $x \in c_\pi(\ell)$. If there exists ℓ' such that $x \in p_\pi(\ell')$, then by Fact 12 there exists $y \in c_\pi(\ell')$ and by Fact 9 $s(f_\pi(y)) > s(f_\pi(x))$ which contradicts that $s(f_\pi(x))$ is maximal, so $\forall \ell' \in L_\pi, x \notin p_\pi(\ell')$.

If $t_\pi(\ell) \in \{\mathtt{unit}, \mathtt{tensor}, \mathtt{par}, \mathtt{prec}\}$, then from Definition 4 $\#c_\pi(\ell) = 1$ so $c_\pi(\ell) = \{x\}$, hence since $\forall \ell' \in L_\pi, x \notin p_\pi(\ell')$ the conclusion holds.

Otherwise if $t_\pi(\ell) \in \{\mathtt{axiom}, \mathtt{FO\text{-}axiom}\}$, then from Definition 4 $c_\pi(\ell) = \{x', y'\}$ for some $x', y' \in W_\pi$, and $f_\pi(x') = A$ and $f_\pi(y') = A^*$ for some $A \in \mathcal{V}$ or $f_\pi(x') = a$ and $f_\pi(y') = a^*$ for some $a \in \mathcal{V}_{FO}$, hence $s(f_\pi(x')) = s(f_\pi(y')) = 1$. So, since $x \in c_\pi(\ell) = \{x', y'\}$, $s(f_\pi(x)) = 1$. If there exist $z' \in c_\pi(\ell)$ and $\ell' \in L_\pi$ such that $z' \in p_\pi(\ell')$, then by Fact 12 there exists $y \in c_\pi(\ell')$ and by Fact 9 $s(f_\pi(y)) > s(f_\pi(z')) = 1 = s(f_\pi(x))$ which contradicts that $s(f_\pi(x))$ is maximal, so $\forall z' \in c_\pi(\ell), \forall \ell' \in L_\pi, z' \notin p_\pi(\ell')$. $\square$

Proposition 16. *Let π be a proof-structure and Γ be the multiset of conclusions of π. The following statements are equivalent:*

- $L_\pi = \emptyset$,
- $W_\pi = \emptyset$,
- $\Gamma = []$,

Proof. By Fact 14, $L_\pi = \emptyset$ if and only if $W_\pi = \emptyset$. If $W_\pi = \emptyset$ then from Definition 4 $\Gamma = []$. If $L_\pi \neq \emptyset$, then by Proposition 15 there exists $\ell \in L_\pi$ such that $\forall x \in c_\pi(\ell), \forall \ell' \in L_\pi, x \notin p_\pi(\ell)$. By Fact 12 $\#c_\pi(\ell) \geq 1$ hence there exists $x \in c_\pi(\ell)$ so $\forall \ell' \in L_\pi, x \notin p_\pi(\ell)$, so $f_\pi(x) \in \Gamma$ so $\Gamma \neq []$. $\square$

Proposition 17. *Let π be a proof-structure and let $\ell \in L_\pi$. Then, $p_\pi(\ell) \cap c_\pi(\ell) = \emptyset$.*

Proof. For a contradiction, assume that there exists $x \in p_\pi(\ell) \cap c_\pi(\ell)$. By Fact 9, $s(f_\pi(x)) < s(f_\pi(x))$ which is absurd. $\square$

Proposition 18. *Let $\Gamma \neq []$ be a multiset of formulas and let π be a proof-structure of Γ. Then, there exists a link $\ell \in L_\pi$ such that $c_\pi(\ell) \subset \mathrm{Op}_\pi$.*

Proof. Since $\Gamma \neq []$, by Proposition 16 $\#L_\pi > 0$, so by Proposition 15 there exists $\ell \in L_\pi$ such that $\forall x \in c_\pi(\ell), \forall \ell' \in L_\pi, x \notin p_\pi(\ell')$. $\square$

Proposition 19. *Let π and π' be proof-structures, let $\ell \in L_\pi$ and $\ell' \in L_{\pi'}$ such that there exist $x \in c_\pi(\ell)$ and $x' \in c_{\pi'}(\ell')$ such that $f_\pi(x) = f_{\pi'}(x')$. Then,*

- $t_\pi(\ell) = t_{\pi'}(\ell')$,
- $\#c_\pi(\ell) = \#c_{\pi'}(\ell')$ *and for each conclusion y of ℓ, denoting y' the corresponding conclusion of ℓ', $f_\pi(y) = f_{\pi'}(y')$,*
- $\#p_\pi(\ell) = \#p_{\pi'}(\ell')$ *and for each premisse y of ℓ, denoting y' the corresponding premisse of ℓ', $f_\pi(y) = f_{\pi'}(y')$.*

Proof. Since $f_\pi(y) = f_\pi(y')$, from Definition 4

- if $f_\pi(x)$ is of one of the forms $\mathcal{V}$ and $\mathcal{V}^*$ then $t_\pi(\ell) = t_{\pi'}(\ell') = \texttt{axiom}$, so $\#p_\pi(\ell) = \#p_{\pi'}(\ell') = 0$ and $\#c_\pi(\ell) = \#c_{\pi'}(\ell') = 2$. Let us denote $c_\pi(\ell) = \{\tilde{x}, \tilde{y}\}$ and $c_{\pi'}(\ell') = \{\tilde{x'}, \tilde{y'}\}$. Then, there exist $A, B \in \mathcal{V}$ such that $f_\pi(\tilde{x}) = A$ and $f_\pi(\tilde{y}) = A^*$ and $f_{\pi'}(\tilde{x'}) = B$ and $f_{\pi'}(\tilde{y'}) = B^*$. Since $x \in \{\tilde{x}, \tilde{y}\}$ and $x' \in \{\tilde{x'}, \tilde{y'}\}$ and $f_\pi(x) = f_{\pi'}(x')$, $A = B$ or $A^* = B^*$, in both cases the equality between variables holds: $A = B$. So, $f_\pi(\tilde{x}) = f_{\pi'}(\tilde{x'})$ and $f_\pi(\tilde{y}) = f_{\pi'}(\tilde{y'})$.

- otherwise if $f_\pi(x)$ is of one of the forms $\mathcal{V}_{FO}$ and $\mathcal{V}_{FO}^*$ then $t_\pi(\ell) = t_{\pi'}(\ell') = \texttt{FO-axiom}$, so $\#p_\pi(\ell) = \#p_{\pi'}(\ell') = 0$ and $\#c_\pi(\ell) = \#c_{\pi'}(\ell') = 2$. Let us denote $c_\pi(\ell) = \{\tilde{x}, \tilde{y}\}$ and $c_{\pi'}(\ell') = \{\tilde{x'}, \tilde{y'}\}$. There exist $a, b \in \mathcal{V}_{FO}$ such that $f_\pi(\tilde{x}) = a$, $f_\pi(\tilde{y}) = a^*$, $f_{\pi'}(\tilde{x'}) = b$ and $f_{\pi'}(\tilde{y'}) = b^*$. Since $x \in \{\tilde{x}, \tilde{y}\}$ and $x' \in \{\tilde{x'}, \tilde{y'}\}$ and $f_\pi(x) = f_{\pi'}(x')$, $a = b$ or $a^* = b^*$, so $a = b$. So, $f_\pi(\tilde{x}) = f_{\pi'}(\tilde{x'})$ and $f_\pi(\tilde{y}) = f_{\pi'}(\tilde{y'})$.

- otherwise if $f_\pi(x) = I$ then $t_\pi(\ell) = t_{\pi'}(\ell') = \texttt{unit}$, $\#p_\pi(\ell) = \#p_\pi(\ell') = 0$ and $\#c_\pi(\ell) = \#c_{\pi'}(\ell') = 1$, so $c_\pi(\ell) = \{x\}$ and $c_{\pi'}(\ell') = \{x'\}$ and $f_\pi(x) = f_{\pi'}(x') = I$.

- otherwise if $f_\pi(x)$ if of the form $F \otimes G$ then $t_\pi(\ell) = t_{\pi'}(\ell') = \texttt{tensor}$, $\#p_\pi(\ell) = \#p_{\pi'}(\ell') = 2$ and $\#c_\pi(\ell) = \#c_{\pi'}(\ell')$. Hence, $c_\pi(\ell) = \{x\}$ and $c_{\pi'}(\ell') = \{x'\}$ and by assumption $f_\pi(x) = f_{\pi'}(x')$. Denoting $f_\pi(x) = F \otimes G$ and $p_\pi(\ell) = \{y, z\}$ and $p_{\pi'}(\ell') = \{y', z'\}$, $f_\pi(y) = f_{\pi'}(y') = F$ and $f_\pi(z) = f_{\pi'}(z') = G$.

- otherwise if $f_\pi(x)$ if of the form $F \parr G$ then $t_\pi(\ell) = t_{\pi'}(\ell') = \texttt{par}$, $\#p_\pi(\ell) = \#p_{\pi'}(\ell') = 2$ and $\#c_\pi(\ell) = \#c_{\pi'}(\ell') = 1$. Hence, $c_\pi(\ell) = \{x\}$ and $c_{\pi'}(\ell') = \{x'\}$, and by assumption $f_\pi(x) = f_{\pi'}(x')$. Denoting $f_\pi(x) = F \parr G$ and $p_\pi(\ell) = \{y, z\}$ and $p_{\pi'}(\ell') = \{y', z'\}$, $f_\pi(y) = f_{\pi'}(y') = F$ and $f_\pi(z) = f_{\pi'}(z') = G$.

- otherwise if $f_\pi(x)$ if of the form $F < G$ then $t_\pi(\ell) = t_{\pi'}(\ell') = \texttt{prec}$, $\#p_\pi(\ell) = \#p_{\pi'}(\ell') = 2$ and $\#c_\pi(\ell) = \#c_{\pi'}(\ell') = 1$. Hence, $c_\pi(\ell) = \{x\}$ and $c_{\pi'}(\ell') = \{x'\}$, and by assumption $f_\pi(x) = f_{\pi'}(x')$. Denoting $f_\pi(x) = F < G$ and $p_\pi(x) = \{y, z\}$ and $p_{\pi'}(x') = \{y', z'\}$, $f_\pi(y) = f_{\pi'}(y') = F$ and $f_\pi(z) = f_{\pi'}(z') = G$.

□

Proposition 20. *Let π and π' be proof-structures, let $\ell \in L_\pi$ and $\ell' \in L_{\pi'}$ such that there exists $x \in c_\pi(\ell)$ and $x' \in c_{\pi'}(\ell')$ such that $f_\pi(x) = f_{\pi'}(x')$. Then,*

- $[f_\pi(y), y \in c_\pi(\ell)] = [f_{\pi'}(y'), y' \in c_{\pi'}(\ell')]$,
- $[f_\pi(y), y \in p_\pi(\ell)] = [f_{\pi'}(y'), y' \in p_{\pi'}(\ell')]$.

Proof. By Proposition 19, $\#c_\pi(\ell) = \#c_\pi(\ell')$ and for each conclusion y of ℓ, denoting y' the corresponding conclusion of ℓ', $f_\pi(y) = f_{\pi'}(y')$, and $\#p_\pi(\ell) = \#p_{\pi'}(\ell')$ and for each premisse y or ℓ, denoting y' the corresponding premisse of ℓ', $f_\pi(y) = f_{\pi'}(y')$. So, $[f_\pi(y), y \in c_\pi(\ell)] = [f_{\pi'}(y'), y' \in c_{\pi'}(\ell')]$ and $[f_\pi(y), y \in p_\pi(\ell)] = [f_{\pi'}(y'), y' \in p_{\pi'}(\ell')]$. □

4.3 Wires above one another

Definition 21. *Let π be a proof-structure and let $n \geq 0$. A downward sequence of length n is a pair $((x_0, \ldots, x_n), (\ell_1, \ldots, \ell_n))$ of sequences of wires and links such that for each $i \in \{1, \ldots, n\}$, $x_{i-1} \in p_\pi(\ell_i)$ and $x_i \in c_\pi(\ell_i)$.*

For $x, y \in W_\pi$, we say that x is above y when there exists a downward sequence $((x_0, \ldots, x_n), (\ell_1, \ldots, \ell_n))$ such that $x_0 = x$ and $x_n = y$.

Proposition 22. *Let π be a proof-structure, let $x \in W_\pi$ and let $n \geq 0$. A downward sequence $((x_0, \ldots, x_n), (\ell_1, \ldots, \ell_n))$ of length n such that $x = x_0$ is unique given x and n.*

Proof. We prove this statement by induction on n. Let $((x_0, \ldots, x_n), (\ell_1, \ldots, \ell_n))$ be a downward sequence of length n such that $x = x_0$.

If $n = 0$, then $(x_0, \ldots, x_n) = (x)$ and $(\ell_1, \ldots, \ell_n)$ is the empty sequence, which are unique given x and n.

Otherwise if $n > 0$, by the induction hypothesis the downward sequence $((x_0, \ldots, x_{n-1}), (\ell_1, \ldots, \ell_{n-1}))$ of length $n - 1$ is unique. From Definition 4, x_{n-1} is a premisse of at most one link, hence since $x_{n-1} \in p_\pi(\ell_n)$ and x_{n-1} is unique, ℓ_n is unique. Since $x_{n-1} \in p_\pi(\ell_n)$, $\#p_\pi(\ell_n) > 0$, hence by Fact 12 $\#c_\pi(\ell_n) = 1$. So, since $x_n \in c_\pi(\ell_n)$ and ℓ_n is unique, x_n is unique. Since $((x_0, \dots, x_{n-1}), (\ell_1, \dots, \ell_{n-1}))$, ℓ_n and x_n are unique, the downward sequence $((x_0, \dots, x_n), (\ell_1, \dots, \ell_n))$ is unique given x and n. □

Proposition 23. *Let π be a proof-structure, let $x \in \mathrm{Op}_\pi$, and let y such that x is above y. Then, $x = y$.*

Proof. Since x is above y, there exists a downward sequence $((x_0, \dots, x_n), (\ell_1, \dots, \ell_n))$ for some $n \geq 0$ such that $x_0 = x$ and $x_n = y$. If $n \geq 1$, then $x = x_0 \in p_\pi(\ell_1)$ which contradicts $\forall \ell \in L_\pi, x \notin p_\pi(\ell)$, hence $n = 0$ and $x = x_0 = x_n = y$. □

Proposition 24. *Let π be a proof-structure and let $((x_0, \dots, x_n), (\ell_1, \dots, \ell_n))$ be a downward sequence. For each $i, j \in \{0, \dots, n\}$ such that $i < j$, $s(f_\pi(x_i)) < s(f_\pi(x_j))$.*

Proof. We prove this statement by induction on n. Let $((x_0, \dots, x_n), (\ell_1, \dots, \ell_n))$ be a downward sequence and let $i, j \in \{0, \dots, n\}$ such that $i < j$. Since $0 \leq i < j \leq n$, $i \leq n-1$ and $n > 0$. Since $n > 0$, $((x_0, \dots, x_{n-1}), (\ell_1, \dots, \ell_{n-1}))$ is a downward sequence of length $n - 1$.

If $j \leq n-1$, then $i, j \in \{0, \dots, n-1\}$ so by the induction hypothesis applied to the downward sequence $((x_0, \dots, x_{n-1}), (\ell_1, \dots, \ell_{n-1}))$, $s(f_\pi(x_i)) < s(f_\pi(x_j))$.

Otherwise if $j = n$, then $i \leq n-1$, so by the induction hypothesis $s(f_\pi(x_i)) \leq s(f_\pi(x_{n-1}))$. Since $x_{n-1} \in p_\pi(\ell_n)$ and $x_j = x_n \in c_\pi(\ell_n)$, by Fact 9 $s(f_\pi(x_{n-1})) < s(f_\pi(x_j))$, hence $s(f_\pi(x_i)) < s(f_\pi(x_j))$. □

Fact 25. *Let π be a proof-structure and let $((x_0, \dots, x_n), (\ell_1, \dots, \ell_n))$ be a downward sequence. For each $i \in \{1, \dots, n\}$, $t_\pi(\ell_i) \in \{\mathtt{tensor}, \mathtt{par}, \mathtt{prec}\}$.*

Proof. $x_{i-1} \in p_\pi(\ell_i)$ hence $p_\pi(\ell_i) \neq \emptyset$ which proves the result. □

Proposition 26. *Let π be a proof-structure and let $((x_0, \dots, x_n), (\ell_1, \dots, \ell_n))$ be a downward sequence.*

- *For each $i, j \in \{0, \dots, n\}$ such that $i \neq j$, $x_i \neq x_j$.*
- *For each $i, j \in \{1, \dots, n\}$ such that $i \neq j$, $\ell_i \neq \ell_j$.*

Proof. Since $i \neq j$, $i < j$ or $j < i$, so by Proposition 24 $s(f_\pi(x_i)) < s(f_\pi(x_j))$ or $s(f_\pi(x_j)) < s(f_\pi(x_i))$, so $s(f_\pi(x_i)) \neq s(f_\pi(x_j))$, so $x_i \neq x_j$.

By Fact 25, $t_\pi(\ell_i), t_\pi(\ell_j) \in \{\mathtt{tensor}, \mathtt{par}, \mathtt{prec}\}$ so $c_\pi(\ell_i) = \{x_{i-1}\}$ and $c_\pi(\ell_j) = \{x_{j-1}\}$. Since $i \neq j$, by the previous argument $x_{i-1} \neq x_{j-1}$ so $\ell_i \neq \ell_j$. □

Proposition 27. *Let π be a proof-structure and let $x \in W_\pi$. There exists a unique wire $y \in \mathrm{Op}_\pi$ such that x is above y.*

Proof. $((x), ())$ is a downward sequence $((x_0, \dots, x_n), (\ell_1, \dots, \ell_n))$ such that $x = x_0$. From Definition 4 W_π is finite, so by Proposition 26 for each downward sequence $((x_0, \dots, x_n), (\ell_1, \dots, \ell_n))$ such that $x = x_0$, $n < \#W_\pi$. So, there exists a downward sequence $((x_0, \dots, x_n), (\ell_1, \dots, \ell_n))$ such that $x = x_0$ where n is maximal, hence x is above x_n.

We prove that $\forall \ell \in L_\pi, x_n \notin p_\pi(\ell)$. For a contradiction, assume that there exists $\ell \in L_\pi$ such that $x \in p_\pi(\ell)$. By Fact 12, there exists $y \in c_\pi(\ell)$. Hence, $((x_0, \dots, x_n, y), (\ell_1, \dots, \ell_n, \ell))$ is a downward sequence of length $n+1$ such that $x_0 = x$, which contradicts that n is maximal, so $\forall \ell \in L_\pi, x_n \notin p_\pi(\ell)$.

By Proposition 22 $((x_0, \dots, x_n, y), (\ell_1, \dots, \ell_n, \ell))$ is unique given x and n, hence x_n is unique given x and n. So, since n is unique given x, x_n is unique given x. □

Fact 28. *Let π be a proof-structure, let $\ell \in L_\pi$, let $x \in p_\pi(\ell)$ and let $y \in c_\pi(\ell)$. $f_\pi(x)$ is a direct subformula of $f_\pi(y)$.*

Proof. From Definition 4, since $x \in p_\pi(\ell)$, $\#p_\pi(\ell) > 0$ hence $t_\pi(\ell) \in \{\texttt{tensor}, \texttt{par}, \texttt{axiom}\}$ so ℓ has exactly two premisses x', y' and exactly one conclusion z' and there exist two formulas F, G such that $f_\pi(x') = F$ and $f_\pi(y') = G$ and $f_\pi(z') \in \{F \otimes G, F \parr G, F < G\}$. So, $f_\pi(x')$ and $f_\pi(y')$ are direct subformulas of $f_\pi(z')$. Since $y \in c_\pi(\ell)$, $y = z'$ and since $x \in p_\pi(\ell)$, $x \in \{x', y'\}$ so $f_\pi(x)$ is a direct subformula of $f_\pi(z)$. ☐

Proposition 29. *Let π be a proof-structure and $x, y \in W_\pi$ such that x is above y. Then, $f_\pi(x)$ is a subformula of $f_\pi(y)$.*

Proof. Since x is above y there exists a downward sequence $((x_0, \ldots, x_n), (\ell_1, \ldots, \ell_n))$ such that $x_0 = x$ and $y_0 = y$. We prove this statement by induction on n (for x and y not fixed).

If $n = 0$ then $x = x_0 = x_n = y$ so $f_\pi(x) = f_\pi(y)$ so $f_\pi(x)$ is a subformula of $f_\pi(y)$.

