Efficient quantum-circuit simulation of classical control of causal order

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We show that quantum circuits with classical control of causal order admit an efficient simulation by fixed-order quantum circuits: any such transformation acting on N independent channels can be deterministically, exactly, and universally simulated using at most N^2 channel queries. For quantum circuits with quantum control of order, we prove a similar simulation result, but restricted to unitary input channels (hence non-universal). We further generalize these constructions to extended, bipartite quantum channels and demonstrate that they can be implemented catalytically, without consuming the additional $N^2 - N$ queries required for the simulation. These results show the limitations of the computational power of quantum circuits with classical control of causal order in general, and of quantum circuits with quantum control of causal order when only their action on unitary channels is considered.

1. Introduction.— The quantum circuit model is one of the most widely applied models of quantum computing. It describes a computation applied onto an input quantum state via a sequence of quantum gates, also called quantum channels, and quantum measurements. A well-known such example is the quantum circuit that implements Shor's algorithm, a paramount result in quantum computing about the efficient factoring of integer numbers. The more recent formalism of higher-order quantum computing [1] places quantum channels center stage, not only as the means to execute a computation through a quantum circuit, but *as the input and output of the computation itself.*

Formally, a higher-order order transformation called a *superchannel* is the most general linear map that transforms several quantum channels into another quantum channel. Hence, a quantum circuit with open slots, where arbitrary quantum channels can be plugged in, can be seen as a higher-order order transformation. At the same time, since quantum channels carry an inherent input–output structure, the question of causal order naturally emerges when transforming multiple channels. Strikingly, this formalism allows superchannels that act on the input channels in a manner that goes beyond what can be described with the standard paradigm of quantum circuits, where quantum gates are implemented on a fixed, pre-defined order. These transformations play with the order in which the input channels are acted upon: they can be described with classical control of orders [2, 3] or even with quantum control of order [2, 3], leading to transformations that can be understood as carried out in a dynamical (i.e. not pre-defined) and indefinite causal order.

Theoretical advantages of quantum control of orders have been confirmed in a wide range of quantum information processing tasks, such as quantum channel discrimination [4, 5], quantum metrology [6, 7], quantum computational and query complexity [8, 9], among others. As for classical control, a so far less explored class of transformations, an advantage has been recently reported [10] with respect to fixed-order strategies in a task of simulating quantum control. Such information-processing advantages have mostly been shown via comparison with fixed-order quantum circuits that act on a single call of each input channel. However, their true practical significance hinges on the extent to which these advantages would still hold when comparing superchannels with quantum and classical control of causal order against fixed-order quantum circuits that use a larger number of queries to the input channels. Recent results have shown that a universal, deterministic and exact simulation of the quantum switch, a superchannel with *quantum control* of orders, is computationally hard, if at all possible [10].

In contrast, in this work, we show that quantum circuits with a *classical control* of orders can be simulated by quantum circuits with a fixed order in query-efficient manner: we prove that any quantum circuit with classical control of orders that acts on N independent quantum channels can be deterministically, exactly, and universally simulated by a fixed-order quantum circuit that acts on at most N^2 queries to the input channels. Our proof is constructive and builds on top of a known quantum circuit that can simulate the action of the quantum N-switch on unitary channels [8]. Moreover, we show that quantum circuits with quantum control of orders can also be simulated in a similar manner, however, not universally: their action on only *unitary* channels can by deterministically and exactly simulated with at most N^2 queries to the input unitary channels. We generalize these results to show that, at no extra cost in terms of channel queries, both simulations work for extended quantum channels and can be made catalytically, i.e., without consuming the

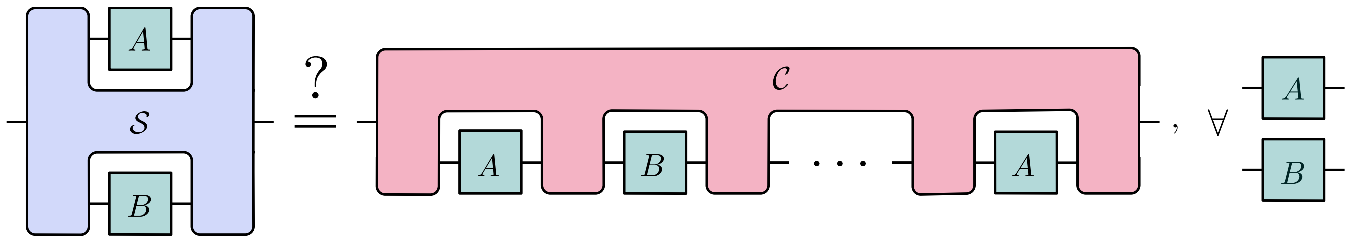

Figure 1. A fixed-order quantum circuit $\mathcal{C}$, here represented by a quantum comb, is able to simulate a given ($N = 2$)-slot superchannel $\mathcal{S}$ if, for every input channels A and B, the output channel of $\mathcal{S}$ can be reproduced by a $\mathcal{C}$ that has access to a finite number of queries to the input channels, i.e., if $\exists\ n_A, n_B \in \mathbb{N}$ such that $\mathcal{S}(A, B) = \mathcal{C}(A^{\otimes n_A}, B^{\otimes n_B})$, $\forall\ A, B$.

$N^2 - N$ extra queries of the input channels.

2. Framework.— A *superchannel* is a higher-order transformation that maps every input set of quantum channels into an output quantum channel. In this work, we study three main kinds of superchannels:

- **Quantum circuits with a fixed order (QC-FO).** These higher-order transformations act on their input channels in a fixed and pre-defined order. We will denote the set of all such transformations as QCFO.
- **Quantum circuits with classical control of order (QC-CC).** These higher-order transformations act on their input channels in a order that is determined by the state of a classical control system. This order may be dynamical, and hence not pre-established, but it can nevertheless be interpreted as a definite causal order. We will denote the set of all such transformations as QCCC.
- **Quantum circuits with quantum control of order (QC-QC).** These higher-order transformations act on their input channels in a order that is determined by the state of a quantum control system. This order may be dynamical and, moreover, it may be causally indefinite. We will denote the set of all such transformations as QCQC.

Their precise definitions are given in the technical manuscript. Notice that QCFO $\subset$ QCCC $\subset$ QCQC. The problem of simulating a superchannel with a fixed-order superchannel is formalized in the following way:

Definition 1 (Simulation of a superchannel)**.** *The action of a superchannel $\mathcal{S}$ on its input channels can be universally, deterministically, and exactly simulated by a fixed-order superchannel $\mathcal{C} \in$ QCFO if there exist some finite numbers of queries $n_1, \ldots, n_N \in \mathbb{N}$ such that*

$$\mathcal{S}(\mathcal{A}_1, \ldots, \mathcal{A}_N) = \mathcal{C}(\mathcal{A}_1^{\otimes n_1}, \ldots, \mathcal{A}_N^{\otimes n_N}) \tag{1}$$

for all possible input channels $\mathcal{A}_1, \ldots, \mathcal{A}_N$. When the action of the superchannel $\mathcal{S}$ on only a subset of possible input channels can be simulated (e.g. only on unitary channels), the simulation is called non-universal.

In this work, we investigate the cost of simulating superchannels $\mathcal{S} \in$ QCCC and $\mathcal{S} \in$ QCQC with superchannels $\mathcal{C} \in$ QCFO, that is, the cost of simulating a computation with a dynamical causal order, that is either classically or quantumly controlled, using computations with a fixed causal order. Here, this cost is measured via a notion of *quantum query complexity*: the minimum amount of queries to the input channels $n_1 + \ldots + n_N$ that a fixed-order quantum circuit $\mathcal{C}$ requires in order to simulate a given superchannel $\mathcal{S}$.

Furthermore, we consider the two following forms of simulation, each of which is a "harder" simulation requirement than that of Definition 1: simulation of the action of a superchannel on extended quantum channels and catalytic simulation. The existence of a simulation of a given superchannel that satisfies the following requirements immediately implies that a simulation of the form of Definition 1 also exists:

– **Simulation of the action of superchannels on extended quantum channels:** A simulation of the action of a given superchannel $\mathcal{S}$ on extended quantum channels is a simulation that holds in the case where the superchannel $\mathcal{S}$ acts only on part of *extended*, bipartite input quantum channels, while the simulation $\mathcal{C}$ acts on the entire extended quantum channels, hence satisfying, for some $n_1, \ldots, n_N \in \mathbb{N}$,

$$\mathcal{S} \otimes \mathcal{I}(\mathcal{A}_1, \ldots, \mathcal{A}_N) = \mathcal{C}(\mathcal{A}_1^{\otimes n_1}, \ldots, \mathcal{A}_N^{\otimes n_N}) \tag{2}$$

for all input quantum channels $\mathcal{A}_1, \ldots, \mathcal{A}_N$, where $\mathcal{I}$ is the identity superchannel.

– **Catalytic simulation of superchannels:** A fixed-order quantum-circuit simulation $\mathcal{C}$ of a given superchannel $\mathcal{S}$ is deemed catalytic if it works in such a way that the extra queries of the input channels are preserved throughout the simulation. That is, if

$$\mathcal{C}(\mathcal{A}_1^{\otimes n_1}, \ldots, \mathcal{A}_N^{\otimes n_N}) = \mathcal{S}(\mathcal{A}_1, \ldots, \mathcal{A}_N) \otimes \mathcal{A}_1^{\otimes(n_1-1)} \otimes \ldots \otimes \mathcal{A}_N^{\otimes(n_N-1)}, \tag{3}$$

for all input quantum channels $\mathcal{A}_1, \ldots, \mathcal{A}_N$. This transformation is deemed catalytic in the sense that, for the set of allowed simulators QCFO,

$$\mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_N \overset{\mathsf{QCFO}}{\not\longmapsto} \mathcal{S}(\mathcal{A}_1, \ldots, \mathcal{A}_N) \quad \text{but} \tag{4}$$

$$\mathcal{A}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{A}_N^{\otimes n_N} \overset{\mathsf{QCFO}}{\longmapsto} \mathcal{S}(\mathcal{A}_1, \ldots, \mathcal{A}_N) \otimes \mathcal{A}_1^{\otimes(n_1-1)} \otimes \ldots \otimes \mathcal{A}_N^{\otimes(n_N-1)}. \tag{5}$$

That is, without the resource of the extra queries, the simulation is not possible with the set of simulators QCFO; however, when given access to the resource of the extra queries, the simulation becomes possible and, moreover, the extra required queries are not consumed in the process of the simulation. More details on the interpretation of this catalytic transformation and on the reusability of the recovered queries of the input channels are given in the technical manuscript.

Notice that a simulation $\mathcal{C}$ may, in principle, satisfy both of the above requirements at once.

3. Results.— Our main results are the construction of explicit quantum circuits that perform the following simulations:

Theorem 1 (Universal QC-CC simulation)**.** *For every $\mathcal{S} \in \mathsf{QCCC}$ that acts on N quantum channels, there exists a fixed-order quantum circuit $\mathcal{C} \in \mathsf{QCFO}$ using at most N queries of each input channel, hence N^2 total queries, that simulates $\mathcal{S}$ universally.*

Theorem 2 (Non-universal QC-QC simulation)**.** *For every $\mathcal{S} \in \mathsf{QCQC}$ that acts on N quantum channels, there exists a fixed-order quantum circuit $\mathcal{C} \in \mathsf{QCFO}$ using at most N queries of each input channel, hence N^2 total queries, that simulates the action of $\mathcal{S}$ on unitary channels.*

Furthermore, we prove that both of our main results hold also in a stronger form: both Theorems 1 and 2 hold, for the same number of total queries, N^2, also in the scenarios of extended channels and catalytic simulations. We provide explicit construction for the fixed-order quantum circuits that perform these simulations in all cases and figures depicting these circuits in the technical manuscript.

For our explicit catalytic simulations, the preserved extra $N^2 - N$ queries to the input channels comes at a cost: the fixed-order quantum circuits required for the implementation of the catalytic simulation requires extra auxiliary systems, and hence, a larger quantum memory dimension.

4. Discussion.— We showed that any QC-CC and any QC-QC acting on unitary channels can be simulated by fixed-order quantum circuits with at most a quadratic cost in quantum query complexity, hence, efficiently. We conjecture that our simulations are optimal, in the required number of queries, in the regime of large N. That is, we conjecture that our provably attainable upper bound of $\mathcal{O}(N^2)$ is asymptotically tight.

While we show that the computational power of QC-CCs is limited, at least in terms of quantum query complexity, to at most a quadratic advantage over fixed-order quantum circuits, it remains an open question to understand for which kinds of computational and information-processing tasks this quadratic advantage can be achieved, if any. At the same time, our results do not forbid greater advantages with respect to other notions of cost/performance. For example, when comparing superchannels of a fixed number of slots in a given task, QC-CCs can provide advantages over fixed-order quantum circuits in terms of measures such as probability of success or precision, one such example being reported in Ref. [10]. It would be interesting to find out for which other tasks such advantages are possible and how they scale in, e.g., the dimension of the input quantum channels. Our results imply, nevertheless, that any such advantages can always be matched by fixed-order circuits with at most a quadratic cost in number of queries.

It also remains an open question whether a universal simulation of QC-QCs is possible using fixed-order quantum circuits.

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We show that quantum circuits with classical control of causal order admit an efficient simulation by fixed-order quantum circuits: any such transformation acting on N independent channels can be deterministically, exactly, and universally simulated using at most N^2 channel queries. For quantum circuits with quantum control of order, we prove a similar simulation result, but restricted to unitary input channels (hence non-universal). We further generalize these constructions to extended, bipartite quantum channels and demonstrate that they can be implemented catalytically, without consuming the additional $N^2 - N$ queries required for the simulation. These results show the limitations of the computational power of quantum circuits with classical control of causal order in general, and of quantum circuits with quantum control of causal order when only their action on unitary channels is considered.