Otherwise if $n \geq 1$ then $((x_0, \ldots, x_{n-1}), (\ell_1, \ldots, \ell_{n-1}))$ is a downward sequence, so by the induction hypothesis $f_\pi(x_0)$ is a subformula of $f_\pi(x_{n-1})$. Since $x_{n-1} \in p_\pi(\ell_n)$ and $x_n \in c_\pi(\ell_n)$, by Fact 28 $f_\pi(x_{n-1})$ is a subformula of $f_\pi(x_n)$. Since $f_\pi(x) = f_\pi(x_0)$ is a subformula of $f_\pi(x_{n-1})$ and $f_\pi(x_{n-1})$ is a subformula of $f_\pi(x_n) = f_\pi(y)$, $f_\pi(x)$ is a subformula of $f_\pi(y)$. ☐

Proposition 30. *Let π be a proof-structure and $x, y \in W_\pi$ such that x is above y. If $s(f_\pi(x)) = s(f_\pi(y))$ then $x = y$.*

Proof. By Proposition 24. ☐

Proposition 31. *Let π be a proof-structure and let $((x_0, \ldots, x_n), (\ell_1, \ldots, \ell_n))$ and $((x'_0, \ldots, x'_m), (\ell'_0, \ldots, \ell'_m))$ be downward sequences such that $x_n = x'_m$. Then, there exists $k_1 \in \{0, \ldots, n\}$ and $k_2 \in \{0, \ldots, m\}$ such that $x_{k_1} = x'_{k_2}$, and if $k_1 > 0$ and $k_2 > 0$ then $x_{k_1 - 1} \neq x'_{k_2 - 1}$.*

Proof. This is a consequence of Proposition 22. ☐

Fact 32. *Let π be a proof-structure, let $x \in W_\pi$ and let F be a formula. Then, F is a subformula of $f_\pi(x)$ if and only if there exists $x' \in W_\pi$ such that $f_\pi(x') = F$ and x' is above x. In that case, denoting $((x_0, \ldots, x_n), (\ell_1, \ldots, \ell_n))$ the downward sequence such that $x_0 = x'$ and $x_n = x$,*

- *$n > 0$ if and only if F is a proper subformula of $f_\pi(x)$, and*
- *$n = 1$ if and only if F is a direct subformula of $f_\pi(x)$.*

Proof. This is proven by induction on F. ☐

4.4 Balanced multiset of formulas

Definition 33. *We define recursively the multiset of atoms $\mathrm{At}(F)$ of a formula F as:*

$$\forall A \in \mathcal{V}, \mathrm{At}(A) = [A],\ \mathrm{At}(A^*) = [A^*]\ , \tag{13}$$

$$\forall a \in \mathcal{V}_{FO}, \mathrm{At}(a) = [a],\ \mathrm{At}(a^*) = [a^*]\ , \tag{14}$$

$$\mathrm{At}(I) = [I]\ , \tag{15}$$

$$\mathrm{At}(F \otimes G) = \mathrm{At}(F \parr G) = \mathrm{At}(F < G) = \mathrm{At}(F) + \mathrm{At}(G)\ . \tag{16}$$

Given a multiset of formulas Γ, we define $\mathrm{At}(\Gamma) = \sum_{F \in \Gamma} \mathrm{At}(F)$.

Fact 34. *Let H be an atomic formula and let F be a formula. Then, H is a subformula of F if and only if $H \in \mathrm{At}(F)$.*

Proof. This is proven by induction on F. ☐

Definition 35. *A multiset of formulas Γ is called balanced when:*

- *for each variable $A \in \mathcal{V}$, $\mathrm{At}(\Gamma)(A) = \mathrm{At}(\Gamma)(A^*) \leq 1$,*
- *for each variable $a \in \mathcal{V}_{FO}$, $\mathrm{At}(\Gamma)(a) = \mathrm{At}(\Gamma)(a^*) \leq 1$.*

A formula F is called balanced when the multiset $[F]$ is balanced.

Fact 36. *Let Γ be a balanced multiset of formulas such that $I \notin \mathrm{At}(\Gamma)$ and let $F, G \in \Gamma$. If $\mathrm{At}(F) \cap \mathrm{At}(G) \neq []$, then $F = G$.*

Proof. Since $\mathrm{At}(F) \cap \mathrm{At}(G) \neq []$ there exists $H \in \mathrm{At}(F) \cap \mathrm{At}(G)$. Since $H \in \mathrm{At}(F)$, H is an atomic formula. Assume that $F \neq G$. Then, $\mathrm{At}(\Gamma)(H) = \sum_{\tilde{F} \in \Gamma} \mathrm{At}(\tilde{F})(H) \geq \mathrm{At}(F)(H) + \mathrm{At}(G)(H) \geq 2$. If $H = I$, then $H \notin \mathrm{At}(\Gamma)$ which contradicts $\mathrm{At}(\Gamma)(H) \geq 2$. Otherwise if $H \in \{A, A^*\}$ for some $A \in \mathcal{V}$ or if $H \in \{a, a^*\}$ for some $a^* \in \mathcal{V}_{FO}$, then since Γ is balanced, $\mathrm{At}(\Gamma)(H) \leq 1$ which contradicts $\mathrm{At}(\Gamma)(H) \geq 2$. Hence, $F = G$. □

Fact 37. *Let Γ be a balanced multiset of formulas such that $I \notin \mathrm{At}(\Gamma)$. Then, $\Gamma \leq 1$.*

Proof. Let $F \in \Gamma$. From Definition 33 there exists an atomic formula $H \in \mathrm{At}(F)$. Since $\mathrm{At}(\Gamma) = \sum_{\tilde{F} \in \Gamma} \mathrm{At}(\tilde{F}) \geq \mathrm{At}(F)$, $H \in \mathrm{At}(\Gamma)$ so $H \neq I$. So $F \in \{A, A^*\}$ for some $A \in \mathcal{V}$ or $F \in \{a, a^*\}$ for some $a \in \mathcal{V}_{FO}$, hence since Γ is balanced, $\mathrm{At}(\Gamma)(H) \leq 1$. So, $1 \geq \mathrm{At}(\Gamma)(H) = \sum_{\tilde{F} \in \Gamma} \mathrm{At}(\tilde{F})(H) = \sum_{\text{formula } \tilde{F}} \Gamma(\tilde{F})\mathrm{At}(\tilde{F})(H) \geq \Gamma(F)\mathrm{At}(F)(H) \geq \Gamma(F)$ since $H \in \mathrm{At}(F)$, so $\Gamma(F) \leq 1$. □

4.5 Induction on proof-structures

4.5.1 Induced substructure of a proof-structure after removing a link

Definition 38. *Let π be a proof-structure, let $\ell \in L_\pi$ such that $c_\pi(\ell) \subset \mathrm{Op}_\pi$. We call the induced substructure of π after removing ℓ the following structure $\overline{\pi}$ defined with*

- *the set of wires $W_{\overline{\pi}} = W_\pi \setminus c_\pi(\ell)$ and set of links $L_{\overline{\pi}} = L_\pi \setminus \{\ell\}$,*
- *the formula labels of wires $f_{\overline{\pi}} = (f_\pi)_{|W_{\overline{\pi}}}$,*
- *the types of links $t_{\overline{\pi}} = (t_\pi)_{|L_{\overline{\pi}}}$, premisses of links $p_{\overline{\pi}} = (p_\pi)_{|L_{\overline{\pi}}}$ and conclusions of links $c_{\overline{\pi}} = (c_\pi)_{|L_{\overline{\pi}}}$.*

Given a proof-structure π and a structure $\overline{\pi}$, we say that $\overline{\pi}$ is an induced substructure of π when $\overline{\pi}$ is the induced substructure of π after removing a link $\ell \in L_\pi$ such that $c_\pi(\ell) \subset \mathrm{Op}_\pi$.

Proposition 39. *Let π be a proof-structure and let $\overline{\pi}$ be an induced substructure of π. Then, $\overline{\pi}$ is a proof-structure.*

Proof. This is proven by checking Definition 4. □

(In the following, Proposition 39 will be used implicitely each time we consider an induced substructure of a proof-structure.)

Fact 40. *Let π be a proof-structure and let $\overline{\pi}$ be the induced substructure of π after removing a link $\ell \in L_\pi$. Then,*

- *$\mathrm{Op}_{\overline{\pi}} \cap c_\pi(\ell) = \emptyset$,*
- *$\mathrm{Op}_\pi \cap p_\pi(\ell) = \emptyset$,*
- *$\mathrm{Op}_{\overline{\pi}} \cup c_\pi(\ell) = \mathrm{Op}_\pi \cup p_\pi(\ell)$.*

Proof. Since $\overline{\pi}$ is the induced substructure of π after removing ℓ, from Definition 38 $c_\pi(\ell) \subset \mathrm{Op}_\pi$ and $L_{\overline{\pi}} = L_\pi \setminus \{\ell\}$ and $W_{\overline{\pi}} = W_\pi \setminus c_\pi(\ell)$.

Let $x \in W_{\overline{\pi}}$ such that $\forall \tilde{\ell} \in L_{\overline{\pi}}, x \notin p_{\overline{\pi}}(\tilde{\ell})$. Since $x \in W_{\overline{\pi}} = W_\pi \setminus c_\pi(\ell)$, $x \notin c_\pi(\ell)$.

Let $x \in p_\pi(\ell)$. Since $\ell \in L_\pi$, x doesn't satisfy $\forall \tilde{\ell} \in L_\pi, \tilde{x} \notin p_\pi(\tilde{\ell})$.

Let $x \in W_{\overline{\pi}}$ such that $\forall \tilde{\ell} \in L_{\overline{\pi}}, x \notin p_{\overline{\pi}}(\tilde{\ell})$. If there exists $\ell' \in L_\pi$ such that $x \in p_\pi(\ell')$, then since $L_{\overline{\pi}} = L_\pi \setminus \{\ell\}$, $\ell' \in L_{\overline{\pi}}$ or $\ell' = \ell$. If $\ell' \in L_{\overline{\pi}}$, then $x \notin p_{\overline{\pi}}(\ell') = p_\pi(\ell')$, which contradicts $x \in p_\pi(\ell')$. So, $\ell' = \ell$, hence $x \in p_\pi(\ell)$. Otherwise if for each $\ell' \in L_\pi$, $x \notin p_\pi(\ell')$, then $x \in \mathrm{Op}_\pi$. So, $\mathrm{Op}_{\overline{\pi}} \subset \mathrm{Op}_\pi \cup p_\pi(\ell)$.

Let $x \in c_\pi(\ell)$. Then, $\forall \tilde{\ell} \in L_\pi, x \notin p_\pi(\tilde{\ell})$. So, $c_\pi(\ell) \subset \mathrm{Op}_\pi$.

Let $x \in W_\pi$ such that for each $\ell' \in L_\pi, x \notin p_\pi(\ell')$. Since $W_{\overline{\pi}} = W_\pi \setminus c_\pi(\ell)$, $x \in W_{\overline{\pi}}$ or $x \in c_\pi(\ell)$. If $x \in W_{\overline{\pi}}$, then for each $\ell' \in L_{\overline{\pi}} \subset L_\pi$, $x \notin p_\pi(\ell')$. Otherwise, $x \in c_\pi(\ell)$. So, $\mathrm{Op}_{\overline{\pi}} \cup c_\pi(\ell) \supset \mathrm{Op}_\pi$.

Let $x \in p_\pi(\ell)$. If $x \notin c_\pi(\ell)$, then $x \in W_{\overline{\pi}}$. If there exists $\ell' \in L_{\overline{\pi}}$ such that $x \in p_{\overline{\pi}}(\ell')$, then $\ell' \in L_\pi$ and $x \in p_{\overline{\pi}}(\ell') = p_\pi(\ell')$. From Definition 4, x is a premisse of at most one link in π, hence since $x \in p_\pi(\ell)$ and $x \in p_\pi(\ell')$, $\ell' = \ell$ which contradicts $\ell' \in L_{\overline{\pi}} = L_\pi \setminus \{\ell\}$, so for each $\ell' \in L_{\overline{\pi}}$, $x \notin p_{\overline{\pi}}(\ell')$. Otherwise, $x \in c_\pi(\ell)$. So, $\mathrm{Op}_{\overline{\pi}} \cup c_\pi(\ell) \supset p_\pi(\ell)$. □

Fact 41. *Let Γ be a multiset of formulas, let π be a proof-structure of Γ, let $\overline{\pi}$ be the induced substructure of π after removing a link $\ell \in L_\pi$ and let $\overline{\Gamma}$ be the multiset of conclusion formulas of $\overline{\pi}$. Then, $\Gamma + [f_\pi(x), x \in p_\pi(\ell)] = \overline{\Gamma} + [f_\pi(x), x \in c_\pi(\ell)]$.*

Proof. By Fact 40, $\mathrm{Op}_{\overline{\pi}} \uplus c_\pi(\ell) = \mathrm{Op}_\pi \uplus p_\pi(\ell)$. So, $[f_\pi(\tilde{x}), \tilde{x} \in \mathrm{Op}_{\overline{\pi}}] + [f_\pi(\tilde{x}), \tilde{x} \in c_\pi(\ell)] = [f_\pi(\tilde{x}), \tilde{x} \in \mathrm{Op}_\pi] + [f_\pi(\tilde{x}), \tilde{x} \in p_\pi(\ell)]$. From Definition 4, $\Gamma = [f_\pi(\tilde{x}), \tilde{x} \in \mathrm{Op}_\pi]$ and $\overline{\Gamma} = [f_{\overline{\pi}}(\tilde{x}), \tilde{x} \in \mathrm{Op}_{\overline{\pi}}] = [f_\pi(\tilde{x}), \tilde{x} \in \mathrm{Op}_{\overline{\pi}}]$ since for each $x \in \mathrm{Op}_{\overline{\pi}} \subset W_{\overline{\pi}}$, $f_\pi(x) = f_{\overline{\pi}}(x)$, so $\Gamma + [f_\pi(x), x \in p_\pi(\ell)] = \overline{\Gamma} + [f_\pi(x), x \in c_\pi(\ell)]$. □

Proposition 42. *Let Γ be a multiset of formulas, let π and π' be two proof-structures of Γ, let $\overline{\pi}$ and $\overline{\pi'}$ be the induced substructures of π and π' after removing links $\ell \in L_\pi$ and $\ell' \in L_{\pi'}$ respectively and let $\overline{\Gamma}$ and $\overline{\Gamma'}$ be the multisets of conclusion formulas of $\overline{\pi}$ and $\overline{\pi'}$ respectively. If there exist $x \in c_\pi(\ell)$ and $x' \in c_{\pi'}(\ell')$ such that $f_\pi(x) = f_{\pi'}(x')$, then $\overline{\Gamma} = \overline{\Gamma'}$.*

Proof. Since $f_\pi(x) = f_{\pi'}(x')$, by Proposition 20, $[f_\pi(x), x \in c_\pi(\ell)] = [f_{\pi'}(x'), x' \in c_{\pi'}(\ell')]$ and $[f_\pi(x), x \in p_\pi(\ell)] = [f_{\pi'}(x'), x' \in p_{\pi'}(\ell')]$. By Fact 41, $\Gamma + [f_\pi(x), x \in p_\pi(\ell)] = \overline{\Gamma} + [f_\pi(x), x \in c_\pi(\ell)]$ and $\Gamma + [f_{\pi'}(x'), x' \in p_{\pi'}(\ell')] = \overline{\Gamma'} + [f_{\pi'}(x'), x' \in c_{\pi'}(\ell')]$. So, $\overline{\Gamma} + [f_\pi(x), x \in c_\pi(\ell)] = \overline{\Gamma'} + [f_\pi(x), x \in c_\pi(\ell)]$, so $\overline{\Gamma} = \overline{\Gamma'}$. □

4.5.2 Induction on the number of links

Theorem 43. *Let π be a proof-structure such that $\#L_\pi = 0$. Then, $W_\pi = \emptyset$ and $L_\pi = \emptyset$.*

Proof. By Proposition 16. □

Theorem 44. *Let π be a proof-structure such that $\#L_\pi > 0$. Then, there exists an induced substructure $\overline{\pi}$ of π.*

Proof. By Proposition 15. □

Theorem 45. *Let π be a proof-structure and let $\overline{\pi}$ be an induced substructure of π. Then, $\#L_{\overline{\pi}} < \#L_\pi$.*

Proof. Since $\overline{\pi}$ is an induced substructure of π, from Definition 38 there exists $\ell \in L_\pi$ such that $\overline{\pi}$ is the induced substructure of π after removing ℓ and $L_{\overline{\pi}} = L_\pi \setminus \{\ell\}$, so $\#L_{\overline{\pi}}(\ell) = \#L_\pi - 1$. □

4.5.3 Induction on the sum of formulas size of a multiset of formulas

Definition 46. *Let Γ be a multiset of formulas. We define $s(\Gamma) = \sum_{F \in \Gamma} s(F)$.*

Fact 47. *Let Γ be a multiset of formulas. Then,*

- *$s(\Gamma) \geq 0$,*
- *$s(\Gamma) > 0$ if and only if $\Gamma \neq []$.*

Proof. For each formula F, $s(F) > 0$, hence $s(\Gamma) = \sum_{F \in \Gamma} s(F) \geq 0$, and $s(\Gamma) = 0$ if and only if $\Gamma = []$. □

Theorem 48. *Let Γ be a multiset of formulas such that $s(\Gamma) = 0$, and let π be a proof-structure of Γ. Then, $W_\pi = \emptyset$ and $L_\pi = \emptyset$.*

Proof. Since $s(\Gamma) = 0$, by Fact 47 $\Gamma = []$ so by Proposition 16 $W_\pi = \emptyset$ and $L_\pi = \emptyset$. □

Proposition 49. *Let Γ be a multiset of formulas such that $s(\Gamma) > 0$ and let π be a proof-structure of Γ. Then, there exists $\ell \in L_\pi$ such that $c_\pi(\ell) \subset \mathrm{Op}_\pi$.*

Proof. Since $s(\Gamma) > 0$, by Fact 47 $\Gamma \neq []$ so by Proposition 16 $\#L_\pi > 0$ so by Proposition 15 there exists a link $\ell \in L_\pi$ such that $\forall x \in c_\pi(\ell), \forall \ell' \in L_{\overline{\pi}}, x \notin p_\pi(\ell)$. $\square$

Theorem 50. *Let Γ be a multiset of formulas such that $s(\Gamma) > 0$ and let π be a proof-structure of Γ. Then, there exists an induced substructure $\overline{\pi}$ of π.*

Proof. Since $s(\Gamma) > 0$, by Proposition 49 there exists a link $\ell \in L_\pi$ such that $c_\pi(\ell) \subset \mathrm{Op}_\pi$, so the induced substructure of π after removing ℓ is an induced substructure of π. $\square$

Fact 51. *Let π be a proof-structure and let $\ell \in L_\pi$. Then, $s([f_\pi(x), x \in p_\pi(\ell)]) < s([f_\pi(x), x \in c_\pi(\ell)])$.*

Proof. If $t_\pi(\ell) \in \{\texttt{axiom}, \texttt{FO-axiom}, \texttt{unit}\}$, then by Fact 12 $\#c_\pi(\ell) \geq 1$ and from Definition 4 $\#p_\pi(\ell) = 0$. Hence, since $s(F) > 0$ for each formula F, $s([f_\pi(x), x \in c_\pi(\ell)]) > 0 = s([f_\pi(x), x \in p_\pi(\ell)])$.

Otherwise if $t_\pi(\ell) \in \{\texttt{tensor}, \texttt{par}, \texttt{prec}\}$, from Definition 4 there exist $x, y, z \in W_\pi$ such that $p_\pi(\ell) = \{x, y\}$ and $c_\pi(\ell) = \{z\}$ and there exist formulas F and G such that $f_\pi(x) = F$, $f_\pi(y) = G$ and $f_\pi(z) \in \{F \otimes G, F \parr G, F < G\}$. So, $s([f_\pi(\tilde{x}), \tilde{x} \in p_\pi(\ell)]) = s(f_\pi(x)) + s(f_\pi(y)) = s(F) + s(G) < s(F) + s(G) + 1 = s(F \otimes G) = s(F \parr G) = s(F < G) = s(f_\pi(z)) = s([f_\pi(\tilde{x}), \tilde{x} \in c_\pi(\ell)])$. $\square$

Theorem 52. *Let Γ be a multiset of formulas, let π be a proof-structure of Γ, let $\overline{\pi}$ be an induced substructure of π and let $\overline{\Gamma}$ be the multiset of conclusion formulas of $\overline{\pi}$. Then, $s(\overline{\Gamma}) < s(\Gamma)$.*

Proof. There exists $\ell \in L_\pi$ such that $\overline{\pi}$ is the induced substructure of π after removing ℓ. Let us denote $U = [f_\pi(x), x \in p_\pi(\ell)]$ and $V = [f_\pi(x), x \in c_\pi(\ell)]$. By Fact 41, $\Gamma + U = \overline{\Gamma} + V$, so $s(\Gamma + U) = s(\overline{\Gamma} + V)$. By Fact 51, $s(U) < s(V)$ hence $s(\Gamma) + s(U) - s(V) < s(\Gamma)$. Since $s(\Gamma + U) = s(\Gamma) + s(U)$ and $s(\overline{\Gamma} + V) = s(\overline{\Gamma}) + s(V)$, $s(\overline{\Gamma}) = s(\Gamma) + s(U) - s(V) < s(\Gamma)$. $\square$

4.5.4 Induction on the size of a balanced multiset of formulas

Fact 53. *Let π be a proof-structure and let $\ell \in L_\pi$ such that $t_\pi(\ell) \in \{\texttt{axiom}, \texttt{FO-axiom}, \texttt{unit}\}$. Then:*

- $[f_\pi(x), x \in c_\pi(\ell)] = \mathrm{At}([f_\pi(x), x \in c_\pi(\ell)])$,
- $[f_\pi(x), x \in c_\pi(\ell)]$ *is balanced.*

Proof. Denote $U = [f_\pi(x), x \in c_\pi(\ell)]$.

If $t_\pi(\ell) = \texttt{axiom}$, then $U = [A, A^*]$ for some $A \in \mathcal{V}$. So, $\mathrm{At}(U) = \mathrm{At}(A) + \mathrm{At}(A^*) = [A] + [A^*] = U$. For $B \in \mathcal{V}$, if $B = A$ then $\mathrm{At}(U)(B) = \mathrm{At}(U)(B^*) = 1$, otherwise if $B \neq A$ then $\mathrm{At}(U)(B) = \mathrm{At}(U)(B^*) = 0$ and for $a \in \mathcal{V}_{FO}$, $\mathrm{At}(U)(a) = \mathrm{At}(U)(a) = 0$, so U is balanced.

Otherwise if $t_\pi(\ell) = \texttt{FO-axiom}$, then $U = [a, a^*]$ for some $a \in \mathcal{V}_{FO}$. So, $\mathrm{At}(U) = \mathrm{At}(a) + \mathrm{At}(a^*) = [a] + [a^*] = U$, and U is balanced.