CONTENTS

I. INTRODUCTION

The quantum circuit model is one of the most widely applied models of quantum computing. It describes a computation applied onto an input quantum state via a sequence of quantum gates, also called quantum channels, and quantum measurements. A well-known such example is the quantum circuit that implements Shor's algorithm, a paramount result in quantum computing about the efficient factoring of integer numbers. The more recent formalism of higher-order quantum computing [1] places quantum channels center stage, not only as the means to execute a computation through a quantum circuit, but *as the input and output of the computation itself.*

Formally, a higher-order order transformation called a *superchannel* is the most general linear map that transforms several quantum channels into another quantum channel [2–5]. Hence, a quantum circuit with open slots, where arbitrary quantum channels can be plugged in, can be seen as a higher-order order transformation. At the same time, since quantum channels carry an inherent input–output structure, the question of causal order naturally emerges when transforming multiple channels. Strikingly, this formalism allows superchannels that act on the input channels in a manner that goes beyond what can be described with the standard paradigm of quantum circuits, where quantum gates are implemented on a fixed, pre-defined order. These transformations play with the order in which the input channels are acted upon: they can be described with classical control of orders [6, 7] or even with quantum control of order [6, 7], leading to transformations that can be understood as carried out in a dynamical (i.e. not pre-defined) and indefinite causal order [7–10]. Such transformations raise immediate fundamental questions about role of causality in quantum theory, as well as more practical questions, concerning their implementability and potential advantages.

Theoretical advantages of quantum control of orders have been confirmed in a wide range of quantum information processing tasks, such as quantum channel discrimination [11, 12], quantum metrology [13, 14], quantum computational and query complexity [15, 16], among others. As for classical control, a so far less explored class of transformations, an advantage has been recently reported [17] with respect to fixed-order strategies in a task of simulating quantum control. Such information-processing advantages have mostly been shown via comparison with fixed-order quantum circuits that act on a single call of each input channel. However, their true practical significance hinges on the extent to which these advantages would still hold when comparing superchannels with quantum and classical control of causal order against fixed-order quantum circuits that use a larger number of queries to the input channels. Recent results have shown that a universal, deterministic and exact simulation of the quantum switch, a superchannel with *quantum control* of orders, is computationally hard, if at all possible [17].

In contrast, in this work, we show that quantum circuits with a *classical control* of orders can be simulated by quantum circuits with a fixed order in query-efficient manner: we prove that any quantum circuit with classical control of orders that acts on N independent quantum channels can be deterministically, exactly, and universally simulated by a fixed-order quantum circuit that acts on at most N^2 queries to the input channels. Our proof is constructive and builds on top of a known quantum circuit that can simulate the action of the quantum N-switch on unitary channels [15]. Moreover, we show that quantum circuits with quantum control of orders can also be simulated in a similar manner, however, not universally: their action on only *unitary* channels can by deterministically and exactly simulated with at most N^2 queries to the input unitary channels. We generalize these results to show that, at no extra cost in terms of channel queries, both simulations work for extended quantum channels and can be made catalytically, i.e., without consuming the $N^2 - N$ extra queries of the input channels.

The paper is organised as follows. In Sec. II we introduce some preliminary notions concerning the simulation of the N-classical switch. In Sec. III we present a fixed order circuit that simulates efficiently a QC-CC. In Sec. IV we present the analogous circuit for the QC-QC simulation when input quantum channels are unitaries. In Sec. V we discuss the optimality of the simulation schemes previously introduced in terms of the number of calls to the parties operations. Then, in Sec. VI, we conclude with a discussion of our results and some related open questions.

II. PRELIMINARIES

A. Mathematical tools

A handy tool that we will use throughout this paper is the Choi–Jamiołkowski isomorphism [18, 19]. Its "pure" version allows to describe any operator $V : \mathcal{H}^X \to \mathcal{H}^Y$ by its so-called Choi vector as the "double-ket" vector

$$|V\rangle\rangle := (\mathbb{1}^X \otimes V)|\mathbb{1}\rangle\rangle^{XX} = \sum_i |i\rangle^X \otimes V|i\rangle^X \ \in \mathcal{H}^{XY}, \tag{1}$$

with $\{|i\rangle^X\}_i$ denoting the computational basis of $\mathcal{H}^X$ and $|\mathbb{1}\rangle\rangle^{XX} := \sum_i |i\rangle^X \otimes |i\rangle^X$, and where we use the short-hand notations $\mathcal{H}^{XY} := \mathcal{H}^X \otimes \mathcal{H}^Y$. A quantum circuit being defined as the composition of various maps, we now introduce a tool that we will employ extensively in what follows to describe composition of maps in the Choi picture. Given two maps $U : \mathcal{H}^X \to \mathcal{H}^Y$ and $V : \mathcal{H}^Y \to \mathcal{H}^Z$ that share a common space $\mathcal{H}^Y$, described by their Choi vectors $|U\rangle\!\rangle \in \mathcal{H}^{XY}$ and $|V\rangle\!\rangle \in \mathcal{H}^{YZ}$, we describe their composition with the so-called "pure" link product [2, 4, 7] as

$$|U\rangle\!\rangle * |V\rangle\!\rangle := \big(\mathbb{1}^{XZ} \otimes \langle\!\langle \mathbb{1}|^{YY}\big)\big(|U\rangle\!\rangle \otimes |V\rangle\!\rangle^{YZ}\big) \ \in \mathcal{H}^{XZ}. \tag{2}$$

The link product is commutative (up to a re-ordering of the tensor products) and associative. It reduces to a tensor product if $\mathcal{H}^Y$ is trivial, or to a scalar product if $\mathcal{H}^X$ and $\mathcal{H}^Z$ are trivial.

In the mixed case, one can still define the Choi–Jamiołkowski isomorphism and the link product as follows. For any given map $\mathcal{M} : \mathcal{L}(\mathcal{H}^X) \to \mathcal{L}(\mathcal{H}^Y)$, one can define its Choi matrix M as

$$M := \sum_{i,j} |i\rangle\langle j|^X \otimes \mathcal{M}\big(|i\rangle\langle j|^X\big) \ \in \mathcal{L}\big(\mathcal{H}^Y\big), \tag{3}$$

where we use the short-hand notations $|i\rangle\langle j|^X := |i\rangle^X \langle j|^X$. For two maps $\mathcal{M}_1 : \mathcal{L}(\mathcal{H}^X) \to \mathcal{L}(\mathcal{H}^Y)$ and $\mathcal{M}_2 : \mathcal{L}(\mathcal{H}^Y) \to \mathcal{L}(\mathcal{H}^Z)$, their link product is then defined as

$$\begin{aligned} M_1 * M_2 :=& \big(\mathbb{1}^{XZ} \otimes \langle\!\langle \mathbb{1}|^{YY}\big)(M_1 \otimes M_2)\big(\mathbb{1}^{XZ} \otimes |\mathbb{1}\rangle\!\rangle^{YY}\big) \\ =& \operatorname{Tr}_Y \big[\big((M_1)^{T_Y} \otimes \mathbb{1}^Z\big)\big(\mathbb{1}^X \otimes M_2\big)\big] \quad \in \mathcal{L}\big(\mathcal{H}^{XZ}\big), \end{aligned} \tag{4}$$

where Tr_Y and ^{T_Y} denote the partial trace and partial transpose over $\mathcal{H}^Y$, respectively.

B. Simulation of higher-order transformations with quantum circuits with fixed order

Throughout this work we will be interested in the simulation of different higher-order transformations within quantum theory [2, 4], also known as quantum supermaps [3], given access to quantum circuits and extra calls to the input channels on which the higher-order transformations are applied. Formally a supermap $\mathcal{C}$ transforms any N input completely positive (CP) maps $\mathcal{A}_k : \mathcal{L}(\mathcal{H}^{A_k^I}) \to \mathcal{L}(\mathcal{H}^{A_k^O})$, for $k = 1, \ldots, N$, into a valid output CP map $\mathcal{C}(\mathcal{A}_1, \ldots, \mathcal{A}_N) : \mathcal{L}(\mathcal{H}^P) \to \mathcal{L}(\mathcal{H}^F)$, defined as a transformation from a global past Hilbert space $\mathcal{H}^P$ to a global future Hilbert space $\mathcal{H}^F$. That the output map is a valid CP map for any choice of the $\mathcal{A}_k$'s imposes some validity constrains on the supermap $\mathcal{C}$, that can for instance be expressed in the Choi picture using the process matrix formalism [5]. In this work we will be interested in valid supermaps that transform quantum channels into a quantum channel, which we then refer to as superchannels. The valid superchannels capture various kinds of causal structures between the operations $\mathcal{A}_k$, from a fixed causal order as in quantum circuits with fixed order [2], to indefinite causal order in the quantum switch [6], or dynamical causal order in QC-CCs and QC-QCs [7].

We start by quickly defining the quantum circuits with fixed order that will be used later to simulate QC-CCs and QC-QCs, and then define the general notion of simulation of a superchannel with such quantum circuits with fixed order.

Using the formalism of higher-order quantum circuits, also known as "quantum combs" [2, 4], "quantum channels with memory" [20] or "quantum strategies" [21], one can describe the most general way to transform N quantum operations $\mathcal{A}_k : \mathcal{L}(\mathcal{H}^{A_k^I}) \to \mathcal{L}(\mathcal{H}^{A_k^O})$ into a valid quantum channel defined from a global past Hilbert space $\mathcal{H}^P$ to a global future Hilbert space $\mathcal{H}^F$, where the operations are applied sequentially in a fixed causal order $\mathcal{A}_{k_1} \prec \mathcal{A}_{k_2} \prec \ldots \prec \mathcal{A}_{k_N}$. The architecture of the resulting quantum channel consists of an alternation between the operations $\mathcal{A}_k$ and internal completely positive trace-preserving (CPTP) operations, applied jointly on the output of the previous operation $\mathcal{A}_k$ and some potential quantum memory. In what follows we will refer to this class of superchannels as the quantum circuits with fixed order, or simply quantum circuits.

In this work, we investigate the cost of simulating QC-CC and QC-QC superchannels with quantum circuits with fixed order, that is, the cost of simulating a computation with a dynamical causal order, that is either classically or quantumly controlled, using computations with a fixed causal order. Here, this cost is measured via a notion of *quantum query complexity*: the minimum amount of queries to the input channels that a quantum circuit with fixed order $\mathcal{C}$ requires in order to simulate a given superchannel $\mathcal{S}$. We will be interested in different forms of simulation, that are formalised in what follows.

Definition 1 (Simulation of a superchannel). *A universal, deterministic and exact simulation of a given superchannel $\mathcal{S}$ is a quantum circuit with fixed order $\mathcal{C}$ that acts on finite numbers $n_1, \ldots, n_N \in \mathbb{N}$ of calls to the respective quantum channels $\mathcal{A}_1 : \mathcal{L}(\mathcal{H}^{A_1^I}) \to \mathcal{L}(\mathcal{H}^{A_1^O}), \ldots, \mathcal{A}_N : \mathcal{L}(\mathcal{H}^{A_N^I}) \to \mathcal{L}(\mathcal{H}^{A_N^O})$, in such a way that the channel $\mathcal{C}(\mathcal{A}_1^{\otimes n_1}, \ldots, \mathcal{A}_N^{\otimes n_N}) : \mathcal{L}(\mathcal{H}^P) \to \mathcal{L}(\mathcal{H}^F)$ resulting from this transformation satisfies*

$$\mathcal{C}(\mathcal{A}_1^{\otimes n_1}, \ldots, \mathcal{A}_N^{\otimes n_N}) = \mathcal{S}(\mathcal{A}_1, \ldots, \mathcal{A}_N), \tag{5}$$

for all possible input channels $\mathcal{A}_1, \ldots, \mathcal{A}_N$. When the action of the superchannel $\mathcal{S}$ on only a subset of possible input channels can be simulated (e.g. only on unitary channels), the simulation is called non-universal.

Furthermore, we consider the following two additional forms of simulation, each of which is a "harder" simulation requirement than that of Definition 1: simulation of the action of a superchannel on extended quantum channels and catalytic simulation [22, 23]. The existence of a simulation of a given superchannel that satisfies either of the requirements immediately implies that a simulation of the form of Definition 1 also exists.

Definition 2 (Simulation of a superchannel on extended channels). *A universal, deterministic and exact simulation of a given superchannel $\mathcal{S}$ on extended quantum channels is a quantum circuit with fixed order $\mathcal{C}'$ that acts on finite numbers $n_1, \ldots, n_N \in \mathbb{N}$ of calls to the respective extended quantum channels $\mathcal{A}'_1 : \mathcal{L}(\mathcal{H}^{A_1^I A_1^{I'}}) \to \mathcal{L}(\mathcal{H}^{A_1^O A_1^{O'}}), \ldots, \mathcal{A}'_N : \mathcal{L}(\mathcal{H}^{A_N^I A_N^{I'}}) \to \mathcal{L}(\mathcal{H}^{A_N^O A_N^{O'}})$, in such a way that the channel $\mathcal{C}'(\mathcal{A}_1'^{\otimes n_1}, \ldots, \mathcal{A}_N'^{\otimes n_N}) : \mathcal{L}(\mathcal{H}^P) \to \mathcal{L}(\mathcal{H}^F)$ resulting from this transformation satisfies*

$$\mathcal{C}'(\mathcal{A}_1'^{\otimes n_1}, \ldots, \mathcal{A}_N'^{\otimes n_N}) = \mathcal{S} \otimes \mathcal{I}(\mathcal{A}'_1, \ldots, \mathcal{A}'_N), \tag{6}$$

for all possible input channels $\mathcal{A}'_1, \ldots, \mathcal{A}'_N$, where $\mathcal{I}$ is the identity superchannel. When the action of the superchannel $\mathcal{S}$ on only a subset of possible input extended channels can be simulated (e.g. only on extended unitary channels), the simulation is called non-universal.