Otherwise if $t_\pi(\ell) = \texttt{unit}$, then $\mathrm{At}(U) = \mathrm{At}(I) = [I] = U$. For $A \in \mathcal{V}$, $\mathrm{At}(U)(A) = \mathrm{At}(U)(A^*) = 0$ and for $a \in \mathcal{V}_{FO}$, $\mathrm{At}(U)(a) = \mathrm{At}(U)(a^*) = 0$, so U is balanced. $\square$

Proposition 54. *Let Γ be a multiset of formulas, let π be a proof-structure of Γ, let $\ell \in L_\pi$ such that $c_\pi(\ell) \subset \mathrm{Op}_\pi$, let $\overline{\pi}$ be the induced substructure of π after removing ℓ and let $\overline{\Gamma}$ be the multiset of conclusion formulas of $\overline{\pi}$.*

- *If $t_\pi(\ell) \in \{\texttt{axiom}, \texttt{FO-axiom}, \texttt{unit}\}$, then $\mathrm{At}(\overline{\Gamma}) + [f_\pi(x), x \in c_\pi(\ell)] = \mathrm{At}(\Gamma)$.*
- *Otherwise if $t_\pi(\ell) \in \{\texttt{tensor}, \texttt{par}, \texttt{prec}\}$, then $\mathrm{At}(\overline{\Gamma}) = \mathrm{At}(\Gamma)$.*

Proof. By Fact 41, $\Gamma + [f_\pi(x), x \in p_\pi(\ell)] = \overline{\Gamma} + [f_\pi(x), x \in c_\pi(\ell)]$. So, $\mathrm{At}(\Gamma) + \mathrm{At}([f_\pi(x), x \in p_\pi(\ell)]) = \mathrm{At}(\overline{\Gamma}) + \mathrm{At}([f_\pi(x), x \in c_\pi(\ell)])$

If $t_\pi(\ell) \in \{\texttt{axiom}, \texttt{FO-axiom}, \texttt{unit}\}$, then from Definition 4 $p_\pi(\ell) = \emptyset$ so $\mathrm{At}(\Gamma) = \mathrm{At}(\overline{\Gamma}) + \mathrm{At}([f_\pi(x), x \in c_\pi(\ell)])$. By Fact 53, $\mathrm{At}([f_\pi(x), x \in c_\pi(\ell)]) = [f_\pi(x), x \in c_\pi(\ell)]$, hence $\mathrm{At}(\Gamma) = \mathrm{At}(\overline{\Gamma}) + [f_\pi(x), x \in c_\pi(\ell)]$.

Otherwise if $t_\pi(\ell) \in \{\texttt{tensor}, \texttt{par}, \texttt{prec}\}$, then from Definition 4 $p_\pi(\ell) = \{x, y\}$ and $c_\pi(\ell) = \{z\}$ for some $x, y, z \in W_\pi$ and $f_\pi(x) = F$ and $f_\pi(y) = G$ and $f_\pi(z) \in \{F \otimes G, F \parr G, F < G\}$.

So, $\mathrm{At}([f_\pi(\tilde{x}), \tilde{x} \in p_\pi(\ell)]) = \mathrm{At}(F) + \mathrm{At}(G)$. From Definition 33, $\mathrm{At}(f_\pi(x)) = \mathrm{At}(F \otimes G) = \mathrm{At}(F \parr G) = \mathrm{At}(F < G) = \mathrm{At}(F) + \mathrm{At}(G)$, hence $\mathrm{At}([f_\pi(\tilde{x}), \tilde{x} \in c_\pi(\ell)]) = \mathrm{At}(F) + \mathrm{At}(G)$. So, $\mathrm{At}(\Gamma) + \mathrm{At}(F) + \mathrm{At}(G) = \mathrm{At}(\overline{\Gamma}) + \mathrm{At}(F) + \mathrm{At}(G)$, hence $\mathrm{At}(\Gamma) = \mathrm{At}(\overline{\Gamma})$. □

Theorem 55. *Let Γ be a multiset of formulas, let π be a proof-structure of Γ, let $\overline{\pi}$ be an induced substructure of π and let $\overline{\Gamma}$ be the multiset of conclusion formulas of $\overline{\pi}$.*

- *If Γ is balanced, then $\overline{\Gamma}$ is also balanced.*
- *If $I \notin \mathrm{At}(\Gamma)$, then $I \notin \mathrm{At}(\overline{\Gamma})$.*

Proof. Since $\overline{\pi}$ is an induced substructure of π, from Definition 38 there exists $\ell \in L_\pi$ such that $c_\pi(\ell) \subset \mathrm{Op}_\pi$ and $\overline{\pi}$ is the induced substructure of π after removing ℓ.

If $t_\pi(\ell) \in \{\texttt{axiom}, \texttt{FO-axiom}, \texttt{unit}\}$, then by Proposition 54 $\mathrm{At}(\overline{\Gamma}) = \mathrm{At}(\Gamma) - [f_\pi(x), x \in c_\pi(\ell)]$, and by Fact 53 $\mathrm{At}([f_\pi(x), x \in c_\pi(\ell)]) = [f_\pi(x), x \in c_\pi(\ell)]$ and $[f_\pi(x), x \in c_\pi(\ell)]$ is balanced. Denote $U = [f_\pi(x), x \in c_\pi(\ell)]$, then $\mathrm{At}(\overline{\Gamma}) = \mathrm{At}(\Gamma) - U$ and U is balanced and $\mathrm{At}(U) = U$. Since $\mathrm{At}(\overline{\Gamma}) = \mathrm{At}(\Gamma) - U$, $\mathrm{At}(\overline{\Gamma}) \leq \mathrm{At}(\Gamma)$.

- Assume that Γ is balanced. For $A \in \mathcal{V}$, $\mathrm{At}(\overline{\Gamma})(A) = \mathrm{At}(\Gamma)(A) - U(A)$. Since Γ is balanced, $\mathrm{At}(\Gamma)(A) = \mathrm{At}(\Gamma)(A^*) \leq 1$. Since U is balanced and $\mathrm{At}(U) = U$, $U(A) = U(A^*)$. So, $\mathrm{At}(\overline{\Gamma})(A) = \mathrm{At}(\Gamma)(A^*) - U(A^*) = \mathrm{At}(\overline{\Gamma})(A^*) \leq 1$. For $a \in \mathcal{V}_{FO}$, $\mathrm{At}(\overline{\Gamma})(a) = \mathrm{At}(\Gamma)(a) - U(a)$. Since Γ and U are balanced and $\mathrm{At}(U) = U$, $\mathrm{At}(\Gamma)(a) = \mathrm{At}(\Gamma)(a^*) \leq 1$ and $U(a) = U(a^*)$. So, $\mathrm{At}(\overline{\Gamma})(a) = \mathrm{At}(\Gamma)(a^*) - U(a^*) = \mathrm{At}(\overline{\Gamma})(a^*) \leq 1$. So, $\overline{\Gamma}$ is balanced.
- Assume that $I \notin \mathrm{At}(\Gamma)$. Then, $\mathrm{At}(\overline{\Gamma})(I) \leq \mathrm{At}(\Gamma)(I) = 0$, so $I \notin \mathrm{At}(\overline{\Gamma})$.

Otherwise if $t_\pi(\ell) \in \{\texttt{tensor}, \texttt{par}, \texttt{prec}\}$, then by Proposition 54 $\mathrm{At}(\overline{\Gamma}) = \mathrm{At}(\Gamma)$, which implies both statements. □

4.6 Some results proven by induction

Theorem 56. *Let Γ be a balanced multiset of formulas such that $I \notin \mathrm{At}(\Gamma)$ and let π be a proof-structure of Γ. For any wires $x, y \in W_\pi$, if $f_\pi(x) = f_\pi(y)$ then $x = y$.*

Proof. Since $s(\Gamma) \geq 0$ for any multiset of formulas Γ, we prove this proposition by induction on $s(\Gamma)$. Let Γ be a balanced multiset of formulas such that $I \notin \mathrm{At}(\Gamma)$. We can use the induction hypothesis: "for any balanced multiset of formulas $\overline{\Gamma}$ such that $I \notin \mathrm{At}(\overline{\Gamma})$ and $s(\overline{\Gamma}) < s(\Gamma)$, for any proof-structure $\overline{\pi}$ of $\overline{\Gamma}$, for any wires $x, y \in W_{\overline{\pi}}$, if $f_{\overline{\pi}}(x) = f_{\overline{\pi}}(y)$ then $x = y$". Let π be a proof-structure of Γ. Let $x, y \in W_\pi$ such that $f_\pi(x) = f_\pi(y)$, we show that $x = y$. By Proposition 27 there exist unique wires $x', y' \in \mathrm{Op}_\pi$ such that x is above x' and y is above y' in π. From Definition 4, $f_\pi(x'), f_\pi(y') \in \Gamma$. There exists an atomic formula H which is a subformula of $f_\pi(x)$. Since x is above x' and y is above y' and since $f_\pi(x) = f_\pi(y)$, by Proposition 29 $f_\pi(x)$ is a subformula of $f_\pi(x')$ and of $f_\pi(y')$. Since H is a subformula of $f_\pi(x)$ and $f_\pi(x)$ is a subformula of $f_\pi(x')$ and of $f_\pi(y')$, H is a subformula of $f_\pi(x')$ and of $f_\pi(y')$. So, by Fact 34 $H \in \mathrm{At}(f_\pi(x')) \cap \mathrm{At}(f_\pi(y'))$ hence $\mathrm{At}(f_\pi(x')) \cap \mathrm{At}(f_\pi(y')) \neq []$. Since Γ is balanced and $I \notin \mathrm{At}(\Gamma)$ and $f_\pi(x'), f_\pi(y') \in \Gamma$ and $\mathrm{At}(f_\pi(x')) \cap \mathrm{At}(f_\pi(y')) \neq []$, by Fact 36 $f_\pi(x') = f_\pi(y')$, and by Fact 37 $\Gamma(f_\pi(x')) \leq 1$. Hence, since $x', y' \in \mathrm{Op}_\pi$ and Γ is the set of conclusion formulas of π, $x' = y'$. Since $f_\pi(x)$ is a subformula of $f_\pi(x')$, by Proposition 11 $s(f_\pi(x')) \geq s(f_\pi(x))$.

If $s(f_\pi(x')) = s(f_\pi(x))$, then since $f_\pi(x) = f_\pi(y)$, $s(f_\pi(x')) = s(f_\pi(y))$. Since x is above x' and $s(f_\pi(x')) = s(f_\pi(x))$ and y is above $y' = x'$ and $s(f_\pi(x')) = s(f_\pi(y))$, by Proposition 30 $x = x'$ and $y = x'$, hence $x = y$.

Otherwise if $s(f_\pi(x')) > s(f_\pi(x))$, then since $s(f_\pi(x)) \geq 1$, $s(f_\pi(x')) > 1$. From Definition 4 there exists ℓ such that $x' \in c_\pi(\ell)$. Since $s(f_\pi(x')) > 1$ and $x \in \mathrm{Op}_\pi$, by Fact 13 $c_\pi(\ell) \subset \mathrm{Op}_\pi$. Let $\overline{\pi}$ be the induced substructure of π after removing ℓ. Let $\overline{\Gamma}$ be the multiset of formulas of $\overline{\pi}$. Since $\overline{\pi}$ is an induced substructure of π, by Theorem 52 $s(\overline{\Gamma}) < s(\Gamma)$. Since $\overline{\pi}$ is an induced substructure of π and Γ is balanced and $I \notin \mathrm{At}(\Gamma)$, by Theorem 55 $\overline{\Gamma}$ is balanced and $I \notin \overline{\Gamma}$. So, by the induction hypothesis, for each $\tilde{x}, \tilde{y} \in W_{\overline{\pi}}$, if $f_{\overline{\pi}}(\tilde{x}) = f_{\overline{\pi}}(\tilde{y})$ then $\tilde{x} = \tilde{y}$. Since $s(f_\pi(x')) > s(f_\pi(x)) = s(f_\pi(y))$ and $x' \in c_\pi(\ell)$, by Fact 10 $x, y \notin c_\pi(\ell)$, hence $x, y \in W_{\overline{\pi}} = W_\pi \setminus c_\pi(\ell)$. Since $x, y \in W_{\overline{\pi}}$ and $f_{\overline{\pi}}(x) = f_\pi(x) = f_\pi(y) = f_{\overline{\pi}}(y)$, $x = y$. □

Proposition 57. *Let Γ be a balanced multiset of formulas such that $I \notin \mathrm{At}(\Gamma)$, let π and π' be two proof-structures of Γ, let $\ell \in L_\pi$ and $\ell' \in L_{\pi'}$ and let $x \in c_\pi(\ell)$ and $x' \in c_{\pi'}(\ell')$ such that $c_\pi(\ell) \subset \mathrm{Op}_\pi$ and $f_\pi(x) = f_{\pi'}(x')$. Then, $c_{\pi'}(\ell') \subset \mathrm{Op}_{\pi'}$.*

Proof. Since $f_\pi(x) = f_\pi(y)$, by Proposition 19 $\#c_\pi(\ell) = \#c_{\pi'}(\ell')$ and for each conclusion y of ℓ, denoting y' the corresponding conclusion of ℓ', $f_\pi(y) = f_{\pi'}(y')$. Let $y' \in c_{\pi'}(\ell')$, we show that $y' \in \mathrm{Op}_{\pi'}$. The conclusion $y \in c_\pi(\ell)$ corresponding to y' satisfies $f_\pi(y) = f_{\pi'}(y')$. Since $y \in c_\pi(\ell) \subset \mathrm{Op}_\pi$, $y \in \mathrm{Op}_\pi$, hence $f_\pi(y) \in \Gamma$. So, since $f_\pi(y) = f_{\pi'}(y')$, $f_{\pi'}(y') \in \Gamma$. Since Γ is balanced and $I \notin \mathrm{At}(\Gamma)$, by Theorem 56 y' is unique given $f_{\pi'}(y')$. So, if $y' \notin \mathrm{Op}_{\pi'}$ then $f_{\pi'}(y') \notin \Gamma$ which is absurd, so $y' \in \mathrm{Op}_{\pi'}$. □

4.7 Equivalent proof-structures

Definition 58. *Two proof structures π and π' are called equivalent when there exist bijections $\mu : W_\pi \simeq W_{\pi'}$ and $\nu : L_\pi \simeq L_{\pi'}$ such that:*

- *for each $x \in W_\pi$,*
$$f_\pi(x) = f_{\pi'}(\mu(x)) \ , \tag{17}$$
- *for each $\ell \in L_\pi$,*
$$t_\pi(\ell) = t_{\pi'}(\nu(\ell)) \ , \tag{18}$$
$$\mu(p_\pi(\ell)) = p_{\pi'}(\nu(\ell)) \ , \tag{19}$$
$$\mu(c_\pi(\ell)) = c_{\pi'}(\nu(\ell)) \ . \tag{20}$$

Proposition 59. *Let π and π' be two proof-structures and let $\overline{\pi}$ and $\overline{\pi'}$ be the induced substructures of π and π' respectively after removing some links $\ell \in L_\pi$ and $\ell' \in L_{\pi'}$ respectively such that $\overline{\pi}$ and $\overline{\pi'}$ are equivalent and there exist $x \in c_\pi(\ell)$ and $x' \in c_{\pi'}(\ell')$ such that $f_\pi(x) = f_{\pi'}(x')$. Then, π and π' are equivalent.*

Proof. Since $\overline{\pi}$ and $\overline{\pi'}$ are equivalent there exists $\overline{\mu} : W_{\overline{\pi}} \simeq W_{\overline{\pi'}}$ and $\overline{\nu} : L_{\overline{\pi}} \simeq L_{\overline{\pi'}}$ which satisfy the conditions of Definition 58. From Definition 38, $W_{\overline{\pi}} = W_\pi \setminus c_\pi(\ell)$, $L_{\overline{\pi}} = L_\pi \setminus \{\ell\}$, $W_{\overline{\pi'}} = W_{\pi'} \setminus c_{\pi'}(\ell')$ and $L_{\overline{\pi'}} = L_{\pi'} \setminus \{\ell'\}$. Since $x \in c_\pi(\ell)$ and $x' \in c_{\pi'}(\ell')$ and $f_\pi(x) = f_{\pi'}(x')$, by Proposition 19 $\#c_\pi(\ell) = \#c_{\pi'}(\ell')$. Let $\mu : W_\pi \simeq W_{\pi'}$ be the bijection obtained by extending $\overline{\mu} : W_{\overline{\pi}} \simeq W_{\overline{\pi'}}$ with $c_\pi(\ell) \simeq c_{\pi'}(\ell')$, by mapping each conclusion of ℓ to the corresponding conclusion of ℓ' from leftmost to rightmost. Let $\nu : L_\pi \simeq L_{\pi'}$ obtained by extending $\overline{\nu} : L_{\overline{\pi}} \simeq L_{\overline{\pi'}}$ with $\{\ell\} \simeq \{\ell'\}$. We show that μ and ν satisfy the conditions of Definition 58, hence that π and π' are equivalent proof-structures, which will conclude the proof.

- Let $y \in W_\pi$, we prove that $f_\pi(y) = f_{\overline{\pi}}(\mu(y))$. If $y \in W_{\overline{\pi}}$, then from the definition of μ above, $\mu(y) = \overline{\mu}(y) \in W_{\overline{\pi'}}$. Since $\overline{\mu}$ and $\overline{\nu}$ form an equivalence between the proof-structures $\overline{\pi}$ and $\overline{\pi'}$, from Definition 58 $f_{\overline{\pi}}(y) = f_{\overline{\pi'}}(\overline{\mu}(y)) = f_{\overline{\pi'}}(\mu(y))$. From Definition 38 $f_{\overline{\pi}} = (f_\pi)_{|W_{\overline{\pi}}}$ and $f_{\overline{\pi'}} = (f_{\pi'})_{|W_{\overline{\pi'}}}$, hence since $y \in W_{\overline{\pi}}$, $f_{\overline{\pi}}(y) = f_\pi(y)$, and since $\mu(y) \in W_{\overline{\pi'}}$, $f_{\overline{\pi'}}(\mu(y)) = f_{\pi'}(\mu(y))$. So, $f_\pi(y) = f_{\overline{\pi}}(y) = f_{\overline{\pi'}}(\mu(y)) = f_{\pi'}(\mu(y))$, so the condition of Definition 58 is satisfied. Otherwise if $y \notin W_{\overline{\pi}}$, then $y \in c_\pi(\ell)$ and from the definitions of μ and ν, $\mu(y) \in c_{\pi'}(\ell') = c_{\pi'}(\nu(\ell))$, $\#c_\pi(\ell) = \#c_{\pi'}(\ell') = \#c_{\pi'}(\nu(\ell))$ and $\mu(y) \in c_{\pi'}(\nu(\ell))$ is the corresponding conclusion to $y \in c_\pi(\ell)$. Since $x \in c_\pi(\ell)$ and $x' \in c_{\pi'}(\ell')$ and $f_\pi(x) = f_{\pi'}(x')$, by Proposition 19 $f_\pi(y) = f_{\pi'}(\mu(y))$. So, condition (17) is satisfied.
- Let $\tilde{\ell} \in L_\pi$, we show that $t_\pi(\tilde{\ell}) = t_{\pi'}(\nu(\tilde{\ell}))$, $\mu(p_\pi(\tilde{\ell})) = p_{\pi'}(\nu(\tilde{\ell}))$ and $\mu(c_\pi(\tilde{\ell})) = c_{\pi'}(\nu(\tilde{\ell}))$.

  If $\tilde{\ell} \in L_{\overline{\pi}}$ then from the definition of ν, $\nu(\tilde{\ell}) = \overline{\nu}(\tilde{\ell}) \in L_{\overline{\pi'}}$. Since $\overline{\mu}$ and $\overline{\nu}$ form an equivalence between the proof-structures $\overline{\pi}$ and $\overline{\pi'}$, from Definition 58 $t_{\overline{\pi}}(\tilde{\ell}) = t_{\overline{\pi'}}(\overline{\nu}(\tilde{\ell})) = t_{\overline{\pi'}}(\nu(\tilde{\ell}))$, $\overline{\mu}(p_{\overline{\pi}}(\tilde{\ell})) = p_{\overline{\pi'}}(\overline{\nu}(\tilde{\ell})) = p_{\overline{\pi'}}(\nu(\tilde{\ell}))$ and $\overline{\mu}(c_{\overline{\pi}}(\tilde{\ell})) = c_{\overline{\pi'}}(\overline{\nu}(\tilde{\ell})) = c_{\overline{\pi'}}(\nu(\tilde{\ell}))$. Since $p_{\overline{\pi}}(\tilde{\ell}) \subset W_{\overline{\pi}}$ and $c_{\overline{\pi}}(\tilde{\ell}) \subset W_{\overline{\pi}}$, from the definition of μ, $\mu(p_{\overline{\pi}}(\tilde{\ell})) = \overline{\mu}(p_{\overline{\pi}}(\tilde{\ell})) = p_{\overline{\pi'}}(\nu(\tilde{\ell}))$ and $\mu(c_{\overline{\pi}}(\tilde{\ell})) = \overline{\mu}(c_{\overline{\pi}}(\tilde{\ell})) = c_{\overline{\pi'}}(\nu(\tilde{\ell}))$. Since $\tilde{\ell} \in L_{\overline{\pi}}$ and $\overline{\pi}$ is the induced substructure of π after removing ℓ, from Definition 38 $t_\pi(\tilde{\ell}) = t_{\overline{\pi}}(\tilde{\ell}) = t_{\overline{\pi'}}(\nu(\tilde{\ell}))$, $\mu(p_\pi(\tilde{\ell})) = \mu(p_{\overline{\pi}}(\tilde{\ell})) = p_{\overline{\pi'}}(\nu(\tilde{\ell}))$ and $\mu(c_\pi(\tilde{\ell})) =$

$\mu(c_{\overline{\pi}}(\tilde{\ell})) = c_{\overline{\pi'}}(\nu(\tilde{\ell}))$. Since $\nu(\tilde{\ell}) \in L_{\overline{\pi'}}$ and $\overline{\pi'}$ is the induced substructure of π' after removing ℓ', from Definition 38 $t_\pi(\tilde{\ell}) = t_{\overline{\pi'}}(\nu(\tilde{\ell})) = t_{\pi'}(\nu(\tilde{\ell}))$, $\mu(p_\pi(\tilde{\ell})) = p_{\overline{\pi'}}(\nu(\tilde{\ell})) = p_{\pi'}(\nu(\tilde{\ell}))$ and $\mu(c_\pi(\tilde{\ell})) = c_{\overline{\pi}}(\nu(\tilde{\ell})) = c_\pi(\nu(\tilde{\ell}))$, so conditions (18), (19) and (20) are satisfied.

Otherwise if $\tilde{\ell} \notin L_{\overline{\pi}}$ then $\tilde{\ell} = \ell$. Since x is a conclusion of ℓ and x' is a conclusion of $\ell' = \nu(\ell)$ and $f_\pi(x) = f_{\pi'}(x')$, by Proposition 19, $t_\pi(\ell) = t_{\pi'}(\nu(\ell))$. Since $x \in c_\pi(\ell)$ and $x' \in c_{\pi'}(\nu(\ell))$ and $f_\pi(x) = f_{\pi'}(x')$, by Proposition 19, $\#p_\pi(\ell) = \#p_{\pi'}(\nu(\ell))$ and for each premisse y of ℓ, denoting y' the corresponding premisse of $\nu(\ell)$, $f_\pi(y) = f_{\pi'}(y')$. Let y be a premisse of ℓ and let y' be the corresponding premisse of $\nu(\ell)$, then $f_\pi(y) = f_{\pi'}(y')$. Since $y \in p_\pi(\ell)$, by Proposition 17 $y \notin c_\pi(\ell)$, so $y \in W_{\overline{\pi}} = W_\pi \setminus c_\pi(\ell)$. So from the definition of μ, $\mu(y) = \overline{\mu}(y) \in W_{\overline{\pi'}}$ and since $\overline{\mu}$ and $\overline{\nu}$ form an equivalence between $\overline{\pi}$ and $\overline{\pi'}$, from Definition 58 $f_{\overline{\pi}}(y) = f_{\overline{\pi'}}(\overline{\mu}(y)) = f_{\overline{\pi'}}(\mu(y))$. Since $y \in W_{\overline{\pi}}$ and $\overline{\pi}$ is the induced substructure of π after removing ℓ, from Definition 38 $f_\pi(y) = f_{\overline{\pi}}(y) = f_{\overline{\pi'}}(\mu(y))$. Since $\mu(y) \in W_{\overline{\pi'}}$ and $\overline{\pi'}$ is the induced substructure of π' after removing ℓ', from Definition 38 $f_{\pi'}(\mu(y)) = f_{\overline{\pi'}}(\mu(y)) = f_\pi(y)$. So $f_{\pi'}(\mu(y)) = f_\pi(y) = f_{\pi'}(y')$, hence by Proposition 56 $y' = \mu(y)$. This holds for any $y \in p_\pi(\ell)$ and for the corresponding $y' \in p_{\pi'}(\nu(\ell))$, hence $\mu(p_\pi(\ell)) = p_{\pi'}(\nu(\ell))$. From the definition of μ, μ maps each conclusion of ℓ to the corresponding conclusion of $\ell' = \nu(\ell)$, i.e. $\mu(c_\pi(\ell)) = c_{\pi'}(\nu(\ell))$. $t_\pi(\ell) = t_{\pi'}(\nu(\ell))$, $\mu(p_\pi(\ell)) = p_{\pi'}(\nu(\ell))$ and $\mu(c_\pi(\ell)) = c_{\pi'}(\nu(\ell))$, so conditions (18), (19) and (20) are satisfied.

□

Theorem 60. *Let Γ be a balanced multiset of formulas such that $I \notin \mathrm{At}(\Gamma)$. Then, all proof-structures of Γ are equivalent.*

Proof. Since for any multiset of formulas Γ, $s(\Gamma) \geq 0$, we prove this property by induction on $s(\Gamma)$. Let Γ be a balanced multiset of formulas such that $I \notin \mathrm{At}(\Gamma)$. We can use the induction hypothesis: "for any balanced multiset of formulas $\overline{\Gamma}$ such that $I \notin \mathrm{At}(\overline{\Gamma})$ and $s(\overline{\Gamma}) < s(\Gamma)$, all proof-structures of $\overline{\Gamma}$ are equivalent." Let π and π' be two proof-structures of Γ. We prove that π and π' are equivalent.

If $s(\Gamma) = 0$, since π and π' are proof-structures of Γ, by Theorem 48 $W_\pi = \emptyset$ and $L_\pi = \emptyset$ and $W_{\pi'} = \emptyset$ and $L_{\pi'} = \emptyset$. Let $\mu : W_\pi \simeq W_{\pi'}$ and $\nu : L_\pi \simeq L_{\pi'}$ be the unique bijection $\emptyset \simeq \emptyset$. μ and ν satisfy the conditions of Definition 58 since these conditions are universally quantified over $\emptyset$. Hence, π and π' are equivalent proof-structures.

Otherwise if $s(\Gamma) > 0$, then since π is a proof-structure of Γ, by Theorem 50 there exists an induced substructure $\overline{\pi}$ of π. Let $\overline{\Gamma}$ be the multiset of conclusion formulas of $\overline{\pi}$. By Theorem 52 $s(\overline{\Gamma}) < s(\Gamma)$. Since Γ is balanced and $I \notin \mathrm{At}(\Gamma)$, by Theorem 55 $\overline{\Gamma}$ is balanced and $I \notin \mathrm{At}(\overline{\Gamma})$. Since $\overline{\pi}$ is an induced substructure of π there exists $\ell \in L_\pi$ such that $c_\pi(\ell) \subset \mathrm{Op}_\pi$ and $\overline{\pi}$ is the induced substructure of π after removing ℓ. By Fact 12 there exists $x \in c_\pi(\ell)$, and hence $x \in \mathrm{Op}_\pi$, so $f_\pi(x) \in \Gamma$. Since π' is a proof-structure of Γ and $f_\pi(x) \in \Gamma$, there exists $x' \in \mathrm{Op}_{\pi'}$ such that $f_{\pi'}(x') = f_\pi(x)$. From Definition 4 there exists $\ell' \in L_{\pi'}$ such that $x' \in c_{\pi'}(\ell')$. Since Γ is balanced and $I \notin \mathrm{At}(\Gamma)$ and $c_\pi(\ell) \subset \mathrm{Op}_\pi$ and $f_\pi(x) = f_{\pi'}(x')$, by Proposition 57 $c_{\pi'}(\ell') \subset \mathrm{Op}_{\pi'}$. Let $\overline{\pi'}$ be the induced substructure of π' after removing ℓ'. Since $f_{\pi'}(x') = f_\pi(x)$, by Proposition 42 the multiset of conclusion formulas of $\overline{\pi'}$ equals the multiset of conclusion formulas of $\overline{\pi}$ i.e. $\overline{\Gamma}$.

Since $\overline{\pi}$ and $\overline{\pi'}$ are both proof-structures of $\overline{\Gamma}$ and $\overline{\Gamma}$ is balanced and $I \notin \mathrm{At}(\overline{\Gamma})$ and $s(\overline{\Gamma}) < s(\Gamma)$, by the induction hypothesis $\overline{\pi}$ and $\overline{\pi'}$ are equivalent. Since $\overline{\pi}$ and $\overline{\pi'}$ are equivalent and $f_\pi(x) = f_{\pi'}(x')$, by Proposition 59 π and π' are equivalent. □

Proposition 61. *Let π and π' be two equivalent proof-structures. If π is a proof-net, then π' is also a proof-net.*

Proof. From Definition 58 there exists bijections $\mu : W_\pi \simeq W_{\pi'}$ and $\nu : L_\pi \simeq L_{\pi'}$ such that for each $x \in W_\pi, f_\pi(x) = f_{\pi'}(\mu(x))$, and for each $\ell \in L_\pi, t_\pi(\ell) = t_{\pi'}(\nu(\ell))$, $\mu(p_\pi(\ell)) = p_{\pi'}(\nu(\ell))$ and $\mu(c_\pi(\ell)) = c_{\pi'}(\nu(\ell))$. Let s' be a switching function of π' and let $\mathcal{G}'$ be the switching graph of π' whose set of arcs is the image of s'. Let $s : L_\pi \to W_\pi \times W_\pi$ be the following partial function: for each link $\ell \in L_\pi$, if s' maps the link $\nu(\ell)$ to an arc $s'(\nu(\ell)) = x' \to y'$ then $s(\ell) = x \to y$ where $x = \mu^{-1}(x')$ and $y = \mu^{-1}(y')$, otherwise if s' doesn't map $\nu(\ell)$ to any arc then s doesn't map ℓ to any arc.

We show that s is a switching function of π. I.e., we show that for each $\ell \in L_\pi$, s satisfies the corresponding condition of Definition 5 depending on $t_\pi(\ell)$. If $t_\pi(\ell) = \texttt{axiom}$, then $t_{\pi'}(\nu(\ell)) = \texttt{axiom}$ so from Definition 5 s' maps $\nu(\ell)$ to an arc $s'(\nu(\ell)) \in \{x' \to y', y' \to x'\}$ where x' and y' respectively denote the left and right conclusions of $\nu(\ell)$. So from the definition of s, denoting $x = \mu^{-1}(x')$ and $y = \mu^{-1}(y')$, s maps ℓ to an arc $s(\ell) \in \{x \to y, y \to x\}$. Since $\mu(c_\pi(\ell)) = c_{\pi'}(\nu(\ell))$ and $x' = \mu(x)$ and $y' = \mu(y)$ are the left and right conclusions of $\nu(\ell)$ respectively, x and y are the left and right conclusions of ℓ respectively, hence since $s(\ell) \in \{x \to y, y \to x\}$, s satisfies the condition of Definition 5. If $t_\pi(\ell) = \texttt{FO-axiom}$, then $t_{\pi'}(\nu(\ell)) = \texttt{FO-axiom}$ so from Definition 5, $s'(\nu(\ell)) = y' \to x'$ where x' and y' denote the left and right conclusions of $\nu(\ell)$ respectively. So from the definition of s, denoting $x = \mu^{-1}(x')$ and $y = \mu^{-1}(y')$, s maps ℓ to the arc $s(\ell) = y \to x$. Since $\mu(c_\pi(\ell)) = c_{\pi'}(\nu(\ell))$ and x' and y' are the left and right conclusions of $\nu(\ell)$ respectively, x and y are the left and right conclusions of ℓ respectively, hence since $s(\ell) = y \to x$, s satisfies the condition of Definition 5. If $t_\pi(\ell) = \texttt{unit}$, then $t_{\pi'}(\nu(\ell)) = \texttt{unit}$ so from Definition 5, s' doesn't map $\nu(\ell)$ to any arc. So from the definition of s, s doesn't map ℓ to any arc, so s satisfies the condition of Definition 5. If $t_\pi(\ell) = \texttt{tensor}$, then $t_{\pi'}(\nu(\ell)) = \texttt{tensor}$ so from Definition 5, denoting x' and y' the left and right premisses of $\nu(\ell)$ respectively and z' the conclusion of $\nu(\ell)$, s' maps $\nu(\ell)$ to an arc $s'(\nu(\ell)) \in \{x' \to z', z' \to x', y' \to z', z' \to y', x' \to y', y' \to x'\}$. So from the definition of s, denoting $x = \mu^{-1}(x')$, $y = \mu^{-1}(y')$ and $z = \mu^{-1}(z')$, s maps ℓ to an arc $s(\ell) \in \{x \to z, z \to x, y \to z, z \to y, x \to y, y \to x\}$. Since $\mu(p_\pi(\ell)) = p_{\pi'}(\nu(\ell))$ and x' and y' are the left and right premisses of $\nu(\ell)$ respectively, x and y are the left and right premisses of ℓ respectively, and since $\mu(c_\pi(\ell)) = c_{\pi'}(\nu(\ell))$ and z' is the conclusion of $\nu(\ell)$, z is the conclusion of ℓ, hence since $s(\ell) \in \{x \to z, z \to x, y \to z, z \to y, x \to y, y \to x\}$, s satisfies the condition of Definition 5. If $t_\pi(\ell) = \texttt{par}$, then $t_{\pi'}(\nu(\ell)) = \texttt{par}$ so from Definition 5, denoting x' and y' respectively the left and right premisses of $\nu(\ell)$ and z' the conclusion of $\nu(\ell)$, s' maps $\nu(\ell)$ to an arc $s'(\nu(\ell)) \in \{x' \to z', z' \to x', y' \to z', z' \to y'\}$. So from the definition of s, denoting $x = \mu^{-1}(x')$, $y = \mu^{-1}(y')$ and $z = \mu^{-1}(z')$, s maps ℓ to an arc $s(\ell) \in \{x \to z, z \to x, y \to z, z \to y\}$. Since $\mu(p_\pi(\ell)) = p_{\pi'}(\nu(\ell))$ and $x' = \mu(x)$ and $y' = \mu(y)$ are the left and right premisses of $\nu(\ell)$ respectively, x and y are the left and right premisses of ℓ respectively, and since $\mu(c_\pi(\ell)) = c_{\pi'}(\nu(\ell))$ and $z' = \mu(z)$ is the conclusion of $\nu(\ell)$, z is the conclusion of ℓ, hence since $s(\ell) \in \{x \to z, z \to x, y \to z, z \to y\}$, s satisfies the condition of Definition 5. If $t_\pi(\ell) = \texttt{prec}$, then $t_{\pi'}(\nu(\ell)) = \texttt{prec}$ so from Definition 5, denoting x' and y' respectively the premisses of $\nu(\ell)$ and z' the conclusion of $\nu(\ell)$, s' maps $\nu(\ell)$ to an arc $s'(\nu(\ell)) \in \{x' \to z', z' \to x', y' \to z', z' \to y', x' \to y'\}$. So from the definition of s, denoting $x = \mu^{-1}(x')$, $y = \mu^{-1}(y')$ and $z = \mu^{-1}(z')$, s maps ℓ to an arc $s(\ell) \in \{x \to z, z \to x, y \to z, z \to y, x \to y\}$. Since $\mu(p_\pi(\ell)) = p_{\pi'}(\nu(\ell))$ and $x' = \mu(x)$ and $y' = \mu(y)$ are the left and right premisses of $\nu(\ell)$ respectively, x and y are the left and right premisses of ℓ respectively, and since $\mu(c_\pi(\ell)) = c_{\pi'}(\nu(\ell))$ and $z' = \mu(z)$ is the conclusion of $\nu(\ell)$, z is the conclusion of ℓ, hence since $s(\ell) \in \{x \to z, z \to x, y \to z, z \to y, x \to y\}$, s satisfies the condition of Definition 5. So s is a switching function of π.

Let $\mathcal{G}$ be the switching graph of π whose set of arcs is the image of s. $\mu : W_\pi \simeq W_{\pi'}$, so μ is a bijection between the sets of nodes of $\mathcal{G}$ and $\mathcal{G}'$. We show that μ is a graph isomorphism between $\mathcal{G}$ and $\mathcal{G}'$. Let $x, y \in W_\pi$. Since the set of arcs of $\mathcal{G}$ is the image of s, $\mathcal{G}$ contains the arc $x \to y$ if and only if there exists $\ell \in L_\pi$ such that $s(\ell) = x \to y$. So from the definition of s, $\mathcal{G}$ contains the arc $x \to y$ if and only if there exists $\ell \in L_\pi$ such that $x = \mu^{-1}(x')$ and $y = \mu^{-1}(y')$ where $x' \to y'$ denotes $s'(\nu(\ell))$. Since μ is bijective, $\mathcal{G}$ contains the arc $x \to y$ if and only if there exists $\ell \in L_\pi$ such that $\mu(x) = x'$ and $\mu(y) = y'$ where $x' \to y'$ denotes $s'(\nu(\ell))$, i.e., if and only if there exists $\ell \in L_\pi$ such that $\mu(x) \to \mu(y) = s'(\nu(\ell))$. Since ν is bijective, $\mathcal{G}$ contains the arc $x \to y$ if and only if there exists $\ell' \in L_{\pi'}$ such that $\mu(x) \to \mu(y) = s'(\ell')$ and since the set of arcs of $\mathcal{G}'$ is the image of s', $\mathcal{G}$ contains the arc $x \to y$ if and only if $\mathcal{G}'$ contains the arc $\mu(x) \to \mu(y)$. Hence, μ is a graph isomorphism between $\mathcal{G}$ and $\mathcal{G}'$. Since π is a proof-net and $\mathcal{G}$ is a switching graph of π, $\mathcal{G}$ is acyclic, and since $\mathcal{G}$ and $\mathcal{G}'$ are isomorphic, $\mathcal{G}'$ is also acyclic. This holds for an arbitrary switching function s' of π', hence π' is a proof-net. □

Definition 62. *Let F be a balanced formula such that I isn't a subformula of F. We call the canonical structure of F the following structure π defined with*

- $W = \{G | G$ is a subformula of $F\}$ *and* $L = \{(A, A^*) | A$ and A^* are subformulas of F, with $A \in \mathcal{V}\} \cup \{(a, a^*) | a$ and a^* are subformulas of F, with $a \in \mathcal{V}_{FO}\} \cup \{(X \otimes Y, X, Y) | X \otimes Y$ is a subformula of $F\} \cup$

$\{(X \parr Y, X, Y) | X \parr Y$ is a subformula of $F\} \cup \{(X < Y, X, Y) | X < Y$ is a subformula of $F\}$

- *for each wire $G \in W$, $f(G) = G$,*
- *for each link $\ell \in L$, if $\ell = (A, A^*)$ for some $A \in \mathcal{V}$ then $t(\ell) = \texttt{axiom}$, $p(\ell) = \emptyset$ and $c(\ell) = \{A(\text{left}), A^*(\text{right})\}$ otherwise if $\ell = (a, a^*)$ for some $a \in \mathcal{V}_{FO}$ then $t(\ell) = \texttt{FO-axiom}$, $p(\ell) = \emptyset$ and $c(\ell) = \{a(\text{left}), a^*(\text{right})\}$ otherwise if $\ell = (X \otimes Y, X, Y)$ for some subformula $X \otimes Y$ of F then $t(\ell) = \texttt{tensor}$, $p(\ell) = \{X(\text{left}), Y(\text{right})\}$ and $c(\ell) = \{X \otimes Y\}$ and otherwise if $\ell = (X \parr Y, X, Y)$ for some subformula $X \parr Y$ of F then $t(\ell) = \texttt{par}$, $p(\ell) = \{X(\text{left}), Y(\text{right})\}$ and $c(\ell) = \{X \parr Y\}$, otherwise if $\ell = (X < Y, X, Y)$ for some subformula $X \parr Y$ of F then $t(\ell) = \texttt{prec}$, $p(\ell) = \{X(\text{left}), Y(\text{right})\}$ and $c(\ell) = \{X < Y\}$.*

Proposition 63. *Let F be a balanced formula such that I isn't a subformula of F. The canonical structure of F is a proof-structure.*

Proof. This is proven by checking Definition 4. □

(In the following we will use this proposition implicitely.)