Definition 3 (Catalytic simulation of a superchannel). *A catalytic simulation of a given superchannel $\mathcal{S}$ is a quantum circuit with fixed order $\hat{\mathcal{C}}$ that acts on finite numbers $n_1, \ldots, n_N \in \mathbb{N}$ of calls to the respective quantum channels $\mathcal{A}_1 : \mathcal{L}(\mathcal{H}^{A_1^I}) \to \mathcal{L}(\mathcal{H}^{A_1^O}), \ldots, \mathcal{A}_N : \mathcal{L}(\mathcal{H}^{A_N^I}) \to \mathcal{L}(\mathcal{H}^{A_N^O})$, in such a way that the channel $\hat{\mathcal{C}}(\mathcal{A}_1^{\otimes n_1}, \ldots, \mathcal{A}_N^{\otimes n_N}) : \mathcal{L}(\mathcal{H}^P) \to \mathcal{L}(\mathcal{H}^F)$ resulting from this transformation satisfies*

$$\hat{\mathcal{C}}(\mathcal{A}_1^{\otimes n_1}, \ldots, \mathcal{A}_N^{\otimes n_N}) = \mathcal{S}(\mathcal{A}_1, \ldots, \mathcal{A}_N) \otimes \mathcal{A}_1^{\otimes (n_1-1)} \otimes \ldots \otimes \mathcal{A}_N^{\otimes (n_N-1)}, \tag{7}$$

for all possible input channels $\mathcal{A}_1, \ldots, \mathcal{A}_N$.

The transformation is deemed catalytic in the sense that without the resource of the extra queries the simulation is not possible with fixed order quantum circuits; however, when given access to the resource of the extra queries, the simulation becomes possible and, moreover, the extra required queries are not consumed in the process of the simulation.

We now consider a first example of simulation of a superchannel, namely the N-classical switch.

C. Simulation of the N-classical switch with quantum circuits with fixed order

We start by recalling the definition of the N-classical switch higher-order transformation. The N-classical switch involves N parties A_k, for $k = 1, \ldots, N$, that each applies a quantum channel $\mathcal{A}_k : \mathcal{L}(\mathcal{H}^{A_k^I}) \to \mathcal{L}(\mathcal{H}^{A_k^O})$ on a target system initialised in the state $\rho_t \in \mathcal{L}(\mathcal{H}^t)$. The order in which the parties operations are applied is controlled by a control system initialised in the state $\rho_c = \sum_{\pi=(k_1,\ldots,k_N)} q_\pi |\pi\rangle\langle\pi| \in \mathcal{L}(\mathcal{H}^c)$, with $\sum_\pi q_\pi = 1$ and each $\pi = (k_1, \ldots, k_N)$ is a permutation of $\mathcal{N} := \{1, \ldots, N\}$ corresponding to the causal order $\mathcal{A}_{k_1} \prec \mathcal{A}_{k_2} \prec \ldots \prec \mathcal{A}_{k_N}$, where $\mathcal{A} \prec B$ means "$\mathcal{A}$ is applied before $\mathcal{B}$". The N-classical switch is then formally defined as the higher-order transformation that acts on the input channels $\mathcal{A}_1, \ldots, \mathcal{A}_N$ and outputs the channel $\mathcal{CS}(\mathcal{A}_1, \ldots, \mathcal{A}_N) : \mathcal{L}(\mathcal{H}^t \otimes \mathcal{H}^c) \to \mathcal{L}(\mathcal{H}^F)$ defined as follows [6, 15]:

$$\mathcal{CS}(\mathcal{A}_1, \ldots, \mathcal{A}_N)[\rho_t \otimes \rho_c] = \sum_\pi q_\pi \mathcal{A}_{\pi(N)} \circ \mathcal{A}_{\pi(N-1)} \circ \cdots \circ \mathcal{A}_{\pi(1)}(\rho_t) \otimes |\pi\rangle\langle\pi|, \tag{8}$$

where $\mathcal{H}^F$ is a global future Hilbert space.

We note that the N-quantum switch is defined analogously but with a control system potentially in a quantum state, so that the process allows for a coherent control of all the different causal orders. Various schemes were proposed in the literature, starting with [24] and then followed by [15, 25, 26], to simulate the classical (resp. quantum) control of N quantum operations with quantum circuits. We present in Fig. 1 such a simulation of the N-classical switch that queries N^2 times the parties operations in a fixed order and uses N auxiliary registers initialised in states $|a_k\rangle \in \mathcal{H}^{\mathrm{aux}_k}$, for $k = 1, \ldots, N$.[1] At each time step, the simulation circuit calls a copy of each quantum channel $\mathcal{A}_k$ applied by

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[1] Throughout this paper all the auxiliary Hilbert spaces, such as $\mathcal{H}^{\mathrm{aux}_k}$ here, are isomorphic to the target Hilbert spaces $\mathcal{H}^{A_k^I}$ and $\mathcal{H}^{A_k^O}$.

the parties to the target system initialised in $|\psi_t\rangle$. At the beginning and at the end of each time step n are applied routing operations $\mathcal{S}_{k_n}$ that swap the target system with the auxiliary state in $\mathcal{H}^{\text{aux}_k}$, so that the operation $\mathcal{A}_{k_n}$ is effectively applied on the target system at this time step, while the $N-1$ other operations are applied in parallel on the corresponding auxiliary systems.

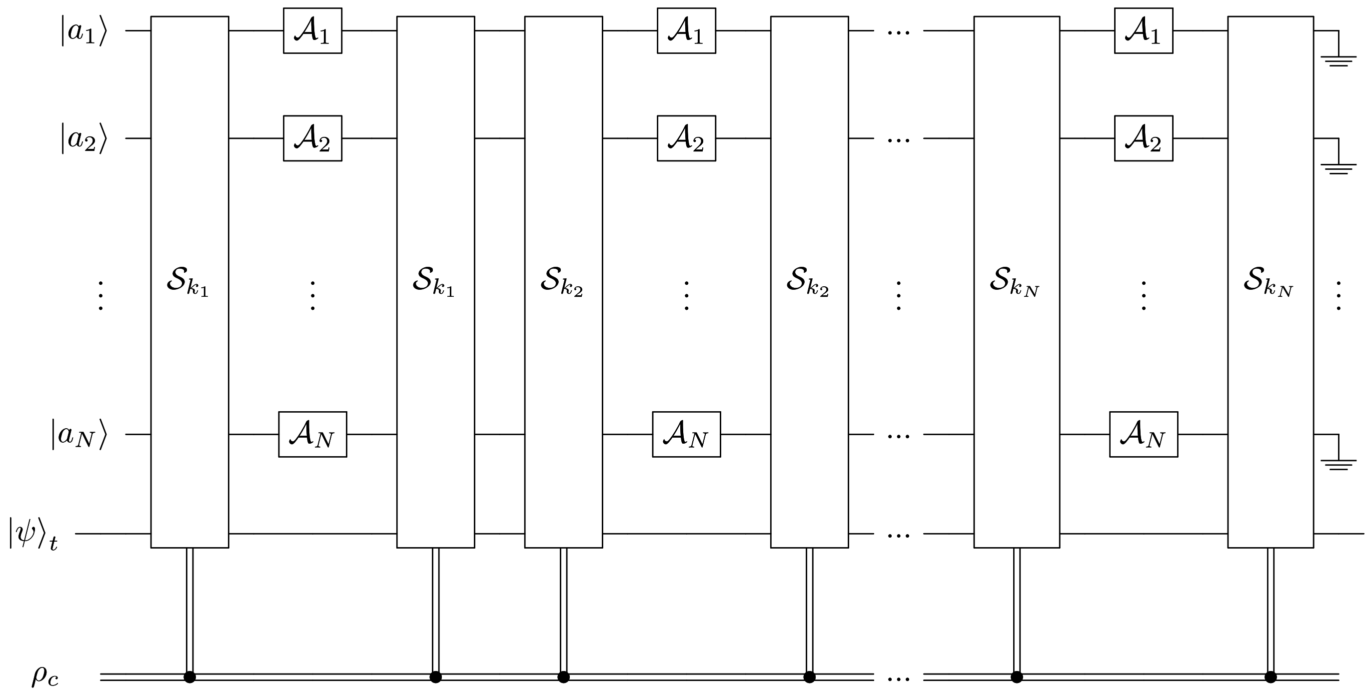

Figure 1. Fixed order simulation of a N-classical switch with general external operations $\mathcal{A}_1, \ldots, \mathcal{A}_N$. The classical control system is initialised in $\rho_c = \sum_{\pi=(k_1,\ldots,k_N)} q_\pi |\pi\rangle\langle\pi|$. N queries of each party's operation $\mathcal{A}_k$ is called, so that the simulation makes N^2 queries to the parties operations.

One can show that the circuit presented in Fig. 1 indeed simulates the action of the N-classical switch by deriving its output state and showing that it corresponds to the output state of the N-classical switch, as defined in Eq. (8), for any choice of target and control states, as well as any choice of the CPTP operations applied by the parties [15, 24–26]. We note that the quantum circuit that simulates the N-quantum switch is defined analogously to the one presented in Fig. 1, with the difference that the control system should be in a quantum state. An analogous simulation result was then derived for the N-quantum switch, but only in the case where the parties apply unitary channels. In the general case of input CPTP channels, it is known that the simulation task is way harder. Indeed, some recent results suggest that the simulation of the 2-quantum switch is exponentially hard when the parties apply general CPTPs [17, 27].

The simulation scheme presented in Fig. 1 achieves an efficient simulation of the N-classical switch as it requires N^2 copies of the parties operations. In what follows we are going to address the simulation complexity of general QC-CCs, of which the N-classical switch is a particular example, with quantum circuits when the parties perform CPTPs. Then we address the simulation complexity of general QC-QCs, of which the N-quantum switch is a particular example, with quantum circuits when the parties perform unitaries.

III. SIMULATING CLASSICAL CONTROL OF CAUSAL ORDER WITH FIXED CAUSAL ORDER

Building on the simulation scheme proposed to simulate the N-classical switch in [15, 24–26] and recalled in Fig. 1, we are going to propose one to simulate any quantum circuit with classical control (QC-CC). For a given QC-CC, the main idea is to incorporate the internal operations that define the QC-CC in the simulation scheme of the N-classical switch. Most of the time when dealing with QC-CCs it is assumed that the parties performs local operations. Yet, in full generality the parties are allowed to perform so-called extended channels which are applied on one part of bipartite states. In what follows we thus start by introducing the formalism of QC-CCs, before proposing a simulation scheme for both types of operations applied by the parties, as well as a catalytic simulation.

A. Quantum circuits with classical control (QC-CCs)

We now briefly introduce the formalism of quantum circuits with classical control (QC-CC) of causal order. As presented above, using the formalism of quantum combs one can describe the most general way to transform N quantum operations $\mathcal{A}_k : \mathcal{L}(\mathcal{H}^{A_k^I}) \to \mathcal{L}(\mathcal{H}^{A_k^O})$ into a valid quantum channel defined from a global past Hilbert space $\mathcal{H}^P$ to a global future Hilbert space $\mathcal{H}^F$, where the operations are applied in a fixed causal order. QC-CCs were introduced in [7] as a generalisation of quantum combs, in an attempt to characterise the most general transformation of N quantum operations $\mathcal{A}_k$ into a valid quantum channel, where the operations are applied in a well-defined causal order. Contrarily to quantum combs, the causal order between the operations $\mathcal{A}_k$ might not be fixed from the beginning and be dynamically established on the fly as the operations $\mathcal{A}_k$ realised by the parties and the internal operations that define the QC-CC are applied, a feature known as dynamical causal order [8–10, 28, 29].

Any QC-CC can be expressed as a generalised quantum circuit with a target system on which the operations $\mathcal{A}_k$ are applied, a potential auxiliary system in $\mathcal{H}^{\alpha_n}$ at time step n, and a classical control register that dictates the order in which the operations $\mathcal{A}_k$ are applied, see an illustration in Fig. 2. The control state is initialised in $[\![(k_1)]\!] := |k_1\rangle\langle k_1| \in \mathcal{L}(\mathcal{H}^{C_1})$ where k_1 encodes the label of the first operation $\mathcal{A}_{k_1}$ to apply, and is then updated at the beginning of each time step with the label of the next operation to apply. At time step n it thus in the state

$$[\![(k_1, \dots, k_n)]\!] := |(k_1, \dots, k_n)\rangle\langle(k_1, \dots, k_n)| \in \mathcal{L}(\mathcal{H}^{C_n}) \tag{9}$$

and controls the order of the n first operations applied to the target system to be $\mathcal{A}_{k_1} \prec \mathcal{A}_{k_2} \prec \dots \prec \mathcal{A}_{k_n}$. A QC-CC is formally defined by the collection of internal maps $\{\mathcal{M}_\emptyset^{\to k_1}, \dots, \mathcal{M}_{(k_1,\dots,k_n)}^{\to k_{n+1}}, \dots, \mathcal{M}_{(k_1,\dots,k_N)}^{\to F}\}$, where the CP maps $\mathcal{M}_\emptyset^{\to k_1} : \mathcal{L}(\mathcal{H}^P) \to \mathcal{L}(\mathcal{H}^{A_{k_1}^I \alpha_1})$ transform a state from the global past space $\mathcal{H}^P$ into a target state in the input space $\mathcal{H}^{A_{k_1}^I}$ of the first operation and an auxiliary state in the space $\mathcal{H}^{\alpha_1}$; for $n = 1, \dots, N-1$ the CP maps $\mathcal{M}_{(k_1,\dots,k_n)}^{\to k_{n+1}} : \mathcal{L}(\mathcal{H}^{A_{k_n}^O \alpha_n}) \to \mathcal{L}(\mathcal{H}^{A_{k_{n+1}}^I \alpha_{n+1}})$ transform the output target state in $\mathcal{H}^{A_{k_n}^O}$ and auxiliary state in $\mathcal{H}^{\alpha_n}$ into the input target state in $\mathcal{H}^{A_{k_{n+1}}^I}$ and the auxiliary state in $\mathcal{H}^{\alpha_{n+1}}$; the CPTP map $\mathcal{M}_{(k_1,\dots,k_N)}^{\to F} : \mathcal{L}(\mathcal{H}^{A_{k_N}^O \alpha_N}) \to \mathcal{L}(\mathcal{H}^F)$ transforms the output target state in $\mathcal{H}^{A_{k_N}^O}$ and auxiliary state in $\mathcal{H}^{\alpha_N}$ into a state in the global future space $\mathcal{H}^F$.