Proposition 64. *Let π be a proof-structure, let s be a switching function of π, let $\mathcal{G}$ be the switching graph of π whose set of arcs is the image of s, and let $y \in W$ be a node of $\mathcal{G}$. If y is the target of an arc $x \to y$ and the source of an arc $y \to z$ in $\mathcal{G}$, then there exist two unique links ℓ_y and ℓ^y such that y is a conclusion of ℓ_y and a premisse of ℓ^y, and either $s(\ell_y) = x \to y$ and $s(\ell^y) = y \to z$ or $s(\ell^y) = x \to y$ and $s(\ell_y) = y \to z$.*

Proof. Since the set of arcs of $\mathcal{G}$ is the image of s there exist two links ℓ and ℓ' such that $s(\ell) = x \to y$ and $s(\ell') = y \to z$. From Definition 5 in each case depending on $t(\ell)$ and $t(\ell')$, $s(\ell) \in (p(\ell) \cup c(\ell)) \times (p(\ell) \cup c(\ell))$ and $s(\ell') \in (p(\ell') \cup c(\ell')) \times (p(\ell') \cup c(\ell'))$. Hence, y is either a premisse or a conclusion of each link ℓ and ℓ'. From Definition 5 in each case depending on $t(\ell)$, the source and the target of the arc $s(\ell)$ are different nodes, i.e. $x \neq y$. Therefore, since the arcs $x \to y$ and $y \to z$ have different source nodes $x \to y \neq y \to z$, i.e. $s(\ell) \neq s(\ell')$ and so $\ell \neq \ell'$. From Definition 4 y is a conclusion of exactly one link and a premisse of at most one link. We denote ℓ_y the unique link of which y is a conclusion. Since y is a premisse or a conclusion of two different links ℓ and ℓ', there exists a unique link ℓ^y of which y is a premisse, and either $\ell = \ell_y$ and $\ell' = \ell^y$ or $\ell = \ell^y$ and $\ell' = \ell_y$. Hence, either $s(\ell_y) = x \to y$ and $s(\ell^y) = y \to z$, or $s(\ell^y) = x \to y$ and $s(\ell_y) = y \to z$. □

Definition 65. *Let π be a proof-structure. We define the height $h(\ell)$ of a link $\ell \in L_\pi$ by induction, as:*

- *if $t(\ell) \in \{\texttt{axiom}, \texttt{FO-axiom}, \texttt{unit}\}$, let $h(\ell) = 0$,*
- *otherwise if $t(\ell) \in \{\texttt{tensor}, \texttt{par}, \texttt{prec}\}$, let $h(\ell) = 1 + \max_{x \in p(\ell)} \max_{\ell' \text{s.t.} x \in c(\ell')} h(\ell')$.*

This is well-defined since:

- *from Definition 4, for each ℓ such that $t_\pi(\ell) \in \{\texttt{tensor}, \texttt{par}, \texttt{prec}\}$, $\#p_\pi(\ell) > 0$ and for each $x \in p_\pi(\ell)$ there exists ℓ' such that $x \in c_\pi(\ell')$,*
- *for each $\ell \in L_\pi$, $\#c_\pi(\ell) > 1$ and for each $x \in p_\pi(\ell)$ and $y \in c_\pi(\ell)$, by Fact 9 $s(f_\pi(x)) < s(f_\pi(y))$. So, $h(\ell)$ is defined recursively for increasing values of $\max_{x \in c_\pi(\ell)} s(f_\pi(x))$.*

Proposition 66. *Let π be a proof-structure, let s be a switching function of π and let $\mathcal{G}$ be the switching graph of π whose set of arcs is the image of s. For any cycle C in $\mathcal{G}$, there exists a link $\ell \in L$ such that $t(\ell) \in \{\texttt{axiom}, \texttt{FO-axiom}\}$ and C contains the arc $s(\ell)$.*

Proof. Let C be a cycle in $\mathcal{G}$. C contains at least one arc. Since the set of arcs of $\mathcal{G}$ is the image of s there exists a link $\ell \in L$ such that $s(\ell) \in C$. We prove that there exists a link ℓ' such that $t(\ell') \in \{\texttt{axiom}, \texttt{FO-axiom}\}$ by induction on $h(\ell)$. For the base case assume that $h(\ell) = 0$. Hence from Definition 65 $t(\ell) \in \{\texttt{axiom}, \texttt{FO-axiom}, \texttt{unit}\}$. From Definition 5 s doesn't map any arc to each link whose type is $\texttt{unit}$, hence $t(\ell) \neq \texttt{unit}$ so $t(\ell) \in \{\texttt{axiom}, \texttt{FO-axiom}\}$, which proves the base case. For the induction case, assume that $h(\ell) > 0$. Hence from Definition 65, $t(\ell) \in \{\texttt{tensor}, \texttt{par}, \texttt{prec}\}$. From Definition 5 in each of these cases, either the source or the target of the arc $s(\ell)$ is a premisse y of ℓ. Since y is a node of C, there exists an arc $x \to y \in C$ whose target is y and an arc $y \to z \in C$ whose source is y. By Proposition 64 there exists $\ell' \in L$ such that y is a conclusion of ℓ' and $s(\ell') \in \{x \to y, y \to z\}$. Since y is a premisse of ℓ and a conclusion of ℓ' from Definition 65 $h(\ell') < h(\ell)$. Since $s(\ell') \in \{x \to y, y \to z\}$, $s(\ell') \in C$ hence we can apply the induction hypothesis, so there exists a link $\ell'' \in L$ such that $t(\ell'') \in \{\texttt{axiom}, \texttt{FO-axiom}\}$ and C contains the arc $s(\ell'')$. ☐

5 Technical results on cycles in switching graphs

5.1 Unique preimages of arcs

Fact 67. *Let π be a proof-structure, let s be a switching function of π, let $\ell \in L_\pi$ and let $x \in c_\pi(\ell)$.*

- *Let $\ell' \in L_\pi$ such that $x \in p_\pi(\ell')$. For $\tilde{\ell} \in L_\pi$, if $s(\tilde{\ell}) \in (W_\pi \to x) \cup (x \to W_\pi)$, then $\tilde{\ell} \in \{\ell, \ell'\}$.*
- *Assume that $x \in \mathrm{Op}_\pi$. For $\tilde{\ell} \in L_\pi$, if $s(\tilde{\ell}) \in (W_\pi \to x) \cup (x \to W_\pi)$, then $\tilde{\ell} = \ell$.*

Proof. This is proven by checking Definitions 4 and 5. ☐

Fact 68. *Let π be a proof-structure, let s be a switching function of π, let $\ell \in L_\pi$ such that $s(\ell)$ is defined. Then,*

- *$s(\ell) \in (p_\pi(\ell) \cup c_\pi(\ell))^2$, and*
- *denoting $s(\ell) = x \to y$, $x \neq y$.*

Proof. From Definition 5. ☐

Proposition 69. *Let π be a proof-structure, let s and s' be switching functions of π, let $\ell, \ell' \in L_\pi$ such that $s(\ell) = s'(\ell')$. Then, $\ell = \ell'$ or $s(\ell) \in (p_\pi(\ell) \cap c_\pi(\ell'))^2 \cup (p_\pi(\ell') \cap c_\pi(\ell))^2$.*

Proof. Let us denote $s(\ell) = x \to y$. By Fact 68 $x \to y \in (p_\pi(\ell) \cup c_\pi(\ell))^2 \cap (p_\pi(\ell') \cup c_\pi(\ell'))^2$. So, $x, y \in (p_\pi(\ell) \cup c_\pi(\ell)) \cap (p_\pi(\ell') \cup c_\pi(\ell'))$. If $p_\pi(\ell) \cap p_\pi(\ell') \neq \emptyset$ or $c_\pi(\ell) \cap c_\pi(\ell') \neq \emptyset$, then from Definition 4 $\ell = \ell'$. Otherwise, $x, y \in (p_\pi(\ell) \cap c_\pi(\ell')) \cup (p_\pi(\ell') \cap c_\pi(\ell))$. For a contradiction, assume that $p_\pi(\ell) \cap c_\pi(\ell') \neq \emptyset$ and $p_\pi(\ell') \cap c_\pi(\ell) \neq \emptyset$, i.e. there exists $z \in p_\pi(\ell) \cap c_\pi(\ell') \neq \emptyset$ and $z' \in p_\pi(\ell') \cap c_\pi(\ell)$. Since $z \in p_\pi(\ell)$ and $z' \in c_\pi(\ell)$, by Fact 9 $s(f_\pi(z)) < s(f_\pi(z'))$. Since $z' \in p_\pi(\ell')$ and $z \in c_\pi(\ell')$, by Fact 9 $s(f_\pi(z)) > s(f_\pi(z'))$ which is absurd. So, $p_\pi(\ell) \cap c_\pi(\ell') = \emptyset$ or $p_\pi(\ell') \cap c_\pi(\ell) = \emptyset$. Hence, $x \to y \in (p_\pi(\ell) \cap c_\pi(\ell'))^2 \cup (p_\pi(\ell') \cap c_\pi(\ell))^2$ which concludes the proof. ☐

Fact 70. *Let π be a proof-structure and let $\ell, \ell', \tilde{\ell} \in L_\pi$. If there exists $x \in W_\pi$ such that $x \in (p_\pi(\ell) \cap c_\pi(\ell')) \cup (p_\pi(\ell') \cap c_\pi(\ell))$ and $x \in (p_\pi(\ell) \cap c_\pi(\tilde{\ell})) \cup (p_\pi(\tilde{\ell}) \cap c_\pi(\ell))$, then $\ell' = \tilde{\ell}$.*

Proof. This is proven by enumerating each case $x \in (p_\pi(\ell) \cap c_\pi(\ell'))$ or $x \in (p_\pi(\ell') \cap c_\pi(\ell))$, and $x \in (p_\pi(\ell) \cap c_\pi(\tilde{\ell}))$ or $x \in (p_\pi(\tilde{\ell}) \cap c_\pi(\ell))$. ☐

Proposition 71. *Let π be a proof-structure, let s and s' be switching functions of π, let $\ell, \ell' \in L_\pi$ such that $s(\ell) = s'(\ell')$ and $s(\ell) \in (p_\pi(\ell) \cap c_\pi(\ell'))^2 \cup (p_\pi(\ell') \cap c_\pi(\ell))^2$. Let $\mathcal{G} = (W_\pi, s(L_\pi))$. If there exists a cycle C of $\mathcal{G}$ such that $s(\ell) \in \mathcal{G}$, then* $\mathrm{len}(C) = 2$.

Proof. Let us denote $s(\ell) = x \to y$. Since $x \to y \in C$ there exist $w, z \in W_\pi$ such that $w \to x \in C$ and $y \to z \in C$ and there exist ℓ_1, ℓ_2 such that $s(\ell_1) = w \to x$ and $s(\ell_2) = y \to z$. By Fact 68, $x \neq y$, so $s(\ell) \neq s(\ell_1)$ and $s(\ell) \neq s(\ell_2)$, hence $\ell \neq \ell_1$ and $\ell \neq \ell_2$.

Since $s(\ell), s(\ell_2) \in (W_\pi \to y) \cup (y \to W_\pi)$ and $\ell \neq \ell_2$, by Fact 67, ($y \in c_\pi(\ell)$ and $y \in p_\pi(\ell_2)$) or ($y \in c_\pi(\ell_2)$ and $y \in p_\pi(\ell)$), i.e., $y \in (p_\pi(\ell) \cap c_\pi(\ell_2)) \cup (p_\pi(\ell_2) \cap c_\pi(\ell))$. Since $s(\ell) \in (p_\pi(\ell) \cap c_\pi(\ell'))^2 \cup (p_\pi(\ell') \cap c_\pi(\ell))^2$, $y \in (p_\pi(\ell) \cap c_\pi(\ell')) \cup (p_\pi(\ell') \cap c_\pi(\ell))$. So, by Fact 70 $\ell' = \ell_2$. Similarly, since $s(\ell_1), s(\ell) \in (W_\pi \to x) \cup (x \to W_\pi)$ and $\ell_1 \neq \ell$, by Fact 67 $x \in (p_\pi(\ell) \cap c_\pi(\ell_1)) \cup (p_\pi(\ell_1) \cap c_\pi(\ell))$, and since $s(\ell) \in (p_\pi(\ell) \cap c_\pi(\ell'))^2 \cup (p_\pi(\ell') \cap c_\pi(\ell))^2$, $x \in (p_\pi(\ell) \cap c_\pi(\ell')) \cup (p_\pi(\ell') \cap c_\pi(\ell))$. So, by Fact 70 $\ell' = \ell_1$.

So, $\ell_1 = \ell_2$ hence $s(\ell_1) = s(\ell_2)$ i.e. $w \to x = y \to z$, so $z = x$. Since $x \neq y$ and $x \to y \in C$ and $y \to x \in C$, $\mathrm{len}(C) = 2$. ☐

Theorem 72. *Let π be a proof-structure, let s be a switching function of π and let $\ell \in L_\pi$. Let $\mathcal{G} = (W_\pi, s(L_\pi))$, and assume that there exists a cycle C of $\mathcal{G}$ such that $s(\ell) \in C$.*

- *Let $\ell' \in L_\pi$. If $s(\ell) = s(\ell')$, then $\ell = \ell'$.*
- *Let s' be a switching function of π and let $\ell' \in L_\pi$. If $s(\ell) = s'(\ell')$ and $\mathrm{len}(C) > 2$, then $\ell = \ell'$.*

Proof. Let $\ell \in L_\pi$ such that $s(\ell) = s(\ell')$. By Proposition 69, $\ell = \ell'$ or $s(\ell) \in (p_\pi(\ell) \cap c_\pi(\ell'))^2 \cup (p_\pi(\ell') \cap c_\pi(\ell))^2$. For a contradiction, we assume that $s(\ell) \in (p_\pi(\ell) \cap c_\pi(\ell'))^2 \cup (p_\pi(\ell') \cap c_\pi(\ell))^2$. Let us denote $s(\ell) = x \to y$. Since $s(\ell) \in C$ there exists $z \in W_\pi$ such that $y \to z \in C$ and there exists ℓ_2 such that $s(\ell_2) = y \to z$. By Fact 68, $x \neq y$, so $s(\ell_2) \neq s(\ell)$, so $\ell_2 \neq \ell$ and $\ell_2 \neq \ell'$. By Fact 68, $s(\ell_2) \in (p_\pi(\ell_2) \cup c_\pi(\ell_2))^2$, so $y \in p_\pi(\ell_2) \cup c_\pi(\ell_2)$. Assume that $s(\ell) \in (p_\pi(\ell) \cap c_\pi(\ell'))^2$. If $y \in p_\pi(\ell_2)$, then $y \in p_\pi(\ell_2) \cap p_\pi(\ell)$ so $\ell_2 = \ell$ which is absurd. Otherwise, $y \in c_\pi(\ell_2)$, hence $y \in c_\pi(\ell_2) \cap c_\pi(\ell')$ so $\ell_2 = \ell'$ which is absurd. So, $s(\ell) \in (p_\pi(\ell') \cap c_\pi(\ell))^2$. By the same argument, we arrive to a contradiction, hence $\ell = \ell'$.

Let s' be a switching function of π and let $\ell' \in L_\pi$ such that $s(\ell) = s'(\ell')$, and assume that $\mathrm{len}(C) > 2$. By Proposition 69, $\ell = \ell'$ or $s(\ell) \in (p_\pi(\ell) \cap c_\pi(\ell')) \cup (p_\pi(\ell') \cap c_\pi(\ell))$. For a contradiction, assume that $s(\ell) \in (p_\pi(\ell) \cap c_\pi(\ell')) \cup (p_\pi(\ell') \cap c_\pi(\ell))$. Then, by Proposition 71 $\mathrm{len}(C) = 2$ which is absurd. So, $\ell = \ell'$. □

Corollary 73. *Let π be a proof-structure, let s be a switching function of π and let $C = (e_1, \ldots, e_n)$ be a cycle of $\mathcal{G} = (W_\pi, s(L_\pi))$. There exists a unique sequence of links $(\ell_1, \ldots, \ell_n)$ such that $s(\ell_i) = e_i$ for each $i \in \{1, \ldots, n\}$.*

Proof. For each arc e_i, ℓ_i is unique by Theorem 72. □

Definition 74. *Let π be a proof-structure, let s be a switching function of π and let $C = (e_1, \ldots, e_n)$ be a cycle of $\mathcal{G} = (W_\pi, s(L_\pi))$. We call the link sequence of C the unique (according to Corollary 73) sequence $(\ell_1, \ldots, \ell_n)$ such that $s(\ell_i) = e_i$ for each $i \in \{1, \ldots, n\}$.*

5.2 Horizontal and (in)direct links

Definition 75. *Let π be a proof-structure, let s be a switching function of π and let $\ell \in L_\pi$.*

- *We say that ℓ is horizontal with respect to s when $s(\ell) \in p_\pi(\ell)^2$.*
- *Denoting $c_\pi(\ell) = x \cdot y$, we say that ℓ is direct or indirect with respect to s when $s(\ell) = y \to x$ or $x \to y$ respectively.*

5.3 Structure of a cycle in a switching graph

Fact 76. *Let π be a proof-structure, let s be a switching function of π, let C be a cycle of $\mathcal{G} = (W_\pi, s(L_\pi))$, let $(x_1, \ldots, x_n)$ be a vertex sequence of C and let $(\ell_1, \ldots, \ell_n)$ be the link sequence of C such that $s(\ell_i) = x_i \to x_{i+1}$ for each $i \in \{1, \ldots, n\}$. Then, for each $i \in \{1, \ldots, n\}$, the following statements are equivalent*

- $x_{i+1} \in p_\pi(\ell_i)$,
- $x_{i+1} \notin c_\pi(\ell_i)$,
- $x_{i+1} \notin p_\pi(\ell_{i+1})$,
- $x_{i+1} \in c_\pi(\ell_{i+1})$.

Proof. By Fact 67 and Proposition 17. □

Proposition 77. *Let π be a proof-structure, let s be a switching function of π, let C be a cycle of $\mathcal{G} = (W_\pi, s(L_\pi))$ and let $(\ell_1, \ldots, \ell_n)$ be a link sequence of C. Let $\ell_{k_1}, \ldots, \ell_{k_m}$ be the links of $(\ell_1, \ldots, \ell_n)$ which are horizontal w.r.t. s and let $\ell_{k_{(m+1)}} = \ell_{k_1}$. Then, for each $j \in \{1, \ldots, m\}$, there exists a unique link ℓ_{q_j} such that $k_j < q_j < k_{(j+1)}$ and $s(\ell_{q_j}) \in c_\pi(\ell_{q_j})^2$. For each $j \in \{1, \ldots, m\}$, $s(\ell_i) \in c_\pi(\ell_i) \times p_\pi(\ell_i)$ for each $i \in \{k_j + 1, \ldots, q_j - 1\}$, and $s(\ell_i) \in p_\pi(\ell_i) \times c_\pi(\ell_i)$ for each $i \in \{q_j + 1, \ldots, k_{(j+1)} - 1\}$.*

Proof. Let us denote $(x_1, \ldots, x_n)$ the vertex sequence of C such that $s(\ell_i) = x_i \to x_{i+1}$ for each $i \in \{1, \ldots, n\}$. Let $j \in \{1, \ldots, m\}$.

Since $\ell_{k_{(j+1)}}$ is horizontal w.r.t. s, $x_{k_{(j+1)}} \in p_\pi(\ell_{k_{(j+1)}})$, so by Fact 76 $x_{k_{(j+1)}} \in c_\pi(\ell_{k_{(j+1)}-1})$. Hence, since $k_j \leq k_{(j+1)} - 1$ there exists a minimum index q_j such that $k_j \leq q_j$ and $x_{q_j+1} \in c_\pi(\ell_{q_j})$. Then, $q_j \leq k_{(j+1)} - 1$ so $q_j < k_{(j+1)}$. Since ℓ_{k_j} is horizontal w.r.t. s, $x_{k_j+1} \in p_\pi(\ell_{k_j})$ so by Fact 76 $x_{k_j+1} \notin c_\pi(\ell_{k_j})$ so $q_j \neq k_j$, hence $k_j < q_j$. Since q_j is minimum, for each $i \in \{k_j, \ldots, q_j - 1\}$, $x_{i+1} \notin c_\pi(\ell_i)$ and so by Fact 76, $x_{i+1} \in p_\pi(\ell_i)$ and $x_{i+1} \in c_\pi(\ell_{i+1})$. Hence, $s(\ell_i) \in c_\pi(\ell_i) \times p_\pi(\ell_i)$ for each $i \in \{k_j + 1, \ldots, q_j - 1\}$, and $x_{q_j} \in c_\pi(\ell_{q_j})$. So, $s(\ell_{q_j}) \in c_\pi(\ell_{q_j})^2$.

Since $\ell_{k_{(j+1)}}$ is horizontal w.r.t. s, $x_{k_{(j+1)}+1} \in p_\pi(\ell_{k_{(j+1)}})$. So, since $q_j \leq k_{(j+1)}$ there exists a minimum index r_j such that $q_j \leq r_j$ and $x_{r_j+1} \in p_\pi(\ell_{r_j})$. Then, $r_j \leq k_{(j+1)}$. Since r_j is minimum, for each $i \in \{q_j, \ldots, r_j - 1\}$, $x_{i+1} \notin p_\pi(\ell_i)$, so by Fact 76 $x_{i+1} \in c_\pi(\ell_i)$ and $x_{i+1} \in p_\pi(\ell_{i+1})$. So, $s(\ell_i) \in p_\pi(\ell_i) \times c_\pi(\ell_i)$ for each $i \in \{q_j + 1, \ldots, r_j - 1\}$, and $x_{r_j} \in p_\pi(r_j)$. So, $s(\ell_{r_j}) \in (p_\pi(r_j))^2$ i.e. ℓ_{r_j} is horizontal w.r.t. s. Hence, since $k_j < r_j$, $k_{(j+1)} \leq r_j$, so $k_{(j+1)} = r_j$.

For each $i \in \{k_j + 1, \ldots, q_j - 1\}, s(\ell_i) \in c_\pi(\ell_i) \times p_\pi(\ell_i)$, so by Fact 76 $s(\ell_i) \notin c_\pi(\ell_i)^2$. For each $i \in \{q_j + 1, \ldots, k_{(j+1)} - 1\}, s(\ell_i) \in p_\pi(\ell_i) \times c_\pi(\ell_i)$, so by Fact 76 $s(\ell_i) \notin c_\pi(\ell_i)^2$, so the index q_j is unique. ☐

Proposition 78. *Let π be an ISOMIX proof-structure, let s be a switching function of π, let C be a cycle of $\mathcal{G} = (W_\pi, s(L_\pi))$, let $(\ell_1, \ldots, \ell_n)$ be a link sequence of C and let k_1, k_2 such that $k_1 < k_2$ and $t_\pi(\ell_{k_1}) = t_\pi(\ell_{k_2}) =$ `axiom` and $t_\pi(\ell_i) \neq$ `axiom` for each $i \in \{k_1 + 1, \ldots, k_2 - 1\}$. Then, there exists $k_3 \in \{k_1 + 1, \ldots, k_2 - 1\}$ such that ℓ_{k_3} is horizontal w.r.t. s and, denoting $s(\ell_{k_j}) = x_j \to y_j$ for $1 \leq j \leq 3$, y_1 is above x_3 and x_2 is above y_3.*

Proof. This is proven by using the fact that each node has total degree at most 2 in a switching graph. ☐

5.4 Cycles with minimum number of horizontal links

Proposition 79. *Let π be a proof-structure and let $((x_0, \ldots, x_n), (\ell_1, \ldots, \ell_n))$ and $((x'_0, \ldots, x'_m), (\ell'_1, \ldots, \ell'_m))$ be downward sequences for some n and m such that $x_n = x'_m$ and $x_{n-1} \neq x'_{m-1}$. Then,*

- *for each $i \in \{0, \ldots, n-1\}$ and each $j \in \{0, \ldots, m-1\}$, $x_i \neq x'_j$,*
- *for each $i \in \{1, \ldots, n-1\}$ and each $j \in \{1, \ldots, m-1\}$, $\ell_i \neq \ell'_j$.*

Proof. For a contradiction, we assume that there exist $i \in \{0, \ldots, n-1\}$ and $j \in \{0, \ldots, m-1\}$ such that $x_i = x'_j$. W.l.o.g. we assume that i is maximum such that such a j exists. If $i < n-1$ and $j < m-1$, then by Proposition 22 $x_{i+1} = x'_{j+1}$ which contradicts that i is maximum. Otherwise if $j = m-1$, then by Proposition 22, $x_{i+1} = x'_m$, hence $x_{i+1} = x_n$ which is absurd by Proposition 26 since $i + 1 < n$. Otherwise, $i = n-1$, hence since $x_{n-1} \neq x'_{m-1}$, $j < m-1$, so by Proposition 22 $x_n = x'_{j+1}$ so $x'_{j+1} = x'_m$ which is absurd by Proposition 26 since $j + 1 < m$. ☐

Theorem 80. *Let π be an ISOMIX proof-structure, let s be a switching function of π, let C be a cycle of $\mathcal{G} = (W_\pi, s(L_\pi))$ such that the number of horizontal links of C is minimum among all cycles and switching functions of π, let $\ell, \ell' \in L_\pi$ such that ℓ and ℓ' are horizontal w.r.t. s and $s(\ell) \in C$ and $s(\ell') \in C$ and let $x \in c_\pi(\ell)$ and $x' \in c_\pi(\ell')$.*

- *If x is above x', then $x = x'$.*
- *Let $((x_0, \ldots, x_n), (\ell_1, \ldots, \ell_n))$ and $((x'_0, \ldots, x'_m), (\ell'_1, \ldots, \ell'_m))$ be two downward sequences such that $n > 0$, $m > 0$, $x_0 = x$, $x'_0 = x'$, $x_n = x'_m$ and $x_{n-1} \neq x'_{m-1}$. Then, $t_\pi(\ell_n) =$ `par`.*

Proof. For a contradiction, assume that x is above x' and $x \neq x'$. There exists a downward sequence $((x_0, \ldots, x_n), (\ell_1, \ldots, \ell_n))$ such that $x_0 = x$ and $x_n = x'$, and since $x' \neq x$, $n > 0$. Since $((x_0, \ldots, x_n), (\ell_1, \ldots, \ell_n))$ is a downward sequence and π is an ISOMIX proof-structure, $t_\pi(\ell_i) \in \{$`tensor`, `par`$\}$ for each $i \in \{1, \ldots, n\}$, so $s(\ell_i)$ is defined. We assume w.l.o.g. that n is minimum given ℓ and x, hence for each $i \in \{1, \ldots, n-1\}$, ℓ_i is not horizontal w.r.t. s or $s(\ell_i) \notin C$.

Since $s(\ell_n) \in C$ and $n \geq 1$, there exist q the minimum index such that $q \geq 1$ and $s(\ell_q) \in C$. Since ℓ' is horizontal, $t_\pi(\ell') =$ `tensor` and there exists $y_1, y_2 \in p_\pi(\ell')$ such that $s(\ell') = y_1 \to y_2$.

By Fact 68, $y_1 \neq y_2$. Hence, since $y_1, y_2 \in p_\pi(\ell')$, from Definition 4 $p_\pi(\ell') = \{y_1, y_2\}$. Since $x' \in c_\pi(\ell')$ and $x' \in c_\pi(\ell_n)$, $\ell' = \ell_n$. Since $x_{n-1} \in p_\pi(\ell_n)$, $x_{n-1} = y_1$ or $x_{n-1} = y_2$. If $x_{n-1} = y_1$, then since $s(\ell_n) \in C$ there exists an arc $z \to x_{n-1} \in C$ and hence by Fact 67 $s(\ell_{n-1}) = z \to x_{n-1}$, so $s(\ell_{n-1}) \in C$. Otherwise $x_{n-1} = y_2$, hence since $s(\ell_n) \in C$ there exists an arc $x_{n-1} \to z$ and by Fact 67 $s(\ell_{n-1}) = x_{n-1} \to z$, so $s(\ell_{n-1}) \in C$. In both cases $s(\ell_{n-1}) \in C$ so $q < n$.

Since q is minimum, for each $i \in \{1, \ldots, q-1\}, s(\ell_i) \notin C$. Since n is minimum and $s(\ell_q) \in C$ and $q < n$, ℓ_q is not horizontal w.r.t. s. Since $t_\pi(\ell_q) \in \{\mathtt{par}, \mathtt{tensor}\}$, $s(\ell_q) \notin c_\pi(\ell_q)^2$. Hence, by Fact 68 $s(\ell_q) \in p_\pi(\ell_q) \times c_\pi(\ell_q)$ or $s(\ell_q) \in c_\pi(\ell_q) \times p_\pi(\ell_q)$. We assume that $s(\ell_q) \in p_\pi(\ell_q) \times c_\pi(\ell_q)$. This is without loss of generality since if $s(\ell_q) \in c_\pi(\ell_q) \times p_\pi(\ell_q)$, we can consider the partial function s^{op} instead of s, since s^{op} is a switching function of π and $s^{op}(\ell_q) \in p_\pi(\ell_q) \times c_\pi(\ell_q)$ and $\mathcal{G}^{op}$ contains the cycle C^{op} and the number of horizontal links of C^{op} is the same as C.

We show by induction that for each $i \in \{q+1, \ldots, n-1\}$, $s(\ell_i) \in C$ and $s(\ell_i) = x_{i-1} \to x_i$. Let $i \in \{q+1, \ldots, n-1\}$. If $i > q+1$, then by the induction hypothesis $s(\ell_{i-1}) = x_{i-2} \to x_{i-1} \in p_\pi(\ell_{i-1}) \times \{x_{i-1}\}$ and $s(\ell_{i-1}) \in C$. Otherwise $i = q+1$ so $i - 1 = q$, so $s(\ell_{i-1}) \in C$ and since $t_\pi(\ell_{i-1}) \in \{\mathtt{tensor}, \mathtt{par}\}$, $c_\pi(\ell_{i-1}) = \{x_{i-1}\}$, so $s(\ell_{i-1}) \in p_\pi(\ell_{i-1}) \times \{x_{i-1}\}$. In both cases, $s(\ell_{i-1}) \in C$ and $s(\ell_{i-1}) \in p_\pi(\ell_{i-1}) \times \{x_{i-1}\}$. Since $s(\ell_{i-1}) \in C$ there exists an arc $x_{i-1} \to y \in C$ for some $y \in W_\pi$, and by Fact 67 $s(\ell_i) = x_{i-1} \to y$. So, $s(\ell_i) \in C$ and by Fact 68, $y \in p_\pi(\ell_i) \cup c_\pi(\ell_i)$. Since $s(\ell_i) \in C$ and $i < n$ and n is minimum, ℓ_i is not horizontal w.r.t. s. Hence since $x_{i-1} \in p_\pi(\ell_i)$, $y \in c_\pi(\ell_i)$ and since $t_\pi(\ell_i) \in \{\mathtt{tensor}, \mathtt{par}\}$, $c_\pi(\ell_i) = \{x_i\}$, so $s(\ell_i) = x_{i-1} \to x_i$.

Since ℓ is horizontal, $t_\pi(\ell) = \mathtt{tensor}$ and there exist $y_3, y_4 \in p_\pi(\ell)$ such that $s(\ell) = y_3 \to y_4$. Since $t_\pi(\ell) = t_\pi(\ell') = \mathtt{tensor}$, from Definition 4 $c_\pi(\ell) = \{x\}$ and $c_\pi(\ell') = \{x'\}$. So, since $x \neq x'$, $\ell \neq \ell'$, and since $s(\ell) \in C$ and $s(\ell') \in C$, by Corollary 73 $s(\ell) \neq s(\ell')$.

Hence, C can be decomposed as a concatenation $C = P_1 \cdot s(\ell) \cdot P_2 \cdot s(\ell_q) \cdot \ldots \cdot s(\ell_n)$ for some pathes P_1 and P_2. Let s_2 be the partial function such that $s_2(\ell_i) = x_{i-1} \to x_i$ for each $i \in \{1, \ldots, q\}$ and $s_2(\ell) = y_1 \to x_0$ and s_2 coincides with s on all other links. Then, s_2 is a switching function of π. Let $\mathcal{G}_2 = (W_\pi, s_2(L_\pi))$. Since for each $i \in \{1, \ldots, q-1\}$, $s(\ell_i) \notin C$, $s(\ell_i) \notin P_1$. Since also $s(\ell) \notin P_1$ and $s(\ell_q) \notin P_1$, $P_1 \subset s_2(L_\pi)$. Since P_1 starts at wire y_2 and ends at wire y_3, $\mathcal{G}_2$ contains the cycle $C_2 = P_1 \cdot s_2(\ell) \cdot s_2(\ell_1) \cdot \ldots \cdot s_2(\ell_n)$. Since $\ell, \ell_1, \ldots, \ell_{n-1}$ are not horizontal w.r.t. s_2, each link of C_2 which is horizontal w.r.t. s_2 is ℓ_n or a link of P_1. Since s and s_2 coincide on these links, each link of C_2 which is horizontal w.r.t. s_2 is also a link of C which is horizontal w.r.t. s. However ℓ is horizontal w.r.t. s and $s(\ell) \in C$ and ℓ is not horizontal w.r.t. s_2. So, C_2 contains strictly less horizontal links than C which contradicts our assumption that C has minimum number of horizontal links. So, $x = x'$.

For a contradiction, assume that $t_\pi(\ell_n) \neq \mathtt{par}$. For each $i \in \{1, \ldots, n\}$, $x_{i-1} \in p_\pi(\ell_i)$ and $x_i \in c_\pi(\ell_i)$, hence $t_\pi(\ell_i) \in \{\mathtt{tensor}, \mathtt{par}\}$, so s is defined on ℓ_i and $t_\pi(\ell_n) = \mathtt{tensor}$. Similarly, for each $j \in \{1, \ldots, m\}$, $t_\pi(\ell_j) \in \{\mathtt{tensor}, \mathtt{par}\}$ and s is defined on ℓ_j. Since the number of horizontal links of C is minimum, by the previous argument, for each $i \in \{1, \ldots, n\}$, $s(\ell_i) \notin C$ or ℓ_i is not horizontal w.r.t. s. Similarly, for each $j \in \{1, \ldots, m\}$, $s(\ell_j) \notin C$ or ℓ_j is not horizontal w.r.t. s.

We show that for each $i \in \{1, \ldots, n\}$, $s(\ell_i) \notin C$. For a contradiction, assume that there exists $i \in \{1, \ldots, n\}$ such that $s(\ell_i) \in C$. Then, ℓ_i is not horizontal w.r.t. s. There exists a link sequence $(\ell_1^C, \ldots, \ell_N^C)$ of C such that $\ell_i = \ell_1^C$. Since $s(\ell) \in C$ and ℓ is horizontal w.r.t. s, there exist k_1 and k_2 the maximum and minimum indices such that $k_1 \leq 1$ and $k_2 \geq 1$ and $\ell_{k_1}^C$ and $\ell_{k_2}^C$ are horizontal w.r.t. s. Since ℓ_i is not horizontal w.r.t. s, $k_1 < 1$ and $k_2 > 1$ so $k_1 < k_2$. Since k_1 is maximum and k_2 is minimum, for each $j \in \{k_1+1, \ldots, k_2-1\}$, ℓ_j^C is not horizontal w.r.t. s. So by Proposition 77, there exists a unique index $q \in \{k_1+1, \ldots, k_2-1\}$ such that $s(\ell_q^C) \in c_\pi(\ell_q^C)^2$ and for each $j \in \{k_1+1, \ldots, q-1\}, s(\ell_j^C) \in c_\pi(\ell_j^C) \times p_\pi(\ell_j^C)$ and for each $j \in \{q+1, \ldots, k_2-1\}, s(\ell_j^C) \in p_\pi(\ell_j^C) \times c_\pi(\ell_j^C)$. Hence, denoting $s(\ell_i^C) = x_{i-1}^C \to x_i^C$ for each $i \in \{1, \ldots, N\}$, $((x_{q-1}^C, \ldots, x_{k_1}^C), (\ell_{q-1}^C, \ldots, \ell_{k_1+1}^C))$ and $((x_q^C, \ldots, x_{k_2-1}^C), (\ell_{q+1}^C, \ldots, \ell_{k_2-1}^C))$ form downward sequences. Since $\ell_{k_1}^C$ and $\ell_{k_2}^C$ are horizontal w.r.t. s, $t_\pi(\ell_{k_1}^C) = t_\pi(\ell_{k_2}^C) = \mathtt{tensor}$. Hence, there exist $y_1 \in c_\pi(\ell_{k_1}^C)$ and $y_2 \in c_\pi(\ell_{k_2}^C)$, and hence $x_{k_1}^C$ is above y_1 and $x_{k_2-1}^C$ is above y_2. Since $s(\ell_q) \in c_\pi(\ell_q)^2$ and $t_\pi(\ell_1^C) \in \{\mathtt{tensor}, \mathtt{par}\}$, $q \neq 1$. If $q > 1$, then $k_1 < 1 < q$, so $\ell_1^C \in (\ell_{q-1}^C, \ldots, \ell_{k_1+1}^C)$, hence x_i is above y_1 which is absurd by our previous argument since $s(\ell_{k_1}^C) \in C$ and $\ell_{k_1}^C$ is horizontal w.r.t. s and $y_1 \in c_\pi(\ell_{k_1}^C)$. Otherwise $q < 1$, hence $q < 1 < k_2$, so $\ell_1^C \in (\ell_{q+1}^C, \ldots, \ell_{k_2-1}^C)$ hence x_i is above y_2 which is also absurd. By the same argument, $s(\ell_j') \notin C$ for each $j \in \{1, \ldots, m\}$.

Since ℓ' is horizontal, there exist $y_1, y_2 \in p_\pi(\ell')$ such that $s(\ell') = y_1 \to y_2$ and since ℓ is horizontal, there exist $y_3, y_4 \in p_\pi(\ell)$ such that $s(\ell) = y_3 \to y_4$. Hence, $((y_3, x_0, \ldots, x_n), (\ell, \ell_1, \ldots, \ell_n))$ and $((y_2, x'_0, \ldots, x'_m), (\ell', \ell'_1, \ldots, \ell'_m))$ form downward sequences. By Propositions 79 and 26, all the links $\ell, \ell_1, \ldots, \ell_n, \ell', \ell'_1, \ldots, \ell'_{m-1}$ are pairwise different. Since $\ell \neq \ell'$ and $s(\ell), s(\ell') \in C$, by Theorem 72 $s(\ell) \neq s(\ell')$ hence C can be decomposed as a concatenation $C = P_1 \cdot s(\ell) \cdot P_2 \cdot s(\ell')$ for some pathes P_1 and P_2, and P_1 starts at wire y_2 and ends at wire y_3.

Let s_2 be the partial function on π such that $s_2(\ell) = y_3 \to x_0$, $s_2(\ell_i) = x_{i-1} \to x_i$ for each $i \in \{1, \ldots, n-1\}$, $s_2(\ell_n) = x_{n-1} \to x'_{m-1}$, $s_2(\ell'_j) = x'_j \to x'_{j-1}$ for each $j \in \{1, \ldots, m-1\}$, $s_2(\ell') = x'_0 \to y_2$ and s_2 coincides with s on all other links. Since $t_\pi(\ell_n) = \texttt{tensor}$, s_2 is a switching function of π.

Since for each $i \in \{1, \ldots, n\}, s(\ell_i) \notin C$ and for each $j \in \{1, \ldots, m-1\}, s(\ell'_j) \notin C$ and $P_1 \subset C$ and $s(\ell), s(\ell') \notin P_1$ and s_2 and s conincide on all other links, $P_1 \subset s_2(L_\pi)$. So, $\mathcal{G}_2 = (W_\pi, s_2(L_\pi))$ contains the cycle $C_2 = P_1 \cdot s_2(\ell) \cdot s_2(\ell_1) \cdot \ldots \cdot s_2(\ell_{n-1}) \cdot s_2(\ell_n) \cdot s_2(\ell'_{m-1}) \cdot \ldots \cdot s_2(\ell'_1) \cdot s_2(\ell')$. Each link of C_2 which is horizontal w.r.t. s_2 is ℓ_n or belongs to P_1. Since s and s_2 coincide on the links of P_1, each link of P_1 which is horizontal w.r.t. s_2 is also horizontal w.r.t. s. The links ℓ and ℓ' are horizontal w.r.t. s and $s(\ell), s(\ell') \in C$. Hence, the number of links of C which are horizontal w.r.t. s is greater than the number of links of C_2 which are horizontal w.r.t. s_2, which contradicts the assumption that C has minimum number of horizontal links. Hence, $t_\pi(\ell_n) = \texttt{par}$. ☐

6 Mapping FO-ISOMIX to ISOMIX

6.1 Bijection between ISOMIX and FO-ISOMIX proof-structures

We consider a bijective map $\mathcal{HO} : \mathcal{V}_{FO} \to \mathcal{V}$ from the set of FO-ISOMIX variables to the set of ISOMIX variables. We show that this map induces a bijection between the sets FO-ISOMIX and ISOMIX formulas, and also between the sets of FO-ISOMIX and ISOMIX proof-structures.

Definition 81. *Let $\mathcal{HO} : \mathcal{V}_{FO} \simeq \mathcal{V}$ be a bijection; there exists such a bijection since $\mathcal{V}$ and $\mathcal{V}_{FO}$ have the same cardinality.*

Let F be an FO-ISOMIX formula. We define $\mathcal{HO}(F)$ by induction on F as:

$$\forall a \in \mathcal{V}_{FO}, \mathcal{HO}(a^*) = \mathcal{HO}(a)^* \ , \tag{21}$$
$$\mathcal{HO}(I) = I \ , \tag{22}$$
$$\mathcal{HO}(F \otimes G) = \mathcal{HO}(F) \otimes \mathcal{HO}(F) \ , \tag{23}$$
$$\mathcal{HO}(F \parr G) = \mathcal{HO}(F) \parr \mathcal{HO}(G) \ . \tag{24}$$

Proposition 82. *$\mathcal{HO}$ is a bijection between the set of FO-ISOMIX formulas and the set of ISOMIX formulas.*

Proof. This is proven by induction on $s(F)$. ☐

Definition 83 ($\mathcal{HO}$ bijective transformation between proof-structures)**.** *Let π be an FO-ISOMIX proof-structure. We define $\mathcal{HO}(\pi)$ as the following data:*

- *$W_{\mathcal{HO}(\pi)} = W_\pi$ (denoted W) and $L_{\mathcal{HO}(\pi)} = L_\pi$ (denoted L)*
- *for each $x \in W$, $f_{\mathcal{HO}(\pi)}(x) = \mathcal{HO}(f_\pi(x))$,*
- *for each $\ell \in L$, if $t_\pi(\ell) = \texttt{FO-axiom}$, $t_{\mathcal{HO}(\pi)}(\ell) = \texttt{axiom}$, otherwise $t_{\mathcal{HO}(\pi)}(\ell) = t_\pi(\ell)$, the set of premisses is $p_{\mathcal{HO}(\pi)}(\ell) = p_\pi(\ell)$ (denoted $p(\ell)$) and the set of conclusions is $c_{\mathcal{HO}(\pi)}(\ell) = c_\pi(\ell)$ (denoted $c(\ell)$).*

Proposition 84. *For any FO-ISOMIX proof-structure π, $\mathcal{HO}(\pi)$ is an ISOMIX proof-structure. For any ISOMIX proof-structure π', there exists a unique FO-ISOMIX proof-structure π such that $\mathcal{HO}(\pi) = \pi'$, and we write $\pi' = \mathcal{HO}^{-1}(\pi)$.*

Proof. By checking Definition 4. ☐

Proposition 85. *Let π be an FO-ISOMIX proof-structure. $\textsc{Graphs}(\pi) \subset \textsc{Graphs}(\mathcal{HO}(\pi))$.*

Proof. Let $\mathcal{G} \in \text{GRAPHS}(\pi)$. There exists a switching function $s : L \to W \times W$ of π whose image is the set of arcs of $\mathcal{G}$. We show that s is also a switching function of $\mathcal{HO}(\pi)$, which implies that $\mathcal{G} \in \text{GRAPHS}(\mathcal{HO}(\pi))$. Let $\ell \in L$. We show that s satisfies the condition in Definition 5 corresponding to $t_{\mathcal{HO}(\pi)}(\ell)$. If $t_{\mathcal{HO}(\pi)}(\ell) = \texttt{axiom}$, then by Definition 83 $t_\pi(\ell) = \texttt{FO-axiom}$. Hence, denoting x and y the left and right conclusions of ℓ respectively, $s(\ell) = y \to x$, so $s(\ell) \in \{x \to y, y \to x\}$. If $t_{\mathcal{HO}(\pi)}(\ell) \in \{\texttt{unit}, \texttt{tensor}, \texttt{par}\}$, by Definition 83 $t_{\mathcal{HO}(\pi)}(\ell) = t_\pi(\ell)$, hence as s is a switching function of $\mathcal{HO}(\pi)$, s satisfies the condition corresponding to $t_{\mathcal{HO}(\pi)}(\ell)$. ☐

Proposition 86. *Let F be an FO-ISOMIX formula. If $\vdash \mathcal{HO}(F)$ then $\vdash F$.*

Proof. Since $\vdash \mathcal{HO}(F)$ there exists a proof-net π' of $\mathcal{HO}(F)$. Denote $\pi = \mathcal{HO}^{-1}(\pi')$, which is a proof structure of F. Since π' is a proof-net, all graphs in $\text{GRAPHS}(\pi')$ are acyclic. Hence by Proposition 85, all graphs in $\text{GRAPHS}(\pi) \subset \text{GRAPHS}(\pi')$ are acyclic, so π is a proof-net, hence $\vdash F$. ☐

The converse doesn't hold, as there exist FO-ISOMIX formulas F such that $\nvdash \mathcal{HO}(F)$ and $\vdash F$ (and there also exist FO-ISOMIX formulas such that $\nvdash \mathcal{HO}(F)$ and $\nvdash F$).