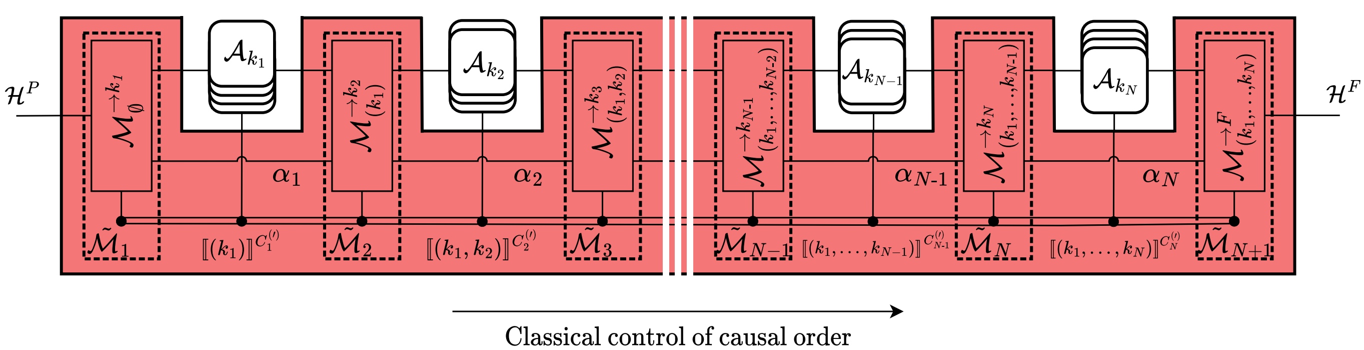

Figure 2. Graphical representation of a QC-CC, reproduced from Ref. [7]

So as to to define a controlled version of the internal and external operations, one assumes w.l.o.g. that all input (resp. output) Hilbert spaces $\mathcal{H}^{A_k^I}$ (resp. $\mathcal{H}^{A_k^O}$) are isomorphic. For each time step n one then introduces "generic" input and output Hilbert spaces $\mathcal{H}^{\tilde{A}_n^I}$ and $\mathcal{H}^{\tilde{A}_n^O}$, isomorphic to all $\mathcal{H}^{A_k^I}$'s and $\mathcal{H}^{A_k^O}$'s respectively. This allows one to write all the external operations that may be applied at time step n as acting in the same generic input and output spaces, and to embed these into some "larger" controlled operation $\tilde{\mathcal{A}}_n$, which applies the desired operation $\tilde{\mathcal{A}}_{k_n}$ conditioning on the state of the control state. In the Choi picture, this larger controlled operation $\tilde{\mathcal{A}}_n$ can be described as:

$$\tilde{A}_n = \sum_{(k_1,\dots,k_n)} \tilde{A}_{k_n} \otimes [\![(k_1, \dots, k_n)]\!]^{C_n} \otimes [\![(k_1, \dots, k_n)]\!]^{C'_n} \in \mathcal{L}(\mathcal{H}^{\tilde{A}_n^I C_n \tilde{A}_n^O C'_n}). \tag{10}$$

Analogously, one can also embed the internal operations in some "larger" controlled operations $\tilde{\mathcal{M}}_{n+1}$ that apply the desired operation $\tilde{\mathcal{M}}_{(k_1,\dots,k_n)}^{\to k_{n+1}}$ depending on the state of the control state. These larger controlled operations are described in the Choi picture as:

$$\tilde{M}_1 = \sum_{k_1} \tilde{M}_{\emptyset}^{\to k_1} \otimes [\![(k_1)]\!]^{C_1} \ \in \mathcal{L}(\mathcal{H}^{P\tilde{A}_1^I\alpha_1 C_1}), \tag{11}$$

$$\tilde{M}_{n+1} = \sum_{(k_1,\ldots,k_n,k_{n+1})} \tilde{M}_{(k_1,\ldots,k_n)}^{\to k_{n+1}} \otimes [\![(k_1,\ldots,k_n)]\!]^{C'_n} \otimes [\![(k_1,\ldots,k_n,k_{n+1})]\!]^{C_{n+1}} \ \in \mathcal{L}(\mathcal{H}^{\tilde{A}_n^O\alpha_n C'_n\tilde{A}_{n+1}^I\alpha_{n+1}C_{n+1}}) \tag{12}$$

$$\tilde{M}_{N+1} = \sum_{(k_1,\ldots,k_N)} \tilde{M}_{(k_1,\ldots,k_N)}^{\to F} \otimes [\![(k_1,\ldots,k_N)]\!]^{C'_N} \ \in \mathcal{L}(\mathcal{H}^{\tilde{A}_N^O\alpha_N C'_N F}). \tag{13}$$

Given the description of the internal operations, one can then evaluate the output state of the QC-CC. For a given input state $\rho_P \in \mathcal{L}(\mathcal{H}^P)$ in the global past of the circuit and N CPTP maps $\mathcal{A}_1,\ldots,\mathcal{A}_N$ realised by the parties, it is expressed by the link product of the successive controlled operations [7]:

$$\begin{aligned}\mathcal{QC}\!-\!\mathcal{CC}(\mathcal{A}_1,\ldots,\mathcal{A}_N)[\rho_P] &= \rho_P * \tilde{M}_1 * \tilde{A}_1 * \tilde{M}_2 * \tilde{A}_2 * \cdots * \tilde{M}_N * \tilde{A}_N * \tilde{M}_{N+1} \\ &= \sum_{(k_1,\ldots,k_N)} \rho_P * M_{\emptyset}^{\to k_1} * A_{k_1} * M_{(k_1)}^{\to k_2} * A_{k_2} * \cdots * M_{(k_1,\ldots,k_{N-1})}^{\to k_N} * A_{k_N} * M_{(k_1,\ldots,k_N)}^{\to F}.\end{aligned} \tag{14}$$

In what follows we show how to simulate the action of a QC-CC with a quantum circuit with fixed causal order and extra copies of the CPTP maps $\mathcal{A}_k$.

B. Local channels performed by the parties

We start with the case where the N parties perform local quantum channels $\mathcal{A}_k : \mathcal{L}(\mathcal{H}^{A_k^I}) \to \mathcal{L}(\mathcal{H}^{A_k^O})$. The first simulation scheme is an adaptation of the scheme proposed in [15, 24–26] to the QC-CC case. We propose a simulation scheme that incorporates the QC-CC architecture, i.e., the internal CP maps $\mathcal{M}_{(k_1,\ldots,k_n)}^{\to k_{n+1}}$ whose order of application is classically controlled, into the simulation scheme of [15, 24–26], see Fig. 3. On top of the N copies of each external operation $\mathcal{A}_k$, the simulation scheme uses N auxiliary registers initialised in state $|a_k\rangle \in \mathcal{H}^{\text{aux}_k}$. We will denote the global auxiliary space $\mathcal{H}^{\text{aux}} := \bigotimes_{k=1}^N \mathcal{H}^{\text{aux}_k}$. At each time step n, controlled on the last label k_n of the control state (see Eq. (9)), a routing operation $\mathcal{S}_{k_n}^I : \mathcal{L}(\mathcal{H}^{\text{aux}A_{k_n}^I}) \to \mathcal{L}(\mathcal{H}^{\text{aux}A_{k_n}^I})$ swaps the incoming target system in $\mathcal{H}^{A_{k_n}^I}$ coming from $\mathcal{M}_{(k_1,\ldots,k_{n-1})}^{\to k_n}$ with the auxiliary system in $\mathcal{H}^{\text{aux}_{k_n}}$, and transmits with an identity channel all the other systems. Then a copy of each quantum channel $\mathcal{A}_k$ is applied on the corresponding auxiliary state in $\mathcal{H}^{\text{aux}_k}$. Finally the routing is inverted by $\mathcal{S}_{k_n}^O : \mathcal{L}(\mathcal{H}^{\text{aux}A_{k_n}^I}) \to \mathcal{L}(\mathcal{H}^{\text{aux}A_{k_n}^O})$ that swaps the auxiliary state in $\mathcal{H}^{\text{aux}_{k_n}}$ with the outgoing target state in $\mathcal{H}^{A_{k_n}^I}$, the latter being eventually output in $\mathcal{H}^{A_{k_n}^O}$ and sent to $\mathcal{M}_{(k_1,\ldots,k_n)}^{\to k_{n+1}}$.

The overall controlled routing operations applied at time step n can thus be described in the Choi picture as

$$S_n^I = \sum_{(k_1,\ldots,k_n)} S_{k_n}^I \otimes [\![(k_1,\ldots,k_n)]\!]^{C_n} \otimes [\![(k_1,\ldots,k_n)]\!]^{C'_n} \tag{15}$$

and

$$S_n^O = \sum_{(k_1,\ldots,k_n)} S_{k_n}^O \otimes [\![(k_1,\ldots,k_n)]\!]^{C'_n} \otimes [\![(k_1,\ldots,k_n)]\!]^{C''_n}, \tag{16}$$

while instead of the operations $\tilde{A}_n$ applied in a QC-CC (cf. Eq. (10)), here are applied in between the internal operations $\tilde{\mathcal{M}}_n$ the operations $\mathcal{R}_n$, described in the Choi picture as

$$R_n = S_n^I * (A_1 \otimes \ldots \otimes A_N \otimes |\mathbb{1}\rangle\!\rangle\langle\!\langle\mathbb{1}|^{\tilde{A}_n^I}) * S_n^O. \tag{17}$$

For any QC-CC, we can then define its simulation by composing alternatively its internal operations $\tilde{\mathcal{M}}_n$ with the operations $\mathcal{R}_n$. For any quantum channels $\mathcal{A}_1,\ldots,\mathcal{A}_N$, the output of the simulation is the map $\mathcal{C}_{\mathcal{QC}-\mathcal{CC}}(\mathcal{A}_1^{\otimes N},\ldots,\mathcal{A}_N^{\otimes N})$, which is defined for any given state $\rho_P \in \mathcal{L}(\mathcal{H}^P)$ as

$$\mathcal{C}_{\mathcal{QC}-\mathcal{CC}}(\mathcal{A}_1^{\otimes N},\ldots,\mathcal{A}_N^{\otimes N})[\rho_P] = \mathrm{Tr}_{\text{aux}}[(\rho_P \otimes \rho_{\text{aux}}) * \tilde{M}_1 * R_1 * \tilde{M}_2 * R_2 * \cdots * \tilde{M}_N * R_N * \tilde{M}_{N+1}], \tag{18}$$

where $\rho_{\text{aux}} = \bigotimes_{i=1}^N |a_i\rangle\langle a_i| \in \mathcal{L}(\mathcal{H}^{\text{aux}})$ is the initial auxiliary state.

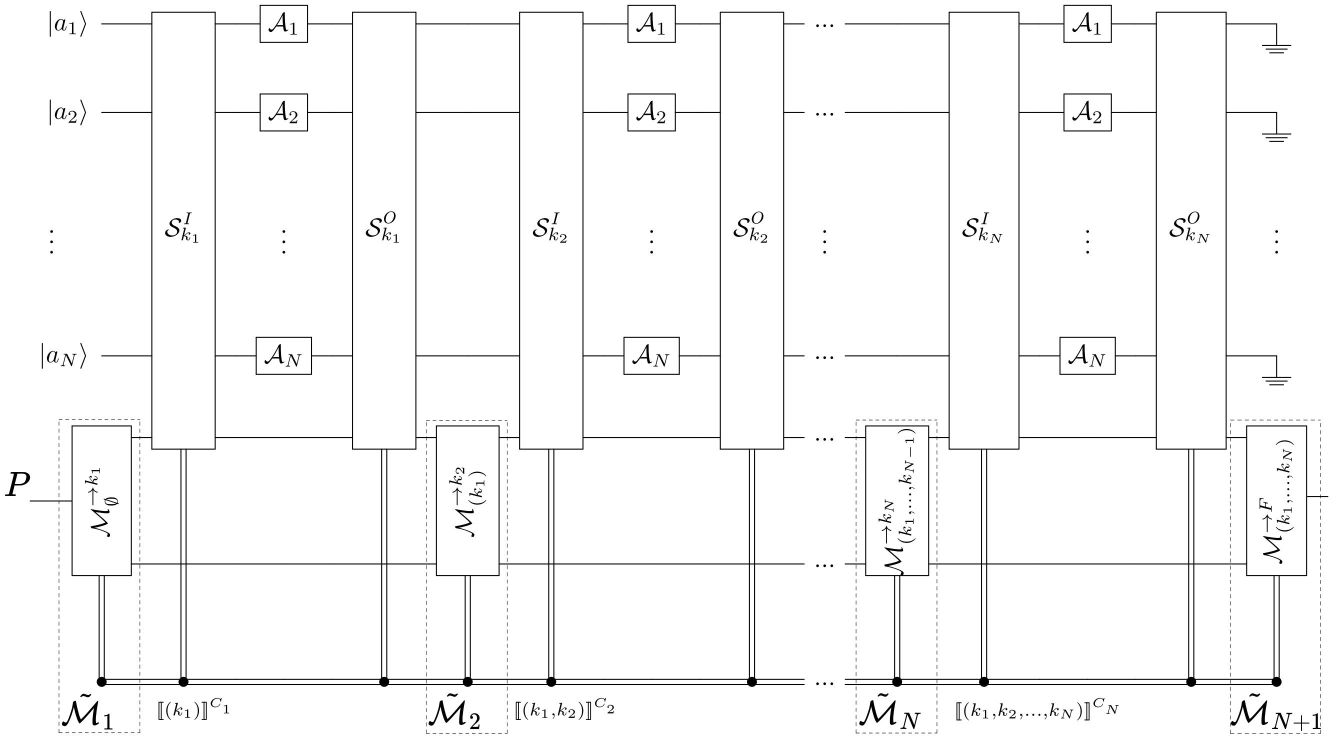

Figure 3. Fixed order simulation of a QC-CC with input quantum channels $\mathcal{A}_1,\ldots,\mathcal{A}_N$ as represented in Fig. 2. N queries of each external operation $\mathcal{A}_k$ is called, so that the simulation makes N^2 queries to the external operations.