Example 87. *Let $a, b \in \mathcal{V}_{FO}$ and let F be the FO-ISOMIX formula $(a \otimes b) \parr (a^* \otimes b^*)$. Then, $\vdash F$ and $\nvdash \mathcal{HO}(F)$.*

Proof. F is balanced and I isn't a subformula of F. Let π be the canonical proof-structure of F. Let s be a switching function of π, and $\mathcal{G}$ be the switching graph of π whose set of arcs is the image of s. We show that $\mathcal{G}$ is acyclic. For a contradiction, assume that $\mathcal{G}$ contains a cycle C. By Proposition 66 there exists a link $\ell \in L$ such that $t(\ell) \in \{\texttt{axiom}, \texttt{FO-axiom}\}$ and C contains the arc $s(\ell)$. From Definition 62 the links of π whose types are in $\{\texttt{axiom}, \texttt{FO-axiom}\}$ are (a, a^*) with $t((a, a^*)) = \texttt{FO-axiom}$ and (b, b^*) with $t((b, b^*)) = \texttt{FO-axiom}$. First assume that C contains $s((a, a^*))$. From Definition 62 (a, a^*) doesn't have any premisses and has left conclusion a and right conclusion a^*, and $t((a, a^*)) = \texttt{FO-axiom}$. So from Definition 5, $s((a, a^*)) = a^* \to a$.

From Definition 62 a is a conclusion of link (a^*, a) and a premisse of $(a \otimes b, a, b)$. Hence by Proposition 64 the arc following $a^* \to a$ in C is $s((a\otimes b, a, b))$. From Definition 62 the link $(a\otimes b, a, b)$ has left premisse a, right premisse b and conclusion $a \otimes b$, and $t((a \otimes b, a, b)) = \texttt{tensor}$. So from Definition 5 $s((a \otimes b, a, b)) \in \{a \to a \otimes b, a \otimes b \to a, b \to a \otimes b, a \otimes b \to b, a \to b, b \to a\}$. Since the source node of $s((a \otimes b, a, b))$ is a, either $s((a \otimes b, a, b)) = a \to a \otimes b$ or $s((a \otimes b, a, b)) = a \to b$.

First assume that $s((a \otimes b, a, b)) = a \to a \otimes b$. From Definition 62, $a \otimes b$ is the conclusion of the link $(a \otimes b, a, b)$ and a premisse of the link $((a \otimes b) \parr (a^* \otimes b^*), a \otimes b, a^* \otimes b^*)$. By Proposition 64 the arc following $s((a \otimes b, a, b))$ is C is $s((a \otimes b) \parr (a^* \otimes b^*), a \otimes b, a^* \otimes b^*)$. From Definition 62 the link $((a \otimes b) \parr (a^* \otimes b^*), a \otimes b, a^* \otimes b^*)$ has left premisse $a \otimes b$, right premisse $a^* \otimes b^*$ and conclusion $(a \otimes b) \parr (a \otimes b^*)$ and $t(((a \otimes b) \parr (a^* \otimes b^*), a \otimes b, a^* \otimes b^*)) = \texttt{par}$. So from Definition 5 $s(((a \otimes b) \parr (a^* \otimes b^*), a \otimes b, a^* \otimes b^*)) \in \{a \otimes b \to (a \otimes b) \parr (a^* \otimes b^*), (a \otimes b) \parr (a^* \otimes b^*) \to a \otimes b, a^* \otimes b^* \to (a \otimes b) \parr (a^* \otimes b^*), (a \otimes b) \parr (a^* \otimes b^*) \to a^* \otimes b^*\}$. Since the source node of $s(((a \otimes b) \parr (a^* \otimes b^*), a \otimes b, a^* \otimes b^*))$ is $a \otimes b$, $s(((a \otimes b) \parr (a^* \otimes b^*), a \otimes b, a^* \otimes b^*)) = a \otimes b \to (a \otimes b) \parr (a^* \otimes b^*)$. From Definition 62, the wire $(a \otimes b) \parr (a^* \otimes b^*)$ is the conclusion of link $((a \otimes b) \parr (a^* \otimes b^*), a \otimes b, a^* \otimes b^*)$ and isn't a premisse of any link. So by Proposition 64 the arc $s(((a \otimes b) \parr (a^* \otimes b^*), a \otimes b, a^* \otimes b^*))$ isn't followed by any other arc in $\mathcal{G}$, which is absurd as C contains $s(((a \otimes b) \parr (a^* \otimes b^*), a \otimes b, a^* \otimes b^*))$.

Now assume that $s((a \otimes b, a, b)) = a \to b$. From Definition 62, b is a conclusion of the link (b, b^*) and a premisse of the link $(a \otimes b, a, b)$. So, by Proposition 64 the arc following $s((a \otimes b, a, b))$ in C is $s((b, b^*))$. From Definition 62 (b, b^*) doesn't have any premisses, has left conclusion b and has right conclusion b^*, and $t((b, b^*)) = \texttt{FO-axiom}$. So from Definition 5 $s((b, b^*)) = b^* \to b$. This is absurd since the source node of arc $s((b, b^*))$ is b, since it follows the previous arc $a \to b$ in C.

So, C doesn't contain $s((a, a^*))$. With the same proof technique we can show that C also doesn't contain $s((b, b^*))$, which is absurd. Hence, the graph $\mathcal{G}$ whose set of arcs is the image of s is acyclic. This holds for any switching function s, so all switching graphs of π are acyclic, hence π is a proof-net of F and $\vdash F$.

$\mathcal{HO}(F)$ is balanced and I isn't a subformula of $\mathcal{HO}(F)$. Let π' be the canonical proof-structure of $\mathcal{HO}(F)$. Let $s' : L_{\pi'} \to W_{\pi'} \times W_{\pi'}$ be the following function: $s'(((a\otimes b) \parr (a^* \otimes b^*), a\otimes b, a^* \otimes b^*)) = (a \otimes b) \parr (a^* \otimes b^*) \to a \otimes b$, $s'((A \otimes B, A, B)) = A \to B$, $s'((A^* \otimes B^*, A^*, B^*)) = B^* \to A^*$,

$s'((A,A^*)) = A^* \to A$ and $s'((B,B^*)) = B \to B^*$. s' satisfies the conditions of Definition 5 accoring to the type $t(\ell)$ of each link $\ell \in L_{\pi'}$ defined in Definition 62, so s' is a switching function of π'. Let $\mathcal{G}'$ be the switching graph of π' whose set of arcs is the image of s'. $\mathcal{G}'$ contains the cycle $A \to B \to B^* \to A^* \to A$. Hence there exists a switching graph of π' which is cyclic so π' is not a proof-net. Since $\mathcal{HO}(F)$ is balanced, so by Proposition 61 and Theorem 60, any proof-structure of $\mathcal{HO}(F)$ is not a proof-net, so $\nvdash \mathcal{HO}(F)$. □

6.2 Switching functions of $\mathcal{HO}(\cdot)$-ed proof-structures

Fact 88. *Let π be an FO-ISOMIX proof-structure and let s be a switching function of $\mathcal{HO}(\pi)$. s is a switching function of π if and only if for each $\ell \in L_\pi$ such that $t_{\mathcal{HO}(\pi)}(\ell) = \texttt{axiom}$, ℓ is direct w.r.t. s.*

Proof. From Definitions 5 and 83. □

Notation 89. *Let $\mathcal{G} = (V,E)$ be a directed graph. For $e = x \to y \in E$, we denote e^{op} the arc $y \to x$. For $P = (e_1,\dots,e_n)$ a sequence of arcs we denote $P^{op} = (e_n^{op},\dots,e_1^{op})$ and for $X \subset E$, we denote $X^{op} = \{e^{op}, e \in X\}$. We denote $\mathcal{G}^{op} = (V, E^{op})$.*

Let π be a proof-structure and let s be a switching function of π. We denote s^{op} the partial function such that $s^{op}(\ell) = (s(\ell))^{op}$ for each ℓ such that $s(\ell)$ is defined.

Proposition 90. *Let π be an FO-ISOMIX proof-net, let s be a switching function of $\mathcal{HO}(\pi)$ and let C be a cycle of $\mathcal{G} = (W_\pi, s(L_\pi))$. Then, there exists $\ell,\ell' \in L_\pi$ such that $s(\ell) \in C$ and $s(\ell') \in C$ and $t_{\mathcal{HO}(\pi)}(\ell) = \texttt{axiom}$ and $t_{\mathcal{HO}(\pi)}(\ell') = \texttt{axiom}$ and ℓ is indirect w.r.t. s and ℓ' is direct w.r.t s.*

Proof. We first prove the existence of ℓ and then use it to prove the existence of ℓ'.

Let

$$s_2 : \ell \mapsto \begin{cases} y \to x & \text{if } t_{\mathcal{HO}(\pi)}(\ell) = \texttt{axiom} \text{ and } s(\ell) \notin C, \text{ denoting } c_\pi(\ell) = x \cdot y \\ s(\ell) & \text{(or undefined if } s \text{ is undefined on } \ell\text{) otherwise} \end{cases} .$$

From Definition 5, s_2 is a switching function of $\mathcal{HO}(\pi)$ and C is a cycle of $\mathcal{G}_2 = (W_\pi, s_2(L_\pi))$. So, since π is a proof-net s_2 is not a switching function of π, hence by Fact 88 there exists $\ell \in L_\pi$ such that $t_{\mathcal{HO}(\pi)}(\ell) = \texttt{axiom}$ and ℓ is indirect w.r.t s_2. By definition of s_2, for each ℓ' such that $t_{\mathcal{HO}(\pi)}(\ell') = \texttt{axiom}$ and $s(\ell') \notin C$, ℓ' is direct w.r.t. s_2, so $s(\ell) \in C$. Since $t_{\mathcal{HO}(\pi)}(\ell) = \texttt{axiom}$, s is defined on ℓ and hence $s_2(\ell) = s(\ell)$, so ℓ is indirect w.r.t s.

Since s is a switching function of $\mathcal{HO}(\pi)$ and $\mathcal{HO}(\pi)$ is an ISOMIX proof-structure, from Definition 5 s^{op} is a switching function of $\mathcal{HO}(\pi)$ and C^{op} is a cycle of $\mathcal{G}^{op}$. So, by the previous argument there exists $\ell' \in L_\pi$ such that $s(\ell') \in C^{op}$ and $t_{\mathcal{HO}(\pi)}(\ell') = \texttt{axiom}$ and ℓ' is indirect w.r.t s^{op}. Hence, $s(\ell') \in C$ and ℓ' is direct w.r.t s. □

Corollary 91. *Let π be an FO-ISOMIX proof-net, let s be a switching function of $\mathcal{HO}(\pi)$, let $C = (e_1,\dots,e_n)$ be a cycle of $\mathcal{G} = (W_\pi, s(L_\pi))$ and let $(\ell_1,\dots,\ell_n)$ be the sequence of links such that $s(\ell_i) = e_i$ for each $i \in \{1,\dots,n\}$. Then,*

- *there exist k_1, k_2 such that $k_1 < k_2$ and $t_{\mathcal{HO}(\pi)}(\ell_{k_1}) = t_{\mathcal{HO}(\pi)}(\ell_{k_2}) = \texttt{axiom}$ and $\forall i \in \{k_1 + 1,\dots,k_2 - 1\}, t_{\mathcal{HO}(\pi)}(\ell_i) \neq \texttt{axiom}$ and ℓ_{k_1} is direct and ℓ_{k_2} is indirect w.r.t. s, and*
- *there exist k_3, k_4 such that $k_3 < k_4$ and $t_{\mathcal{HO}(\pi)}(\ell_{k_3}) = t_{\mathcal{HO}(\pi)}(\ell_{k_4}) = \texttt{axiom}$ and $\forall i \in \{k_3 + 1,\dots,k_4 - 1\}, t_{\mathcal{HO}(\pi)}(\ell_i) \neq \texttt{axiom}$ and ℓ_{k_3} is indirect and ℓ_{k_4} is direct w.r.t. s.*

Proof. By Proposition 90, there exist indices i, i' such that $t_{\mathcal{HO}(\pi)}(\ell_i) = t_{\mathcal{HO}(\pi)}(\ell_{i'}) = \texttt{axiom}$ and ℓ_i is direct and $\ell_{i'}$ is indirect w.r.t s. Let k_2 be the minimum index such that $i < k_2$ and $t_{\mathcal{HO}(\pi)}(\ell_{k_2}) = \texttt{axiom}$ and ℓ_{k_2} is indirect w.r.t. s. Let k_1 be the maximum index such that $k_1 < k_2$ and $t_{\mathcal{HO}(\pi)}(\ell_{k_1}) = \texttt{axiom}$. Then, $t_{\mathcal{HO}(\pi)}(\ell_j) \neq \texttt{axiom}$ for $j \in \{k_1+1,\dots,k_2-1\}$ by maximality of k_1. Since $t_{\mathcal{HO}(\pi)}(\ell_i) = \texttt{axiom}$ and $i < k_2$, $i \leq k_1$. If $i = k_1$ then ℓ_{k_1} is direct w.r.t. s. Otherwise $i < k_1 < k_2$, so since $t_{\mathcal{HO}(\pi)}(\ell_{k_1}) = \texttt{axiom}$, ℓ_{k_1} is direct w.r.t. s by minimality of k_2. We construct k_3, k_4 the same way with i'. □

Proposition 92. *Let π be an FO-ISOMIX proof-net, let s be a switching function of $\mathcal{HO}(\pi)$ and let C be a cycle of $\mathcal{G} = (W_\pi, s(L_\pi))$. Then, there exist $\ell, \ell' \in L_\pi$ such that $\ell \neq \ell'$ and*

- *$s(\ell) \in C$ and ℓ is horizontal w.r.t. s and there exists $x \in c_\pi(\ell)$ such that $f_\pi(x)$ is of the form $H_1 \otimes H_2$ and there exist $a, b \in \mathcal{V}_{FO}$ such that (a, b) is an opposed pair of subformulas of $f_\pi(x)$, and*
- *$s(\ell') \in C$ and ℓ' is horizontal w.r.t. s and there exists $x' \in c_\pi(\ell')$ such that $f_\pi(x')$ is of the form $K_1 \otimes K_2$ and there exist $c, d \in \mathcal{V}_{FO}$ such that (c^*, d^*) is an opposed pair of subformulas of $f_\pi(x')$.*

Proof. Let us denote $C = (e_1, \dots, e_n)$ and let $(\ell_1, \dots, \ell_n)$ be the sequence of links such that $s(\ell_i) = e_i$ for each $i \in \{1, \dots, n\}$. By Corollary 91, there exist k_1, k_2, k_3, k_4, such that $k_1 < k_2$ and $t_{\mathcal{HO}(\pi)}(\ell_{k_1}) = t_{\mathcal{HO}(\pi)}(\ell_{k_2}) = \texttt{axiom}$ and $\forall i \in \{k_1+1, \dots, k_2-1\}, t_{\mathcal{HO}(\pi)}(\ell_i) \neq \texttt{axiom}$ and ℓ_{k_1} is direct and ℓ_{k_2} is indirect w.r.t. s, and $k_3 < k_4$ and $t_{\mathcal{HO}(\pi)}(\ell_{k_3}) = t_{\mathcal{HO}(\pi)}(\ell_{k_4}) = \texttt{axiom}$ and $\forall i \in \{k_3+1, \dots, k_4-1\}, t_{\mathcal{HO}(\pi)}(\ell_i) \neq \texttt{axiom}$ and ℓ_{k_3} is indirect and ℓ_{k_4} is direct w.r.t. s. So, by Proposition 78 there exist $k_5 \in \{k_1+1, \dots, k_2-1\}$ and $k_6 \in \{k_3+1, \dots, k_4-1\}$ such that ℓ_{k_5} and ℓ_{k_6} are horizontal w.r.t. s and, denoting $s(\ell_{k_j}) = x_j \to y_j$ for $j \in \{1, \dots, 6\}$, y_1 is above x_5 and x_2 is above y_5 and y_3 is above x_6 and x_4 is above y_6.

Since ℓ_{k_5} is horizontal w.r.t. s and $\mathcal{HO}(\pi)$ is an ISOMIX proof-structure, $t_{\mathcal{HO}(\pi)}(\ell_{k_5}) = \texttt{tensor}$, so $t_\pi(\ell_{k_5}) = \texttt{tensor}$, hence there exists $z_5 \in c_\pi(\ell_{k_5})$ and H_1, H_2 such that $f_\pi(z_5) = H_1 \otimes H_2$. By Fact 68 $x_5 \neq y_5$, hence since $x_5, y_5 \in p_\pi(\ell_{k_5})$, $(H_1 = f_\pi(x_5)$ and $H_2 = f_\pi(y_5))$ or $(H_1 = f_\pi(y_5)$ and $H_2 = f_\pi(x_5))$. Since ℓ_{k_1} is direct and ℓ_{k_2} is indirect w.r.t. s, there exist $A, B \in \mathcal{V}$ such that $f_{\mathcal{HO}(\pi)}(y_1) = A$ and $f_{\mathcal{HO}(\pi)}(x_2) = B$. So, denoting $a = \mathcal{HO}^{-1}(A)$ and $b = \mathcal{HO}^{-1}(B)$, from Definition 83 $f_\pi(y_1) = a$ and $f_\pi(x_2) = b$. Since y_1 is above x_5 and x_2 is above y_5, by Fact 29 a is a subformula of $f_\pi(x_5)$ and b is a subformula of $f_\pi(y_5)$. So, (a, b) is an opposed pair of subformulas of $f_\pi(z_5)$.

Since ℓ_{k_3} is indirect and ℓ_{k_4} is direct w.r.t. s, by the same argument we prove that there exists $z_6 \in c_\pi(\ell_{k_6})$ and $c, d \in \mathcal{V}_{FO}$ such that $f_\pi(z_6)$ is of the form $K_1 \otimes K_2$ and (c^*, d^*) is an opposed pair of subformulas of z_6.

Since k_1 and k_3 are the maximum indices such that $k_1 < k_5$ and $k_3 < k_6$ and $t_{\mathcal{HO}(\pi)}(\ell_{k_1}) = t_{\mathcal{HO}(\pi)}(\ell_{k_3}) = \texttt{axiom}$. Since ℓ_{k_1} is direct and ℓ_{k_3} is indirect w.r.t. s, $\ell_{k_1} \neq \ell_{k_3}$ so $k_1 \neq k_3$, so $k_5 \neq k_6$ hence since each link happens at most once in the link sequence of C, $\ell_{k_5} \neq \ell_{k_6}$. $\square$

6.3 Forbidden structure

Definition 93. *Let F be a formule of one of the forms $F_1 \otimes F_2$, $F_1 \parr F_2$ and $F_1 < F_2$ and let G, H be formulas. We say that (G, H) is an opposed pair of subformulas of F when (G is a subformula of F_1 and H is a subformula of F_2) or (H is a subformula of F_1 and G is a subformula of F_2).*

Definition 94 (Forbidden structure)**.** *Let F be an FO-ISOMIX formula. We say that F contains a forbidden structure with formulas $(G = G_1 \parr G_2, H = H_1 \otimes H_2, K = K_1 \otimes K_2)$ and FO variables (a, b, c, d) when:*

- *G is a subformula of F,*
- *(H, K) is an opposed pair of subformulas of G,*
- *(a, b) is an opposed pair of subformulas of H,*
- *(c^*, d^*) is an opposed pair of subformulas of K.*

Theorem 95. *Let F be an FO-ISOMIX formula such that $\vdash F$ and $\nvdash \mathcal{HO}(F)$. Then, F contains a forbidden structure.*

Proof. Since $\vdash F$, there exists an FO-ISOMIX proof-net π of F. Since $\nvdash \mathcal{HO}(F)$, $\mathcal{HO}(\pi)$ is not a proof-net, so there exists a switching function s of $\mathcal{HO}(\pi)$ and a cycle C of $\mathcal{G} = (W_\pi, s(L_\pi))$, and we assume w.l.o.g. that the number of horizontal links of C is minimum among all cycles and switching functions of $\mathcal{HO}(\pi)$. By Proposition 92, there exist $\ell, \ell' \in L_\pi$, $x \in c_\pi(\ell)$, $x' \in c_\pi(\ell')$ and $a, b, c, d \in \mathcal{V}_{FO}$ such that $\ell \neq \ell'$ and $s(\ell) \in C$ and ℓ is horizontal w.r.t. s and $f_\pi(x)$ is of the form

$H = H_1 \otimes H_2$ and (a, b) is an opposed pair of subformulas of H and $s(\ell') \in C$ and ℓ' is horizontal w.r.t. s and $f_\pi(x')$ is of the form $K = K_1 \otimes K_2$ and (c^*, d^*) is an opposed pair of subformulas of K.

By Fact 27 there exist downward sequences $((x_0, \dots, x_n), (\ell_1, \dots, \ell_n))$ and $((x'_0, \dots, x'_m), (\ell'_1, \dots, \ell'_m))$ such that $x_0 = x$ and $x'_0 = x'$ and $x_n, x'_m \in \mathrm{Op}_\pi$, and since π is a proof-net of F, $x_n = x'_m$ and $f_\pi(x_n) = F$. By Proposition 31, there exist $k_1 \in \{0, \dots, n\}$ and $k_2 \in \{0, \dots, m\}$ such that $x_{k_1} = x'_{k_2}$ and if $k_1 > 0$ and $k_2 > 0$, then $x_{k_1-1} \neq x'_{k_2-1}$. Since $\ell \neq \ell'$, $x \neq x'$, hence by Theorem 80, x is not above x' and x' is not above x, so $k_2 > 0$ and $k_1 > 0$ and hence $x_{k_1-1} \neq x'_{k_2-1}$. So, by Theorem 80 $t_{\mathcal{HO}(\pi)}(\ell_{k_1}) = \mathtt{par}$, hence from Definition 83 $t_\pi(\ell_{k_1}) = \mathtt{par}$. So, since $x_{k_1} \in c_\pi(\ell_{k_1})$, $f_\pi(x_{k_1})$ is of the form $G = G_1 \parr G_2$, and since $x_{k_1-1} \neq x'_{k_2-1}$ and $x_{k_1-1}, x'_{k_2-1} \in p_\pi(\ell_{k_1})$, $(G_1 = f_\pi(x_{k_1-1})$ and $G_2 = f_\pi(x'_{k_2-1}))$ or $(G_1 = f_\pi(x'_{k_2-1})$ and $G_2 = f_\pi(x_{k_1-1}))$. Since x is above x_{k_1-1} and x' is above x'_{k_2-1}, by Fact 29 H is a subformula of $f_\pi(x_{k_1-1})$ and K is a subformula of $f_\pi(x'_{k_2-1})$, so (H, K) is a pair of opposed subformulas of G. Since x_{k_1} is above x_n, by Fact 29 G is a subformula of F, therefore F contains a forbidden structure. □

7 Tensors of FO maps

7.1 Definition of tensor of FO maps

Definition 96. *A formula U is called a tensor of FO maps when it is of the form:*

$$U = (a_1^* \parr a_2) \otimes \cdots \otimes b_1 \otimes \cdots \otimes c_1^* \otimes \cdots \tag{25}$$

for some parenthesizing between the $\otimes$ connectors and for some $a_1, a_2, \dots, b_1, \dots, c_1, \dots \in \mathcal{V}_{FO}$.

Notation 97. *Given a formula F, we define F^* by induction as:*

$$\forall A \in \mathcal{V}, (A)^* = A^*, (A^*)^* = A \;, \tag{26}$$

$$\forall a \in \mathcal{V}_{FO}, (a)^* = a^*, (a^*)^* = a \;, \tag{27}$$

$$I^* = I \;, \tag{28}$$

$$(F \otimes G)^* = F^* \parr G^* \;, \tag{29}$$

$$(F \parr G)^* = F^* \otimes G^* \;, \tag{30}$$

$$(F < G)^* = F^* < G^* \;. \tag{31}$$

Given two formulas F, G, we define:

$$F \multimap G = F^* \parr G \;. \tag{32}$$

Proposition 98. *Let R be a formula and let U be a tensor of FO maps such that $U \multimap R$ contains a forbidden structure. Then, R contains a forbidden structure.*

Proof. Since $U \multimap R$ contains a forbidden structure there exist formulas $G = G_1 \parr G_2$, $H = H_1 \otimes H_2$ and $K = K_1 \otimes K_2$ and variables $a, b, c, d \in \mathcal{V}_{FO}$ such that G is a subformula of $U \multimap R$, (H, K) is an opposed pair of subformulas of G, (a, b) is an opposed pair of subformulas of H and (c^*, d^*) is an opposed pair of subformulas of K. Since U is a tensor of FO maps there exist $a_1, a_2, \dots, b_1, \dots, c_1, \dots \in \mathcal{V}_{FO}$ such that $U = (a_1^* \parr a_2) \otimes \cdots \otimes b_1 \otimes \cdots \otimes c_1^* \otimes \cdots$ for some parenthesizing between the $\otimes$ connectors. Hence, $U^* = (a_1 \otimes a_2^*) \parr \cdots \parr b_1^* \parr \cdots \parr c_1 \parr \cdots$. Assume that H is a subformula of U^*. Since $H = H_1 \otimes H_2$, there exists some i such that $H = a_i \otimes a_{i+1}^*$. Since a and b are not subformulas of a_{i+1}^*, (a, b) is not an opposed pair of subformulas of $a_i \otimes a_{i+1}^*$, which contradicts $H = a_i \otimes a_{i+1}^*$. So, H is not a subformula of U^*. Similarly, assume that K is a subformula of U^*. Since $K = K_1 \otimes K_2$, there exists some i such that $K = a_i \otimes a_{i+1}^*$. Since c^* and d^* aren't subformulas of a_i, (c^*, d^*) isn't an opposed pair of subformulas of $a_i \otimes a_{i+1}^*$ which contradicts $K = a_i \otimes a_{i+1}^*$, so K is not a subformula of U^*. Since G is a subformula of $U \multimap R$, $G = U \multimap R$ or G is a subformula of U^* or G is a subformula of R.

If G is a subformula of U^*, then since (H, K) is an opposed pair of subformulas of G, H is a subformula of U^* which is absurd.

Otherwise if $G = U \multimap R$ then since (H, K) is an opposed pair of subformulas of G, H is a subformula of U^* or K is a subformula of U^* and both cases are absurd.

So, G is a subformula of R, hence R contains a forbidden structure. □

7.2 Formulas without repeated variables

Definition 99. *Given a formula F, we say that F doesn't have repeated variables when:*

- *for each $A \in \mathcal{V}$, $\mathrm{At}(F)(A) = 0$ or $\mathrm{At}(F)(A^*) = 0$,*
- *for each $a \in \mathcal{V}_{FO}$, $\mathrm{At}(F)(a) = 0$ or $\mathrm{At}(F)(a^*) = 0$.*

7.3 The main theorem with tensors of FO maps

Theorem 100. *Let R be an FO-ISOMIX formula, and let U be a tensor of FO maps such that $\vdash U \multimap R$ and $\not\vdash \mathcal{HO}(U \multimap R)$. Then, R contains a forbidden structure.*

Proof. Since $\vdash U \multimap R$ and $\not\vdash \mathcal{HO}(U \multimap R)$, by Theorem 95 $U \multimap R$ contains a forbidden structure. Since U is a tensor of FO maps and $U \multimap R$ contains a forbidden structure, by Proposition 98 R contains a forbidden structure. $\square$

Fact 101. *Let π be a proof-structure, let s be a switching function of π and let C be a cycle of $\mathcal{G} = (W_\pi, s(L_\pi))$. Then, there exists $\ell \in L_\pi$ such that $s(\ell) \in C$ and $s(\ell) \in c_\pi(\ell)^2$.*

Proof. This is proven by taking a link in the link sequence of C and using Facts 76 and 9. $\square$

Proposition 102. *Let R be an FO-ISOMIX formula without repeated variables, let formulas ($G = G_1 \parr G_2, H = H_1 \otimes H_2, K = K_1 \otimes K_2$) and FO variables (a, b, c, d) forming a forbidden structure of R, and let $U = (c^* \parr a) \otimes (d^* \parr b) \otimes b_1 \otimes \ldots \otimes c_1^* \otimes \ldots$ for each $b_1, \ldots, c_1^*, \ldots \in \mathrm{At}(R) - [a, b, c^*, d^*]$, with any parenthesizing. Then, $\vdash U \multimap R$.*

Proof. U is a tensor of FO-maps, and $\mathrm{At}(R) = \mathrm{At}(U)$, hence since R doesn't have repeated variables, $U \multimap R$ is balanced. Since R and U are FO-ISOMIX formulas, $U \multimap R$ is an FO-ISOMIX formula. Since $U \multimap R$ is balanced, let π be the canonical structure of $U \multimap R$. For a contradiction, assume that there exists a switching function s of π and a cycle C of the graph $\mathcal{G} = (W_\pi, s(L_\pi))$. There exists $(\ell_1^C, \ldots, \ell_n^C)$ a link sequence of C and let us denote $s(\ell_i^C) = x_i^C \to x_{i+1}^C$. By Fact 101, there exists $k_1 \in \{1, \ldots, n\}$ such that $\ell_{k_1}^C$ is horizontal w.r.t. s. Let k_2 be the index of the horizontal link w.r.t. s following $\ell_{k_1}^C$ in C. By Proposition 77, there exists a unique index q_1 such that $k_1 < q_1 < k_2$ and $s(\ell_{q_1}^C) \in c_\pi(\ell_{q_1}^C)^2$ and $((x_{q_1}^C, \ldots, x_{k_1+1}^C), (\ell_{q_1-1}^C, \ldots, \ell_{k_1+1}^C))$ and $((x_{q_1+1}^C, \ldots, x_{k_2}^C), (\ell_{q_1+1}^C, \ldots, \ell_{k_2-1}^C))$ form downward sequences. Since π is an FO-ISOMIX proof-structure, $t_\pi(\ell_{q_1}^C) = \texttt{FO-axiom}$ and hence $\ell_{q_1}^C$ is direct w.r.t. s, hence there exists $a_1 \in \mathcal{V}_{FO}$ such that $a_1 \in \mathrm{At}(U \multimap R)$ and $f_\pi(x_{q_1+1}^C) = a_1$ and $f_\pi(x_{q_1}^C) = a_1^*$. For a contradiction, assume that $a_1 \in \{b_1, \ldots\}$. By definition of U, $U \multimap R = (c \otimes a^*) \parr (d \otimes b^*) \parr b_1^* \parr \ldots \parr c_1 \parr \ldots \parr R$. Hence, each subformula F' of $U \multimap R$ such that a_1^* is a subformula of F' is not of the form $F_1' \otimes F_2'$, hence by Fact 32 there doesn't exist a wire x' such that $f_\pi(x')$ is of the form $F_1' \otimes F_2'$ and $x_{q_1}^C$ is above x'. Since $t_\pi(\ell_{k_1}^C)$ is horizontal w.r.t. s, $t_\pi(\ell_{k_1}^C) = \texttt{FO-axiom}$ and there exists $x_1' \in c_\pi(\ell_{k_1}^C)$ and $f_\pi(x_1')$ is of the form $F_1' \otimes F_2'$. Since $x_{k_1+1}^C \in p_\pi(\ell_{k_1}^C)$ and $x_1' \in c_\pi(\ell_{k_1}^C)$ and $x_{q_1}^C$ is above $x_{k_1+1}^C$, $x_{q_1}^C$ is above x_1', which is absurd.

Similarly, for a contradiction let us assume that $a_1 \in \{c_1, \ldots\}$. Then, by definition of U, each subformula F' of $U \multimap R$ such that a_1 is a subformula of F' is not of the form $F_1' \otimes F_2'$. Hence, by Fact 32 there does not exist a wire x' such that $x_{q_1+1}^C$ is above x' and $f_\pi(x')$ is of the form $F_1' \otimes F_2'$. Since $\ell_{k_2}^C$ is horizontal w.r.t. s, $t_\pi(\ell_{k_2}^C) = \texttt{tensor}$, hence there exists $x_2' \in c_\pi(\ell_{k_2}^C)$ and $f_\pi(x_2')$ is of the form $F_1' \otimes F_2'$, and $x_{q_1+1}^C$ is above x_2', which is absurd. So, $a_1 \in \{a, b, c, d\}$.

Assume that $a_1 = a$. Let k_3 be the index of the first link which is horizontal w.r.t. s appearing after ℓ_{k_2} in C. By Proposition 77 there exists a unique index q_2 such that $k_2 < q_2 < k_3$ and $s(\ell_{q_2}^C) \in c_\pi(\ell_{q_2})^2$ and $((x_{q_2}^C, \ldots, x_{k_2+1}^C), (\ell_{q_2-1}^C, \ldots, \ell_{k_2+1}^C))$ and $((x_{q_2+1}^C, \ldots, x_{k_3}^C), (\ell_{q_2+1}^C, \ldots, \ell_{k_3-1}^C))$ form downward sequences. Since π is an FO-ISOMIX proof-structure, $t_\pi(\ell_{q_2}^C) = \texttt{FO-axiom}$ hence $\ell_{q_2}^C$ is direct w.r.t. s so there exists $a_2 \in \mathcal{V}_{FO}$ such that $f_\pi(x_{q_2}^C) = a_2^*$ and $f_\pi(x_{q_2+1}^C) = a_2$. By the same argument that we used previously for a_1, $a_2 \in \{a, b, c, d\}$. Since $s(\ell_{k_2}^C) = x_{k_2}^C \to x_{k_2+1}^C \in p_\pi(\ell_{k_2}^C)^2$, by Fact 68 $x_{k_2}^C \neq x_{k_2+1}^C$. So, since $x_{q_2}^C$ is above $x_{k_2+1}^C$, $x_{q_2}^C$ is not above $x_{k_2}^C$, and similarly since $x_{q_1+1}^C$ is above $x_{k_2}^C$, $x_{q_1}^C$ is not above $x_{k_2+1}^C$. Hence, a_2^* is a subformula of $f_\pi(x_{k_2+1}^C)$ and not of $f_\pi(x_{k_2}^C)$, and a_1 is a subformula of $f_\pi(x_{k_2}^C)$ and not of $f_\pi(x_{k_2+1}^C)$. Since $x_{k_2}^C, x_{k_2+1}^C \in p_\pi(\ell_{k_2}^C)$

and $x'_2 \in c_\pi(\ell^C_{k_2})$, $f_\pi(x^C_{k_2})$ and $f_\pi(x^C_{k_2+1})$ are direct subformulas of $f_\pi(x'_2)$. Assume that $a_2 = a$. Then, $f_\pi(x'_2) = U \multimap R$ which is absurd since $f_\pi(x'_2)$ is of the form $F'_1 \otimes F'_2$, so $a_2 \neq a$. By the same argument, $a_2 \neq b$. Similarly, if $a_2 = c$ or $a_2 = d$, then $f_\pi(x'_2) = G$ which is absurd since $G = G_1 \parr G_2$ and $f_\pi(x'_2)$ is of the form $F'_1 \otimes F'_2$. Hence, $a_2 \neq c$ and $a_2 \neq d$, which contradicts $a_2 \in \{a, b, c, d\}$. So, $a_1 \neq a$.

By the same argument we can show that $a_1 \neq b$. By a similar argument, where we introduce ℓ_{k_0} the first horizontal link w.r.t. s appearing before ℓ_{k_1} in C instead of ℓ_{k_3}, we can show that $a_1 \neq c$ and $a_1 \neq d$, which contradicts $a_1 \in \{a, b, c, d\}$. Hence, for each switching function s of π, $\mathcal{G}$ doesn't contain a cycle, so π is a proof-net of $U \multimap R$ and hence $\vdash U \multimap R$. □

Proposition 103. *Let R be an FO-ISOMIX formula without repeated variables, let formulas $(G = G_1 \parr G_2, H = H_1 \otimes H_2, K = K_1 \otimes K_2)$ and FO variables (a, b, c, d) forming a forbidden structure of R, and let $U = (c^* \parr a) \otimes (d^* \parr b) \otimes b_1 \otimes \ldots \otimes c^*_1 \otimes \ldots$ for each $b_1, \ldots, c^*_1, \ldots \in \mathrm{At}(R) - [a, b, c^*, d^*]$, with any parenthesizing. Then, $\nvdash \mathcal{HO}(U \multimap R)$.*

Proof. U is a tensor of FO-maps, and $\mathrm{At}(R) = \mathrm{At}(U)$, hence since R doesn't have repeated variables, $U \multimap R$ is balanced, so $\mathcal{HO}(U \multimap R)$ is also balanced. Since R and U are FO-ISOMIX formulas, $\mathcal{HO}(U \multimap R)$ is an ISOMIX formula.

Let π be a proof-structure of $U \multimap R$. $U \multimap R = (c \otimes a^*) \parr (b \otimes d^*) \parr b^*_1 \parr \ldots \parr c_1 \parr \ldots \parr R$. For each formula $F' \in (G, G_1, G_2, H, H_1, H_2, K, K_1, K_2, a, b, c^*, d^*, a^*, b^*, c, d, c \otimes a^*, b \otimes d^*)$, since F' is a subformula of $U \multimap R$ there exists a wire $x_{F'}$ such that $f_\pi(x_{F'}) = F'$. Since $U \multimap R$ is balanced, by Theorem 56 each $x_{F'}$ is unique given F', hence by Fact 32 for each F', G', $x_{F'}$ is above $x_{G'}$ iff F' is a subformula of G'. Let us denote $\ell_{F'}$ such that $x_{F'} \in c_\pi(\ell_{F'})$.

W.l.o.g. let us assume that a is a subformula of H_1 and b is a subformula of H_2, and that c^* is a subformula of K_1 and d^* is a subformula of K_2. Hence, there exist downward sequences $((x_a, \ldots, x_{H_1}), (\ldots, \ell_{H_1}))$, $((x_b, \ldots, x_{H_2}), (\ldots, \ell_{H_2}))$, $((x_{c^*}, \ldots, x_{K_1}), (\ldots, \ell_{K_1}))$ and $((x_{d^*}, \ldots, x_{K_2}), (\ldots, \ell_{K_2}))$. From the expression of $U \multimap R$, all the links among these downward sequences, as well as the links $\ell_H, \ell_K, \ell_a, \ell_b, \ell_c, \ell_d, \ell_{c \otimes a^*}, \ell_{d \otimes b^*}$, are pairwise different. So, there exist a partial function s such that s maps the links $\ldots, \ell_{H_1}$ to arcs forming a path from x_a to x_{H_1}, $s(\ell_H) = x_{H_1} \to x_{H_2}$, s maps the links $\ldots, \ell_{H_2}$ to form a path from x_{H_2} to x_b, $s(\ell_b) = x_b \to x_{b^*}$, $s(\ell_{d \otimes b^*}) = x_{b^*} \to x_d$, $s(\ell_d) = x_d \to x_{d^*}$, s maps the links $\ldots, \ell_{K_2}$ to form a path from x_{d^*} to x_{K_2}, $s(\ell_K) = x_{K_2} \to x_{K_1}$, s maps the links $\ldots, \ell_{K_1}$ to form a path from x_{K_1} to x_{c^*}, $s(\ell_c) = x_{c^*} \to x_c$, $s(\ell_{c \otimes a^*}) = x_c \to x_{a^*}$, $s(\ell_a) = x_{a^*} \to x_a$, and s maps all other links to any allowed arcs.

Since $H = H_1 \otimes H_2$ and $K = K_1 \otimes K_2$, $p_\pi(\ell_H) = \{x_{H_1}, x_{H_2}\}$ and $p_\pi(\ell_K) = \{x_{K_1}, x_{K_2}\}$ and $t_{\mathcal{HO}(\pi)}(\ell_H) = t_{\mathcal{HO}(\pi)}(\ell_K) = \texttt{tensor}$. Since $t_\pi(\ell_a) = t_\pi(\ell_b) = t_\pi(\ell_c) = t_\pi(\ell_d) = \texttt{FO-axiom}$, $t_{\mathcal{HO}(\pi)}(\ell_a) = t_{\mathcal{HO}(\pi)}(\ell_b) = t_{\mathcal{HO}(\pi)}(\ell_c) = t_{\mathcal{HO}(\pi)}(\ell_d) = \texttt{axiom}$. Hence, s is a switching function of $HO(\pi)$ and and $\mathcal{G} = (W_\pi, s(L_\pi))$ contains a cycle, so $\mathcal{HO}(\pi)$ is not a proof-net of $\mathcal{HO}(U \multimap R)$. Since this holds for any proof-structure π of $\mathcal{HO}(U \multimap R)$, $\nvdash \mathcal{HO}(U \multimap R)$. □

Theorem 104. *Let R be an FO-ISOMIX formula without repeated variables. If R contains a forbidden structure, then there exists a tensor of FO maps U such that $\mathrm{At}(U) = \mathrm{At}(R)$ and $\vdash U \multimap R$ and $\nvdash \mathcal{HO}(U \multimap R)$.*

Proof. There exist subformulas $(G = G_1 \parr G_2, H = H_1 \otimes H_2, K = K_1 \otimes K_2)$ and FO variables (a, b, c, d) forming a forbidden structure of R. Let U be the left-parenthesized formula $U = (c^* \parr a) \otimes (d^* \parr b) \otimes b_1 \otimes \ldots \otimes c^*_1 \otimes \ldots$ for each $b_1, \ldots, c^*_1, \ldots \in \mathrm{At}(R) - [a, b, c^*, d^*]$. Then, by Proposition 102 $\vdash U \multimap R$ and by Proposition 103 $\nvdash \mathcal{HO}(U \multimap R)$. □

Theorem 105. *Let R be an FO-ISOMIX formula without repeated variables. R contains a forbidden structure if and only if there exists a tensor of FO maps U such that $\mathrm{At}(U) = \mathrm{At}(R)$ and $\vdash U \multimap R$ and $\nvdash \mathcal{HO}(U \multimap R)$.*

Proof. By Theorems 100 and 104. □

8 Finding forbidden structures in FO-ISOMIX formulas

Definition 106. *We define by induction on F the following predicates:*

$$hasV(a) = 1 \tag{33}$$

$$hasV(a^*) = hasV(I) = 0 \tag{34}$$

$$hasV(F \otimes G) = hasV(F \parr G) = hasV(F) \vee hasV(G) \tag{35}$$

$$hasVStar(a^*) = 1 \tag{36}$$

$$hasVStar(a) = hasVStar(I) = 0 \tag{37}$$

$$hasVStar(F \otimes G) = hasVStar(F \parr G) = hasVStar(F) \vee hasVStar(G) \tag{38}$$

$$hasH(a) = hasH(a^*) = hasH(I) = 0 \tag{39}$$

$$hasH(F \otimes G) = hasH(F \parr G) = hasV(F) \wedge hasV(G) \tag{40}$$

$$hasK(a) = hasK(a^*) = hasK(I) = 0 \tag{41}$$

$$hasK(F \otimes G) = hasK(F \parr G) = hasVStar(F) \wedge hasVStar(G) \tag{42}$$

$$hasG(a) = hasG(a^*) = hasG(I) = 0 \tag{43}$$

$$hasG(F \otimes G) = hasG(F \parr G) = (hasH(F) \wedge hasK(G)) \vee (hasK(F) \wedge hasH(G)) \tag{44}$$

Theorem 107. *Let F be an FO-ISOMIX formula. Then, F contains a forbidden structure if and only if $hasG(F) = 1$.*

Proof. We can prove this by induction on F with Definition 94. □

Acknowledgments

This research was funded in part by l'Agence Nationale de la Recherche (ANR) project ANR-22-CE47-0012 and the PEPR integrated project EPiQ ANR-22-PETQ-0007 as part of Plan France 2030. For the purpose of open access, the authors have applied a CC-BY public copyright license to any Author Accepted Manuscript (AAM) version arising from this submission.

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