Theorem 1. *The action of any QC-CC on N local quantum channels $\{\mathcal{A}_k : \mathcal{L}(\mathcal{H}^{A_k^I}) \to \mathcal{L}(\mathcal{H}^{A_k^O})\}_{k=1}^N$, as described by $\mathcal{QC}\!-\!\mathcal{CC}(\mathcal{A}_1,\ldots,\mathcal{A}_N)$ in Eq. (14), can be simulated by a quantum circuit that has access to N calls of each quantum channel $\mathcal{A}_k$, i.e., N^2 copies of the channels in total, as described by $\mathcal{C}_{\mathcal{QC}-\mathcal{CC}}(\mathcal{A}_1^{\otimes N},\ldots,\mathcal{A}_N^{\otimes N})$ in Eq. (18).*

The proof consists in computing iteratively the output state of the simulation scheme presented in Fig. 3 and showing that it matches the output state of the QC-CC as defined in Eq. (14), see the details in Appendix 1.

C. Extended channels performed by the parties

In full generality, a QC-CC is defined for parties that are allowed to perform extended maps $\mathcal{A}'_k : \mathcal{L}(\mathcal{H}^{A_k^I A_k^{I'}}) \to \mathcal{L}(\mathcal{H}^{A_k^O A_k^{O'}})$, where the primed spaces correspond to auxiliary systems that do not enter the QC-CC. Plugging the operations $\mathcal{A}'_k$ in the QC-CC thus defines a valid map from $\mathcal{L}(\mathcal{H}^{PA'^I_N})$ to $\mathcal{L}(\mathcal{H}^{FA'^O_N})$. In that case the output state of the QC-CC can be written as follows, for $\rho_P \in \mathcal{L}(\mathcal{H}^P)$, $\rho_{A_k} \in \mathcal{L}(\mathcal{H}^{A_k^{I'}})$ and $\mathcal{A}'_k : \mathcal{L}(\mathcal{H}^{A_k^I A_k^{I'}}) \to \mathcal{L}(\mathcal{H}^{A_k^O A_k^{O'}})$,

$$\begin{aligned}\mathcal{QC}\!-\!\mathcal{CC}\otimes\mathcal{I}(\mathcal{A}'_1,\ldots,\mathcal{A}'_N)[\rho_P,\rho_{A_1},\ldots,\rho_{A_N}] &= (\rho_P\otimes\rho_{A_1}\otimes\ldots\otimes\rho_{A_N}) * \tilde{M}_1 * \tilde{A}'_1 * \tilde{M}_2 * \tilde{A}'_2 * \cdots * \tilde{M}_N * \tilde{A}'_N * \tilde{M}_{N+1}\\ &= \sum_{(k_1,\ldots,k_N)} (\rho_P\otimes\rho_{A_1}\otimes\ldots\otimes\rho_{A_N}) * M_\emptyset^{\to k_1} * A'_{k_1} * M_{(k_1)}^{\to k_2} * A'_{k_2} * \cdots * M^{\to k_N}_{(k_1,\ldots,k_{N-1})} * A'_{k_N} * M^{\to F}_{(k_1,\ldots,k_N)}.\end{aligned} \tag{19}$$

We highlight that it is in principle possible that a fixed order simulation as described in Sec. III B, that reproduces the output state of a QC-CC for any choice of initial state and local operations, does not reproduce the output state of the QC-CC when extended maps are applied. We thus update the simulation scheme of Fig. 3 and provide a complete simulation scheme of any QC-CC when extended maps are applied, see an illustration in Fig. 4.

Since in the QC-CCs each party now acts on the target system and on a primed auxiliary system, one considers for each party A_k two auxiliary systems initialised in the states $|a_k\rangle \in \mathcal{H}^{\mathrm{aux}_k}$ and $|a'_k\rangle \in \mathcal{H}^{\mathrm{aux}'_k}$ respectively. We will here denote the global auxiliary space as $\mathcal{H}^{\mathrm{aux}} := \bigotimes_{k=1}^N \mathcal{H}^{\mathrm{aux}_k \mathrm{aux}'_k}$. For each time step n, we define new routing operations $\mathcal{S}'^I_{k_n} : \mathcal{L}(\mathcal{H}^{\mathrm{aux}A^I_{k_n}A^{I'}_{k_n}}) \to \mathcal{L}(\mathcal{H}^{\mathrm{aux}A^I_{k_n}A^{I'}_{k_n}})$ and $\mathcal{S}'^O_{k_n} : \mathcal{L}(\mathcal{H}^{\mathrm{aux}A^I_{k_n}A^{I'}_{k_n}}) \to \mathcal{L}(\mathcal{H}^{\mathrm{aux}A^O_{k_n}A^{O'}_{k_n}})$. The operation $\mathcal{S}'^I_{k_n}$ performs a double swap, one between the target state and the auxiliary state in $\mathcal{H}^{\text{aux}_{k_n}}$, and one between the auxiliary states in $\mathcal{H}^{A^{I'}_{k_n}}$ and $\mathcal{H}^{\text{aux}'_{k_n}}$. The other states are transmitted with an identity channel. Then a copy of each quantum channel $\mathcal{A}'_k$ is applied on the corresponding auxiliary spaces $\mathcal{H}^{\text{aux}_k} \otimes \mathcal{H}^{\text{aux}'_k}$. Finally the routing is inverted by $\mathcal{S}'^{O}_{k_n}$ that performs a double swap, one between the auxiliary state in $\mathcal{H}^{\text{aux}_{k_n}}$ and the outgoing target state in $\mathcal{H}^{A^I_{k_n}}$, and one between the auxiliary states in $\mathcal{H}^{A^{O'}_{k_n}}$ and $\mathcal{H}^{\text{aux}'_{k_n}}$. The overall controlled routing operations $\mathcal{S}'^{I}_n$ and $\mathcal{S}'^{O}_n$ applied at each time step n can thus be described in the Choi picture as

$$S'^{I}_n = \sum_{(k_1,\dots,k_n)} S'^{I}_{k_n} \otimes [\![(k_1,\dots,k_n)]\!]^{C_n} \otimes [\![(k_1,\dots,k_n)]\!]^{C'_n}, \tag{20}$$

and

$$S'^{O}_n = \sum_{(k_1,\dots,k_n)} S'^{O}_{k_n} \otimes [\![(k_1,\dots,k_n)]\!]^{C'_n} \otimes [\![(k_1,\dots,k_n)]\!]^{C''_n}. \tag{21}$$

Instead of the operations $\mathcal{R}_n$ applied in the previous QC-CC simulation scheme (cf. Eq. (17)), here is applied the operations $\mathcal{R}'_n$, described in the Choi picture as

$$R'_n = S'^{I}_n * \left[(|\mathbb{1}\rangle\!\rangle\langle\!\langle\mathbb{1}|^{A^{I'}_1} \otimes A'_1) \otimes \dots \otimes (|\mathbb{1}\rangle\!\rangle\langle\!\langle\mathbb{1}|^{A^{I'}_N} \otimes A'_N) \otimes |\mathbb{1}\rangle\!\rangle\langle\!\langle\mathbb{1}|^{\tilde{A}^{I'}_n} \right] * S'^{O}_n. \tag{22}$$

For any QC-CC, we can then define its simulation by composing alternatively its internal operations $\tilde{\mathcal{M}}_n$ with the operations $\mathcal{R}'_n$. For any extended quantum channels $\mathcal{A}'_1 : \mathcal{L}(\mathcal{H}^{A^I_1 A^{I'}_1}) \to \mathcal{L}(\mathcal{H}^{A^O_1 A^{O'}_1}), \dots, \mathcal{A}'_N : \mathcal{L}(\mathcal{H}^{A^I_N A^{I'}_N}) \to \mathcal{L}(\mathcal{H}^{A^O_N A^{O'}_N})$, the output of the simulation is the map $\mathcal{C}'_{\mathcal{QC-CC}}(\mathcal{A}'^{\otimes N}_1, \dots, \mathcal{A}'^{\otimes N}_N)$, which is defined for any given states $\rho_P \in \mathcal{L}(\mathcal{H}^P)$, $\rho_{A_1} \in \mathcal{L}(\mathcal{H}^{A^{I'}_1}), \dots, \rho_{A_N} \in \mathcal{L}(\mathcal{H}^{A^{I'}_N})$, as

$$\mathcal{C}'_{\mathcal{QC-CC}}(\mathcal{A}'^{\otimes N}_1, \dots, \mathcal{A}'^{\otimes N}_N)[\rho_P, \rho_{A_1}, \dots, \rho_{A_N}] = \mathrm{Tr}_{\text{aux}}[(\rho_P \otimes \rho_{A_1} \otimes \dots \otimes \rho_{A_N} \otimes \rho_{\text{aux}}) * \tilde{M}_1 * R'_1 * \dots * \tilde{M}_N * R'_N * \tilde{M}_{N+1}], \tag{23}$$

where $\rho_{\text{aux}} = \bigotimes_{i=1}^N |a_i\rangle\langle a_i| \otimes |a'_i\rangle\langle a'_i| \in \mathcal{L}(\mathcal{H}^{\text{aux}})$ is the initial auxiliary state.

Theorem 2. *The action of any QC-CC on N extended quantum channels $\{\mathcal{A}'_k : \mathcal{L}(\mathcal{H}^{A^I_k A^{I'}_k}) \to \mathcal{L}(\mathcal{H}^{A^O_k A^{O'}_k})\}_{k=1}^N$, as described by $\mathcal{QC-CC} \otimes \mathcal{I}(\mathcal{A}'_1, \dots, \mathcal{A}'_N)$ in Eq. (19), can be simulated by a quantum circuit that has access to N calls of each quantum channel $\mathcal{A}'_k$, i.e., N^2 copies of the channels in total, as described by $\mathcal{C}'_{\mathcal{QC-CC}}(\mathcal{A}'^{\otimes N}_1, \dots, \mathcal{A}'^{\otimes N}_N)$ in Eq. (23).*

The proof consists in computing iteratively the output state of the simulation scheme presented in Fig. 4 and showing that it matches the output state of the QC-CC as defined in Eq. (19), see the details in Appendix 2.

D. Catalytic simulation

We finally show that a catalytic simulation, as defined in Definition 3, exists for any QC-CC. For the sake of simplicity we will exhibit such a simulation scheme in the case where the parties apply local quantum channels $\mathcal{A}_k : \mathcal{L}(\mathcal{H}^{A^I_k}) \to \mathcal{L}(\mathcal{H}^{A^O_k})$, but we note that the case of extended quantum channels can be treated by straightforwardly combining the simulation scheme presented in this subsection with the one presented just before.

Focusing on the simulation result of QC-CC with local quantum channels presented in Sec. III B, we observe that the output state of the auxiliary systems are of the form $\mathcal{A}_k^{\circ(N-1)}(|a_k\rangle\langle a_k|)$ so that the channel $\mathcal{A}_k$ is composed $N-1$ times on the initial auxiliary state. To make the simulation catalytic in the sense of Definition 3, we have to distribute these applications on $N-1$ individual auxiliary systems so that we can have a tensor product factorisation of the $N-1$ calls to $\mathcal{A}_k$. We thus introduce $N-1$ auxiliary systems for each party, so $N(N-1)$ auxiliary systems in total, each attached to the Hilbert space $\mathcal{H}^{\text{aux}^i_k}$ for $k = 1, \dots, N$, and $i = 1, \dots, N-1$. We will here denote the global auxiliary space as $\mathcal{H}^{\text{aux}} := \bigotimes_{k=1}^N \bigotimes_{i=1}^{N-1} \mathcal{H}^{\text{aux}^i_k}$.

We present in Fig. 5 a catalytic simulation of any given QC-CC. At each time step n, controlled on the last label k_n of the control state, a routing operation $\hat{\mathcal{S}}^I_{k_n} : \mathcal{L}(\mathcal{H}^{\text{aux}A^I_{k_n}}) \to \mathcal{L}(\mathcal{H}^{\text{aux}A^I_{k_n}})$ swaps the incoming target system in $\mathcal{H}^{A^I_{k_n}}$ coming from $\mathcal{M}^{\to k_n}_{(k_1,\dots,k_{n-1})}$ with the auxiliary system in $\mathcal{H}^{\text{aux}^1_{k_n}}$, and transmits with an identity channel all the other

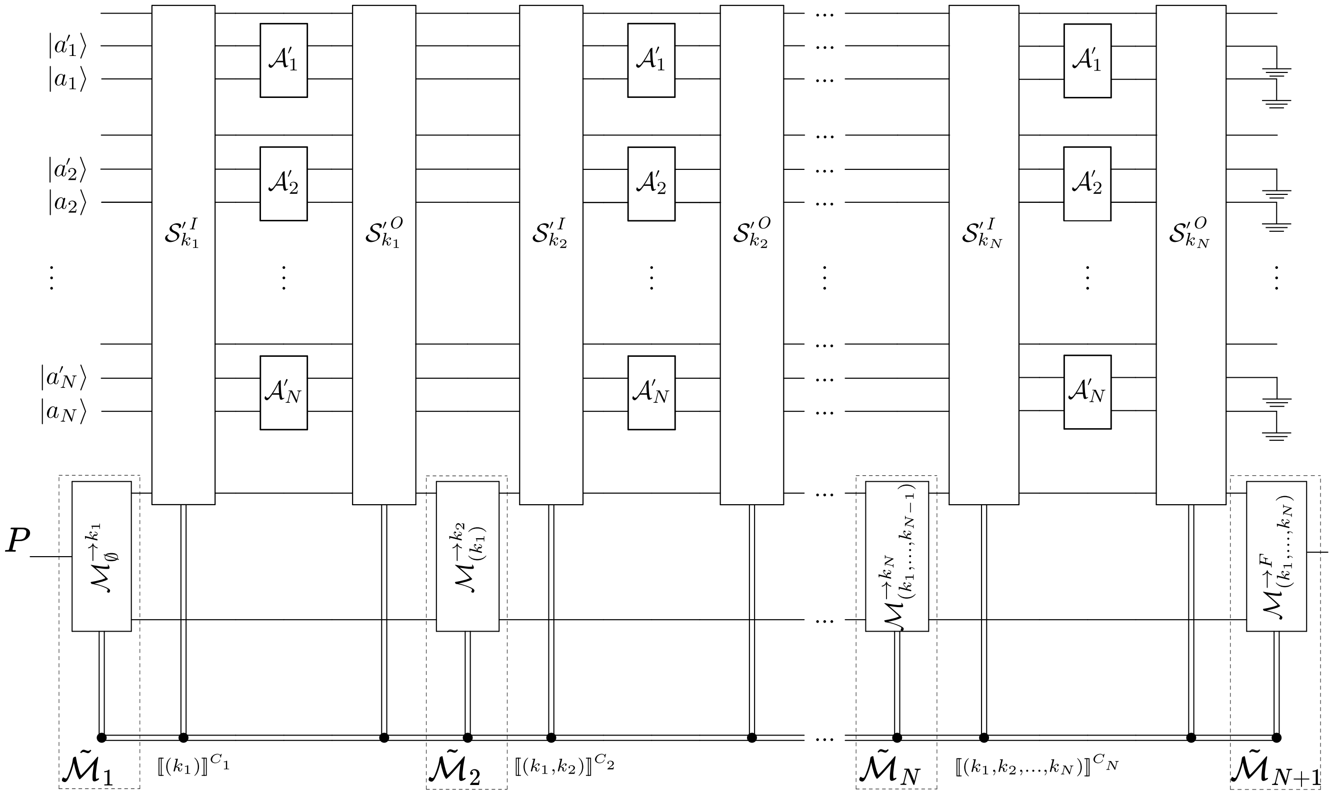

Figure 4. Fixed order simulation of a QC-CC with input extended quantum channels $\mathcal{A}'_1,\ldots,\mathcal{A}'_N$.

auxiliary systems in the subspace $\mathcal{H}^{\text{aux}_{k_n}} := \bigotimes_{i=1}^{N-1}\mathcal{H}^{\text{aux}^i_{k_n}}$. On all the other subspaces $\mathcal{H}^{\text{aux}_{k'_n}}$ for $k'_n \neq k_n$, one of the auxiliary system that has not yet received the action of the corresponding $\mathcal{A}_k$ is swapped with the system in $\mathcal{H}^{\text{aux}^1_{k_n}}$, and all the other subspaces are transmitted with an identity channel. This makes sure that we apply exactly once the operation $\mathcal{A}_k$ on each subspace $\mathcal{H}^{\text{aux}^i_k}$, which will eventually allow to obtain a catalytic simulation. Then a copy of each quantum channel $\mathcal{A}_k$ is applied on the corresponding auxiliary state in $\mathcal{H}^{\text{aux}^1_k}$. Finally the routing is inverted by $\hat{\mathcal{S}}^O_{k_n} : \mathcal{L}(\mathcal{H}^{\text{aux}A^I_{k_n}}) \to \mathcal{L}(\mathcal{H}^{\text{aux}A^O_{k_n}})$, which swaps the auxiliary state in $\mathcal{H}^{\text{aux}^1_{k_n}}$ with the outgoing target state in $\mathcal{H}^{A^I_{k_n}}$, the latter being eventually output in $\mathcal{H}^{A^O_{k_n}}$ and sent to $\mathcal{M}^{\to k_{n+1}}_{(k_1,\ldots,k_n)}$, and which transmits with an identity channel all the other auxiliary systems.

The overall controlled routing operations $\hat{\mathcal{S}}^I_n$ and $\hat{\mathcal{S}}^O_n$ applied at time step n can thus be described analogously to Eq. (15) and Eq. (16) respectively, with the operations $\hat{\mathcal{S}}^I_{k_n}$ and $\hat{\mathcal{S}}^O_{k_n}$ just defined. Instead of the operations $\mathcal{R}_n$ defined in Eq. (17), here is applied the operator $\hat{\mathcal{R}}_n$, described in the Choi picture as

$$\hat{R}_n = \hat{S}^I_n * \Big[(A_1 \otimes |\mathbb{1}\rangle\!\rangle\langle\!\langle\mathbb{1}|^{\text{aux}_1^2} \otimes\ldots\otimes |\mathbb{1}\rangle\!\rangle\langle\!\langle\mathbb{1}|^{\text{aux}_1^{N-1}}) \otimes\ldots\otimes (A_N \otimes |\mathbb{1}\rangle\!\rangle\langle\!\langle\mathbb{1}|^{\text{aux}_N^1} \otimes\ldots\otimes |\mathbb{1}\rangle\!\rangle\langle\!\langle\mathbb{1}|^{\text{aux}_N^{N-1}}) \otimes |\mathbb{1}\rangle\!\rangle\langle\!\langle\mathbb{1}|^{\tilde{A}^I_n}\Big] * \hat{S}^O_n. \tag{24}$$

For any QC-CC, we can then define its catalytic simulation by composing alternatively its internal operations $\tilde{\mathcal{M}}_n$ with the operations $\hat{\mathcal{R}}_n$. For any quantum channels $\mathcal{A}_1 : \mathcal{L}(\mathcal{H}^{A^I_1}) \to \mathcal{L}(\mathcal{H}^{A^O_1}),\ldots,\mathcal{A}_N : \mathcal{L}(\mathcal{H}^{A^I_N}) \to \mathcal{L}(\mathcal{H}^{A^O_N})$, the output of the simulation is the map $\hat{\mathcal{C}}_{\mathcal{QC-CC}}(\mathcal{A}_1^{\otimes N},\ldots,\mathcal{A}_N^{\otimes N})$, which is defined for any states $\rho_P \in \mathcal{L}(\mathcal{H}^P)$, $\rho^i_k \in \mathcal{L}(\mathcal{H}^{\text{aux}^i_k})$ for $k = 1,\ldots,N$ and $i = 1,\ldots,N-1$, as

$$\hat{\mathcal{C}}_{\mathcal{QC-CC}}(\mathcal{A}_1^{\otimes N},\ldots,\mathcal{A}_N^{\otimes N})[\rho_P,\rho^1_1,\ldots,\rho^i_k,\ldots,\rho^{N-1}_N] = (\rho_P \bigotimes_{k=1}^{N}\bigotimes_{i=1}^{N-1}\rho^i_k) * \tilde{M}_1 * \hat{R}_1 * \tilde{M}_2 * \hat{R}_2 * \cdots * \tilde{M}_N * \hat{R}_N * \tilde{M}_{N+1}. \tag{25}$$

Theorem 3. *The action of any QC-CC on N local quantum channels $\{\mathcal{A}_k : \mathcal{L}(\mathcal{H}^{A^I_k}) \to \mathcal{L}(\mathcal{H}^{A^O_k})\}_{k=1}^N$, as described by $\mathcal{QC-CC}(\mathcal{A}_1,\ldots,\mathcal{A}_N)$ in Eq. (14), can be catalytically simulated by a quantum circuit that has access to N calls of each*

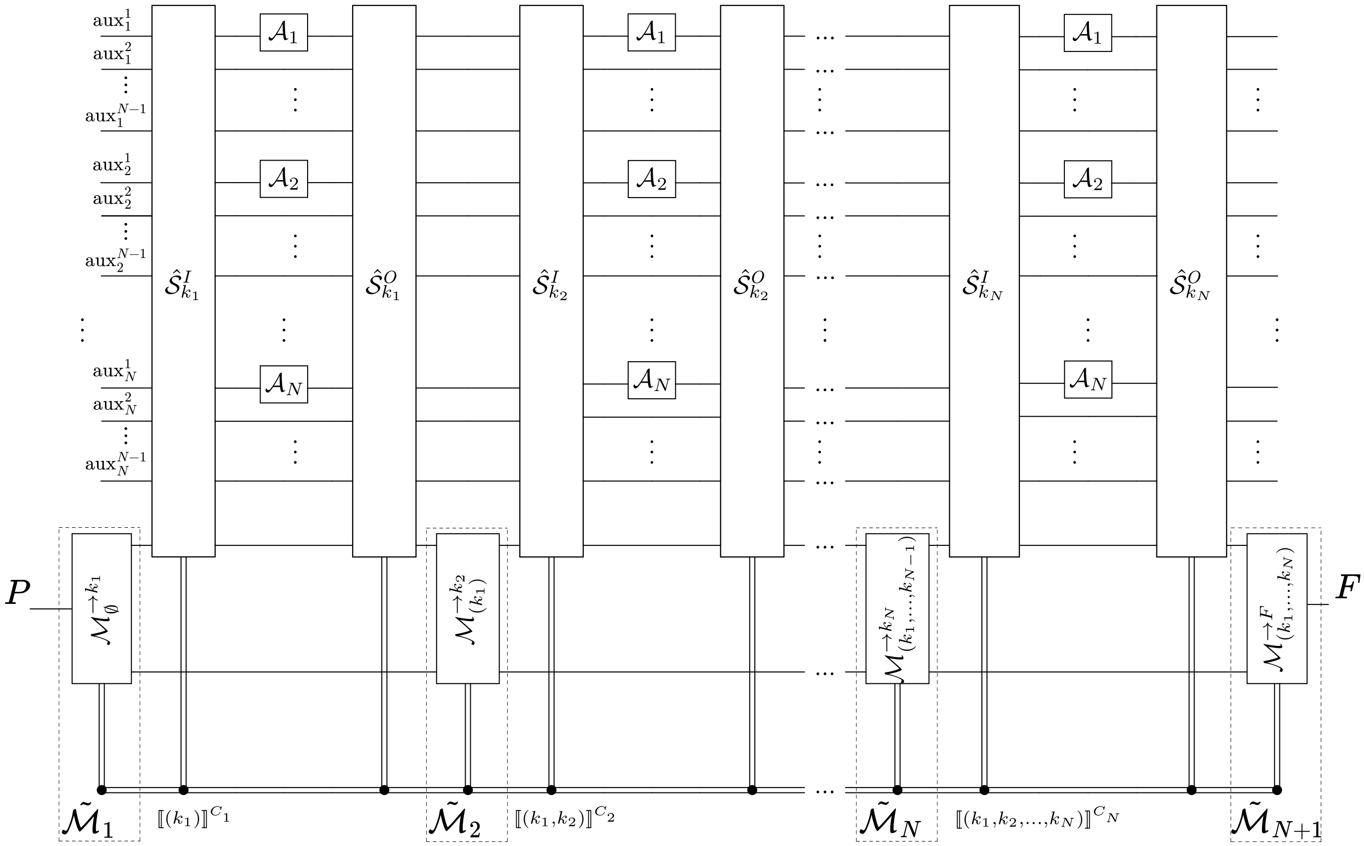

Figure 5. Catalytic fixed order simulation of a QC-CC with input quantum channels $\mathcal{A}_1, \dots, \mathcal{A}_N$.

quantum channel $\mathcal{A}_k$*, i.e.,* N^2 *copies of the channels in total, as described by* $\hat{\mathcal{C}}_{\mathcal{QC}-\mathcal{CC}}(\mathcal{A}_1^{\otimes N}, \dots, \mathcal{A}_N^{\otimes N})$ *in Eq.* (25)*. This amounts to*

$$\hat{\mathcal{C}}_{\mathcal{QC}-\mathcal{CC}}(\mathcal{A}_1^{\otimes N}, \dots, \mathcal{A}_N^{\otimes N}) = \mathcal{QC}-\mathcal{CC}(\mathcal{A}_1, \dots, \mathcal{A}_N) \otimes \mathcal{A}_1^{\otimes(N-1)} \otimes \dots \otimes \mathcal{A}_N^{\otimes(N-1)},$$

for all input quantum channels $\mathcal{A}_1, \dots, \mathcal{A}_N$*.*

Proof. The proof can be obtained in the same way than the proofs of the last two theorems. It consists in computing iteratively the output state of the simulation scheme presented in Fig. 5 and showing its factorises in the tensor product of two states in $\mathcal{H}^F \otimes \mathcal{H}^{\mathrm{aux}}$, where the state in $\mathcal{H}^F$ matches the output state of the QC-CC as defined in Eq. (14), and the state in $\mathcal{H}^{\mathrm{aux}}$ is of the form $\bigotimes_{k=1}^N \bigotimes_{i=1}^{N-1} \mathcal{A}_k(|a_k^i\rangle\langle a_k^i|)$, which indeed shows the catalytic property of the simulation as defined in Definition 3. □

IV. SIMULATING QUANTUM CONTROL OF CAUSAL ORDER WITH FIXED CAUSAL ORDER

We now turn our attention to the simulation of quantum circuits with quantum control (QC-QCs) of causal order, a framework introduced in [7] which capture processes with dynamical causal order, as was also the case for QC-CCs, but also with indefinite causal order like the quantum switch. It was recently shown in [17, 27] that simulating the quantum switch when the parties are performing general quantum channels is computationally hard, if not impossible. This implies that providing a universal simulation of a QC-QC is in general also a hard task. We shall thus focus on non-universal simulation of QC-QCs where parties perform unitary channels, for which efficient simulation results of the N-quantum switch were already obtained [15, 25]. In what follows we generalise this simulation scheme to define one for every QC-QC.

A. Quantum circuits with quantum control (QC-QCs)

We now briefly introduce the formalism of quantum circuits with quantum control (QC-QCs), which generalise QC-CCs by allowing the control of the causal order to be quantum. QC-QCs were introduced in [7] as a broad class of higher-order quantum transformations that act on N input quantum operations and include, in particular, processes with indefinite causal order. A handy way to describe QC-QCs is to work with "pure" target, auxiliary and control quantum states, as well as internal and external operations, see an illustration of a generic QC-QC in Fig. 6. Without loss of generality, we will consider that the internal and external operations have just one Kraus operator, the general case being recovered by summing over potentially multiple Kraus operators. More precisely, external operations realised by the parties will now be described by linear operators $A_k : \mathcal{H}^{A_k^I} \to \mathcal{H}^{A_k^O}$ (that we will later restrict to be unitary channels), and internal operations will be described by $\tilde{V}_1 : \mathcal{H}^P \to \mathcal{H}^{\tilde{A}_1^I \alpha_1 C_1}$, $\tilde{V}_{n+1} : \mathcal{H}^{\tilde{A}_n^O \alpha_n C'_n} \to \mathcal{H}^{\tilde{A}_{n+1}^I \alpha_{n+1} C_{n+1}}$ (for $n = 1, \dots, N-1$), $\tilde{V}_{N+1} : \mathcal{H}^{\tilde{A}_N^O \alpha_N C'_N} \to \mathcal{H}^{F \alpha_F}$, where $\mathcal{H}^{\alpha_F}$ is an additional Hilbert space compared to QC-CCs which allow to purify these operators and write them as one Kraus operator, and which can be traced out to allow for convex mixtures. One other difference with QC-CCs is the structure of the control system of QC-QCs: at time step n, it no longer registers the whole order $(k_1, \dots, k_n)$ of the operations already applied as well as the next one to apply, but rather it keeps track of the unordered set $\mathcal{K}_{n-1} := \{k_1, \dots, k_{n-1}\} \subset \mathcal{N}$ of the $n-1$ operations already applied, as well as the label k_n of the operation applied at time step n. The generic basis states of the control system at time step n thus now takes the form

$$|\mathcal{K}_{n-1}, k_n\rangle^{C_n^{(\prime)}}. \tag{26}$$

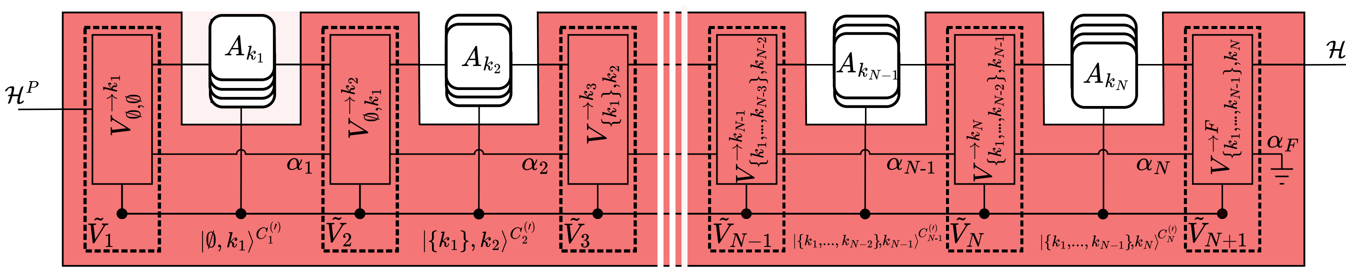

Figure 6. Graphical representation of a QC-QC, reproduced from Ref. [7].

We now describe formally the internal and external controlled operations. They are defined analogously to the QC-CC internal operations (see Eq. (10)–(13)), taking into account the new structure of the control state which involves in particular a coherent control. In the Choi picture, the controlled external operation are described by

$$|\tilde{A}_n\rangle\!\rangle := \sum_{\mathcal{K}_{n-1}, k_n} |\tilde{A}_{k_n}\rangle\!\rangle \otimes |\mathcal{K}_{n-1}, k_n\rangle^{C'_n} \otimes |\mathcal{K}_{n-1}, k_n\rangle^{C_n} \in \mathcal{H}^{\tilde{A}_n^I C_n \tilde{A}_n^O C'_n}, \tag{27}$$

while the internal operations are described by

$$|\tilde{V}_1\rangle\!\rangle := \sum_{k_1} |\tilde{V}_{\emptyset,\emptyset}^{\to k_1}\rangle\!\rangle \otimes |\emptyset, k_1\rangle^{C_1} \in \mathcal{H}^{P \tilde{A}_1^I \alpha_1 C_1}, \tag{28}$$

$$|\tilde{V}_{n+1}\rangle\!\rangle := \sum_{\substack{\mathcal{K}_{n-1}, \\ k_n, k_{n+1}}} |\tilde{V}_{\mathcal{K}_{n-1}, k_n}^{\to k_{n+1}}\rangle\!\rangle \otimes |\mathcal{K}_{n-1} \cup k_n, k_{n+1}\rangle^{C_{n+1}} \otimes |\mathcal{K}_{n-1}, k_n\rangle^{C'_n} \in \mathcal{H}^{\tilde{A}_n^O \alpha_n C'_n \tilde{A}_{n+1}^I \alpha_{n+1} C_{n+1}}, \tag{29}$$

$$|\tilde{V}_{N+1}\rangle\!\rangle := \sum_{k_N} |\tilde{V}_{\mathcal{N} \backslash k_N, k_N}^{\to F}\rangle\!\rangle \otimes |\mathcal{N} \backslash k_N, k_N\rangle^{C'_N} \in \mathcal{H}^{\tilde{A}_N^O \alpha_N C'_N F \alpha_F}. \tag{30}$$

Given the description of the internal operations, one can then evaluate the output state of the QC-QC before tracing out α_F, that we denote $|\psi_{F\alpha_F}\rangle \in \mathcal{H}^{F\alpha_F}$. For a given input state $|\psi_P\rangle \in \mathcal{H}^P$ in the global past of the circuit and N CP maps $A_1, \dots, A_N$ realised by the parties, it is expressed by the link product of the successive controlled operations [7]:

$$\begin{aligned} |\psi_{F\alpha_F}\rangle &= |\psi_P\rangle * |\tilde{V}_1\rangle\!\rangle * |\tilde{A}_1\rangle\!\rangle * |\tilde{V}_2\rangle\!\rangle * \cdots * |\tilde{V}_N\rangle\!\rangle * |\tilde{A}_N\rangle\!\rangle * |\tilde{V}_{N+1}\rangle\!\rangle \\ &= \sum_{(k_1,\dots,k_N)} |\psi_P\rangle * |\tilde{V}_{\emptyset,\emptyset}^{\to k_1}\rangle\!\rangle * |\tilde{A}_{k_1}\rangle\!\rangle * |\tilde{V}_{\emptyset,k_1}^{\to k_2}\rangle\!\rangle * \cdots * |\tilde{V}_{\{k_1,\dots,k_{N-2}\},k_{N-1}}^{\to k_N}\rangle\!\rangle * |\tilde{A}_{k_N}\rangle\!\rangle * |\tilde{V}_{\{k_1,\dots,k_{N-1}\},k_N}^{\to F}\rangle\!\rangle \end{aligned} \tag{31}$$

The action of the QC-QC on a given input state $|\psi_P\rangle \in \mathcal{H}^P$ and N CP maps $A_1, \ldots, A_N$ can then be expressed as

$$\mathcal{QC}-\mathcal{QC}(A_1,\ldots,A_N)[|\psi_P\rangle\!\langle\psi_P|] = \mathrm{Tr}_{\alpha_F}[|\psi_{F\alpha_F}\rangle\!\langle\psi_{F\alpha_F}|]\,. \tag{32}$$

In what follows we show how to simulate the action of a QC-QC when the external operations are unitaries, with a quantum circuit with fixed causal order and extra copies of the external operations.

B. Local unitaries performed by the parties

We start with the case where the N parties perform local unitary channels $A_k : \mathcal{H}^{A_k^I} \to \mathcal{H}^{A_k^O}$. The first simulation scheme is an adaptation of the scheme proposed in Fig. 3 to the QC-QC case. We propose a simulation scheme that incorporates the QC-QC architecture, i.e., the internal operations $V_{\mathcal{K}_{n-1},k_n}^{\to k_{n+1}}$ whose order of application is coherently controlled, into the simulation scheme of [15, 24–26], see Fig. 7. On top of the N copies of each external operation A_k, the simulation scheme uses N auxiliary registers initialised in state $|a_k\rangle \in \mathcal{H}^{\mathrm{aux}_k}$. We will denote the global auxiliary space $\mathcal{H}^{\mathrm{aux}} := \bigotimes_{k=1}^N \mathcal{H}^{\mathrm{aux}_k}$. At each time step n, controlled on the last label k_n of the control state (see Eq. (26)), a routing operation $S_{k_n}^I : \mathcal{H}^{\mathrm{aux}A_{k_n}^I} \to \mathcal{H}^{\mathrm{aux}A_{k_n}^I}$ swaps the incoming target system in $\mathcal{H}^{A_{k_n}^I}$ coming from $V_{\mathcal{K}_{n-2},k_{n-1}}^{\to k_n}$ with the auxiliary system in $\mathcal{H}^{\mathrm{aux}_{k_n}}$, and transmits with an identity channel all the other systems. Then a copy of each unitary channel A_k is applied on the corresponding auxiliary state in $\mathcal{H}^{\mathrm{aux}_k}$. Finally the routing is inverted by $S_{k_n}^O : \mathcal{H}^{\mathrm{aux}A_{k_n}^I} \to \mathcal{H}^{\mathrm{aux}A_{k_n}^O}$ that swaps the auxiliary state in $\mathcal{H}^{\mathrm{aux}_{k_n}}$ with the outgoing target state in $\mathcal{H}^{A_{k_n}^I}$, the latter being eventually output in $\mathcal{H}^{A_{k_n}^O}$ and sent to $V_{\mathcal{K}_{n-1},k_n}^{\to k_{n+1}}$. The overall controlled routing operations applied at time step n can thus be described in the Choi picture as

$$|S_n^I\rangle\!\rangle = \sum_{(k_1,\ldots,k_n)} |S_{k_n}^I\rangle\!\rangle \otimes |\mathcal{K}_{n-1},k_n\rangle^{C_n} \otimes |\mathcal{K}_{n-1},k_n\rangle^{C_n'}, \tag{33}$$

and

$$|S_n^O\rangle\!\rangle = \sum_{(k_1,\ldots,k_n)} |S_{k_n}^O\rangle\!\rangle \otimes |\mathcal{K}_{n-1},k_n\rangle^{C_n'} \otimes |\mathcal{K}_{n-1},k_n\rangle^{C_n''}, \tag{34}$$

while instead of the operations $\mathcal{R}_n$ applied in the QC-CC simulation (cf. Eq. (17)), here are applied in between the internal operations $\tilde{\mathcal{M}}_n$ the operations Q_n, described in the Choi picture as

$$|Q_n\rangle\!\rangle = |S_n^I\rangle\!\rangle * (|A_1\rangle\!\rangle \otimes \ldots \otimes |A_N\rangle\!\rangle \otimes |\mathbb{1}\rangle\!\rangle^{\tilde{A}_n^I}) * |S_n^O\rangle\!\rangle. \tag{35}$$

For any QC-QC, we can compose alternatively its internal operations $\tilde{V}_n$ with the operations Q_n. For given auxiliary states $|a_k\rangle \in \mathcal{H}^{\mathrm{aux}_k}$, for $k = 1, \ldots, N$, and any state $|\psi_P\rangle \in \mathcal{H}^P$, we define the output state before tracing α_F as

$$|\psi_{F\alpha_F}^{\mathrm{sim}}\rangle = (|\psi_P\rangle \otimes |a_1\rangle \otimes \ldots \otimes |a_N\rangle) * |\tilde{V}_1\rangle\!\rangle * |Q_1\rangle\!\rangle * \cdots * |\tilde{V}_N\rangle\!\rangle * |Q_N\rangle\!\rangle * |\tilde{V}_{N+1}\rangle\!\rangle. \tag{36}$$

For any unitary channels $A_1, \ldots, A_N$, the output of the simulation is the map $\mathcal{C}_{\mathcal{QC}-\mathcal{QC}}(A_1^{\otimes N}, \ldots, A_N^{\otimes N})$, which is defined for any state $|\psi_P\rangle \in \mathcal{H}^P$ as

$$\mathcal{C}_{\mathcal{QC}-\mathcal{QC}}(A_1^{\otimes N},\ldots,A_N^{\otimes N})[|\psi_P\rangle\!\langle\psi_P|] = \mathrm{Tr}_{\alpha_F\mathrm{aux}}[|\psi_{F\alpha_F}^{\mathrm{sim}}\rangle\!\langle\psi_{F\alpha_F}^{\mathrm{sim}}|]. \tag{37}$$

Theorem 4. *The action of any QC-QC on N local unitary channels $\{A_k : \mathcal{H}^{A_k^I} \to \mathcal{H}^{A_k^O}\}_{k=1}^N$, as described by $\mathcal{QC}-\mathcal{QC}(A_1,\ldots,A_N)$ in Eq. (32), can be simulated by a quantum circuit that has access to N calls of each unitary A_k, i.e., N^2 copies of the unitary channels in total, as described by $\mathcal{C}_{\mathcal{QC}-\mathcal{QC}}(A_1^{\otimes N},\ldots,A_N^{\otimes N})$ in Eq. (37).*

The proof consists in computing iteratively the output state of the simulation scheme presented in Fig. 7 and showing that it matches the output state of the QC-QC as in defined Eq. (32), see the details in Appendix 3.

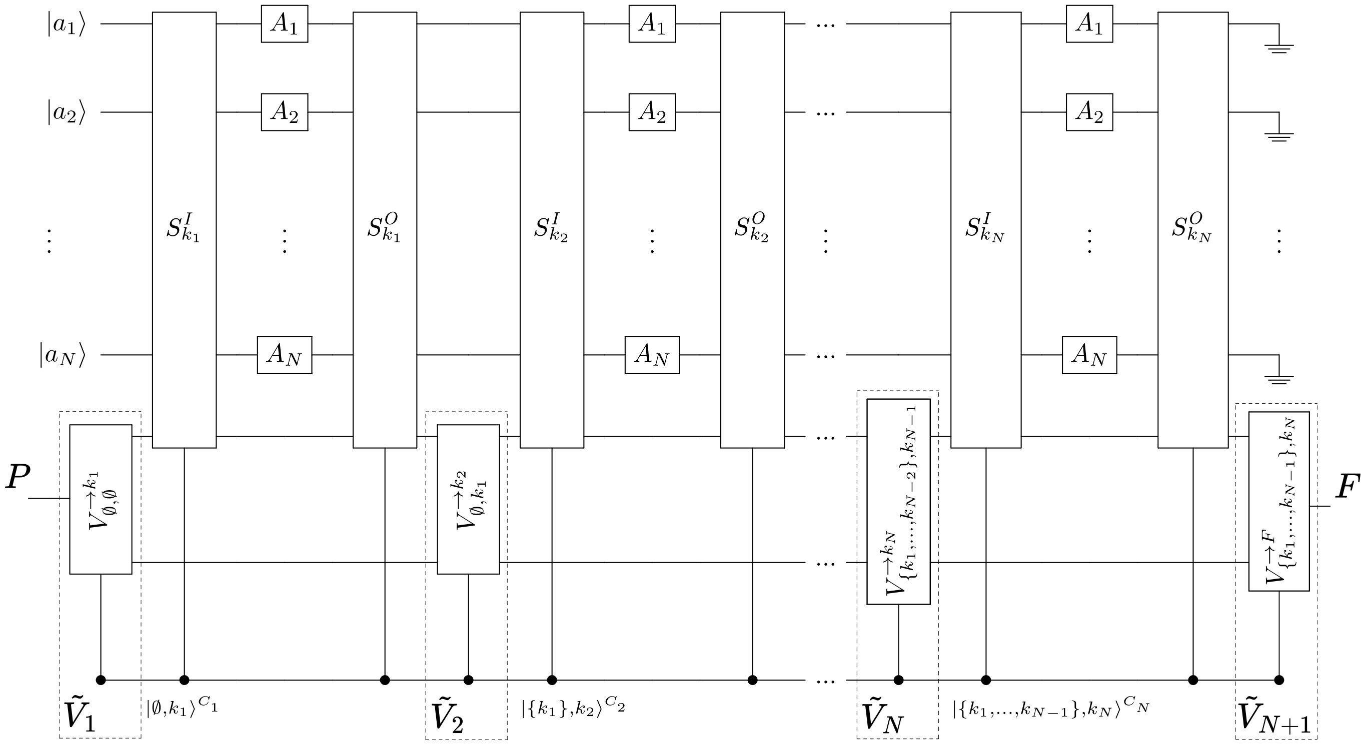

Figure 7. Fixed order simulation of a QC-QC with input unitary channels $A_1, \ldots, A_N$.

C. Extended unitaries performed by the parties

In full generality, a QC-QC is defined for parties that are allowed to perform extended maps $A'_k : \mathcal{H}^{A^I_k A^{I'}_k} \to \mathcal{H}^{A^O_k A^{O'}_k}$, where the primed spaces correspond to auxiliary systems that do not enter the QC-QC. Plugging the operations A'_k in the QC-QC thus defines a valid map from $\mathcal{H}^{PA'^I_{\mathcal{N}}}$ to $\mathcal{H}^{FA'^O_{\mathcal{N}}}$. In that case the output state of the QC-QC before tracing out α_F can be written as follows, for $|\psi_P\rangle \in \mathcal{H}^P$, $|\psi_{A_k}\rangle \in \mathcal{H}^{A^{I'}_k}$,

$$\begin{aligned}|\psi_{F\alpha_F}\rangle &= (|\psi_P\rangle \otimes |\psi_{A_1}\rangle \otimes \ldots \otimes |\psi_{A_N}\rangle) * |\tilde{V}_1\rangle\!\rangle * |\tilde{A}'_1\rangle\!\rangle * |\tilde{V}_2\rangle\!\rangle * \cdots * |\tilde{V}_N\rangle\!\rangle * |\tilde{A}'_N\rangle\!\rangle * |\tilde{V}_{N+1}\rangle\!\rangle \\ &= \sum_{(k_1,\ldots,k_N)} (|\psi_P\rangle \otimes |\psi_{A_1}\rangle \otimes \ldots \otimes |\psi_{A_N}\rangle) * |\tilde{V}^{\to k_1}_{\emptyset,\emptyset}\rangle\!\rangle * |\tilde{A}'_{k_1}\rangle\!\rangle * |\tilde{V}^{\to k_2}_{\emptyset,k_1}\rangle\!\rangle * \cdots * |\tilde{V}^{\to k_N}_{\{k_1,\ldots,k_{N-2}\},k_{N-1}}\rangle\!\rangle * |\tilde{A}'_{k_N}\rangle\!\rangle * |\tilde{V}^{\to F}_{\{k_1,\ldots,k_{N-1}\},k_N}\rangle\!\rangle.\end{aligned} \tag{38}$$

The action of the QC-QC on a given N extended maps $A'_1, \ldots, A'_N$ can then be expressed for $|\psi_P\rangle \in \mathcal{H}^P$, $|\psi_{A_k}\rangle \in \mathcal{H}^{A^{I'}_k}$ as

$$\mathcal{QC}-\mathcal{QC} \otimes \mathcal{I}(A'_1, \ldots, A'_N)[|\psi_P\rangle\langle\psi_P|, |\psi_{A_1}\rangle\langle\psi_{A_1}|, \ldots, |\psi_{A_N}\rangle\langle\psi_{A_N}|] = \mathrm{Tr}_{\alpha_F}[|\psi_{F\alpha_F}\rangle\langle\psi_{F\alpha_F}|]\,, \tag{39}$$

where $|\psi_{F\alpha_F}\rangle$ is defined just above.

We now update the simulation scheme of Fig. 7 and provide a complete simulation scheme of any QC-QC when extended unitary maps are applied, see an illustration in Fig. 8. Since in the QC-QCs each party now acts on the target system and on a primed auxiliary system, one considers for each party A_k two auxiliary systems initialised in the states $|a_k\rangle \in \mathcal{H}^{\mathrm{aux}_k}$ and $|a'_k\rangle \in \mathcal{H}^{\mathrm{aux}'_k}$ respectively. We will here denote the global auxiliary space as $\mathcal{H}^{\mathrm{aux}} := \bigotimes_{k=1}^N \mathcal{H}^{\mathrm{aux}_k \mathrm{aux}'_k}$. For each time step n, we define new routing operations $S'^I_{k_n} : \mathcal{H}^{\mathrm{aux} A^I_{k_n} A^{I'}_{k_n}} \to \mathcal{H}^{\mathrm{aux} A^I_{k_n} A^{I'}_{k_n}}$ and $S'^O_{k_n} : \mathcal{H}^{\mathrm{aux} A^I_{k_n} A^{I'}_{k_n}} \to \mathcal{H}^{\mathrm{aux} A^O_{k_n} A^{O'}_{k_n}}$. The operation $\mathcal{S}'^I_{k_n}$ performs a double swap, one between the target state and the auxiliary state in $\mathcal{H}^{\mathrm{aux}_{k_n}}$, and one between the auxiliary states in $\mathcal{H}^{A^{I'}_{k_n}}$ and $\mathcal{H}^{\mathrm{aux}'_{k_n}}$. The other states are transmitted with an identity channel. Then a copy of each quantum channel $\mathcal{A}'_k$ is applied on the corresponding auxiliary spaces $\mathcal{H}^{\mathrm{aux}_k} \otimes \mathcal{H}^{\mathrm{aux}'_k}$. Finally the routing is inverted by $\mathcal{S}'^O_{k_n}$ that performs a double swap, one between the auxiliary state in $\mathcal{H}^{\mathrm{aux}_{k_n}}$ and the outgoing target state in $\mathcal{H}^{A^I_{k_n}}$, and one between the auxiliary states in $\mathcal{H}^{A^{O'}_{k_n}}$ and $\mathcal{H}^{\mathrm{aux}'_{k_n}}$. The overall controlled routing operations applied at time step n can thus be described in the Choi picture as

$$|S'^I_n\rangle\rangle = \sum_{(k_1,\dots,k_n)} |S'^I_{k_n}\rangle\rangle \otimes |\mathcal{K}_{n-1}, k_n\rangle^{C_n} \otimes |\mathcal{K}_{n-1}, k_n\rangle^{C'_n}, \tag{40}$$

and

$$|S'^O_n\rangle\rangle = \sum_{(k_1,\dots,k_n)} |S'^O_{k_n}\rangle\rangle \otimes |\mathcal{K}_{n-1}, k_n\rangle^{C'_n} \otimes |\mathcal{K}_{n-1}, k_n\rangle^{C''_n}, \tag{41}$$

Instead of the operations Q_n applied in the QC-QC simulation (cf. Eq. (35)), here are applied in between the internal operations $\tilde{V}_n$ the operations Q'_n, described in the Choi picture as

$$|Q'_n\rangle\rangle = |S'^I_n\rangle\rangle * \left[(|\mathbb{1}\rangle\rangle^{A^{I'}_1} \otimes |A'_1\rangle\rangle) \otimes \ldots \otimes (|\mathbb{1}\rangle\rangle^{A^{I'}_N} \otimes |A'_N\rangle\rangle) \otimes |\mathbb{1}\rangle\rangle^{\tilde{A}^I_n} \right] * |S'^O_n\rangle\rangle. \tag{42}$$

For any QC-QC, we can compose alternatively its internal operations $\tilde{V}_n$ with the operations Q'_n. For given auxiliary states $|a^{(\prime)}_k\rangle \in \mathcal{H}^{\mathrm{aux}^{(\prime)}_k}$, for $k = 1, \dots, N$, and any states $|\psi_P\rangle \in \mathcal{H}^P$, $|\psi_{A_1}\rangle \in \mathcal{H}^{A^{I'}_1}, \dots, |\psi_{A_N}\rangle \in \mathcal{H}^{A^{I'}_N}$, we define the output state before tracing α_F as

$$|\psi'^{\mathrm{sim}}_{F\alpha_F}\rangle = (|\psi_P\rangle \bigotimes_{i=1}^N |\psi_{A_i}\rangle \bigotimes_{k=1}^N |a_k\rangle \otimes |a'_k\rangle) * |\tilde{V}_1\rangle\rangle * |Q'_1\rangle\rangle * \cdots * |\tilde{V}_N\rangle\rangle * |Q'_N\rangle\rangle * |\tilde{V}_{N+1}\rangle\rangle. \tag{43}$$

For any unitary extended channels $A'_1 : \mathcal{H}^{A^I_1 A^{I'}_1} \to \mathcal{H}^{A^O_1 A^{O'}_1}, \dots, A'_N : \mathcal{H}^{A^I_N A^{I'}_N} \to \mathcal{H}^{A^O_N A^{O'}_N}$, the output of the simulation is the map $\mathcal{C}'_{\mathcal{QC}-\mathcal{QC}}(A'^{\otimes N}_1, \dots, A'^{\otimes N}_N)$, which is defined for any states $|\psi_P\rangle \in \mathcal{H}^P$, $|\psi_{A_1}\rangle \in \mathcal{H}^{A^{I'}_1}, \dots, |\psi_{A_N}\rangle \in \mathcal{H}^{A^{I'}_N}$, as

$$\mathcal{C}'_{\mathcal{QC}-\mathcal{QC}}(A'^{\otimes N}_1, \dots, A'^{\otimes N}_N)[|\psi_P\rangle\langle\psi_P|, |\psi_{A_1}\rangle\langle\psi_{A_1}|, \dots, |\psi_{A_N}\rangle\langle\psi_{A_N}|] = \mathrm{Tr}_{\alpha_F \mathrm{aux}}[|\psi'^{\mathrm{sim}}_{F\alpha_F}\rangle\langle\psi'^{\mathrm{sim}}_{F\alpha_F}|]. \tag{44}$$

Theorem 5. *The action of any QC-QC on N extended unitary channels $\{A'_k : \mathcal{H}^{A^I_k A^{I'}_k} \to \mathcal{H}^{A^O_k A^{O'}_k}\}_{k=1}^N$, as described by $\mathcal{QC}-\mathcal{QC} \otimes \mathcal{I}(A'_1, \dots, A'_N)$ in Eq. (39), can be simulated by a quantum circuit that has access to N calls of each unitary A'_k, i.e., N^2 copies of the unitaries in total, as described by $\mathcal{C}'_{\mathcal{QC}-\mathcal{QC}}(A'^{\otimes N}_1, \dots, A'^{\otimes N}_N)$ in Eq. (44).*

The proof consists in computing iteratively the output state of the simulation scheme presented in Fig. 8 and showing that it matches the output state of the QC-QC as defined in Eq. (39), see the details in Appendix 4.

D. Catalytic simulation

We finally show that a catalytic simulation, as defined in Definition 3, exists for any QC-QC. For the sake of simplicity we will exhibit such a simulation scheme in the case where the parties apply local unitary channels $A_k : \mathcal{H}^{A^I_k} \to \mathcal{H}^{A^O_k}$, but we note that the case of extended unitary channels can be treated by straightforwardly combining the simulation scheme presented in this subsection with the one presented just before.

Analogously to the catalytic simulation of QC-CCs proposed in Sec. III D, we consider extra auxiliary systems compared to the QC-QC simulation proposed in Sec. IV B, in order to distribute over those auxiliary systems the actions of the $N-1$ calls of each input unitary A_k that are not applied to the target system. We thus introduce $N-1$ auxiliary systems for each party, so $N(N-1)$ auxiliary systems in total, each attached to the Hilbert space $\mathcal{H}^{\mathrm{aux}^i_k}$ for $k = 1, \dots, N$, and $i = 1, \dots, N-1$. We will here denote the global auxiliary space as $\mathcal{H}^{\mathrm{aux}} := \bigotimes_{k=1}^N \bigotimes_{i=1}^{N-1} \mathcal{H}^{\mathrm{aux}^i_k}$.

We present in Fig. 9 a catalytic simulation of any given QC-QC. At each time step n, controlled on the last label k_n of the control state, a routing operation $\hat{S}^I_{k_n} : \mathcal{H}^{\mathrm{aux} A^I_{k_n}} \to \mathcal{H}^{\mathrm{aux} A^I_{k_n}}$ swaps the incoming target system in $\mathcal{H}^{A^I_{k_n}}$ coming from $V^{\to k_n}_{\{k_1,\dots,k_{n-2}\},k_{n-1}}$ with the auxiliary system in $\mathcal{H}^{\mathrm{aux}^1_{k_n}}$, and transmits with an identity channel all the other auxiliary systems in the subspace $\mathcal{H}^{\mathrm{aux}_{k_n}} := \bigotimes_{i=1}^{N-1} \mathcal{H}^{\mathrm{aux}^i_{k_n}}$. On all the other subspaces $\mathcal{H}^{\mathrm{aux}_{k'_n}}$ for $k'_n \neq k_n$, one of the auxiliary system that has not yet received the action of the corresponding A_k is swapped with the system in $\mathcal{H}^{\mathrm{aux}^1_{k_n}}$, and all the other subspaces are transmitted with an identity channel. This makes sure that we apply exactly once the operation A_k on each subspace $\mathcal{H}^{\mathrm{aux}^i_k}$, which will eventually allow to obtain a catalytic simulation. Then

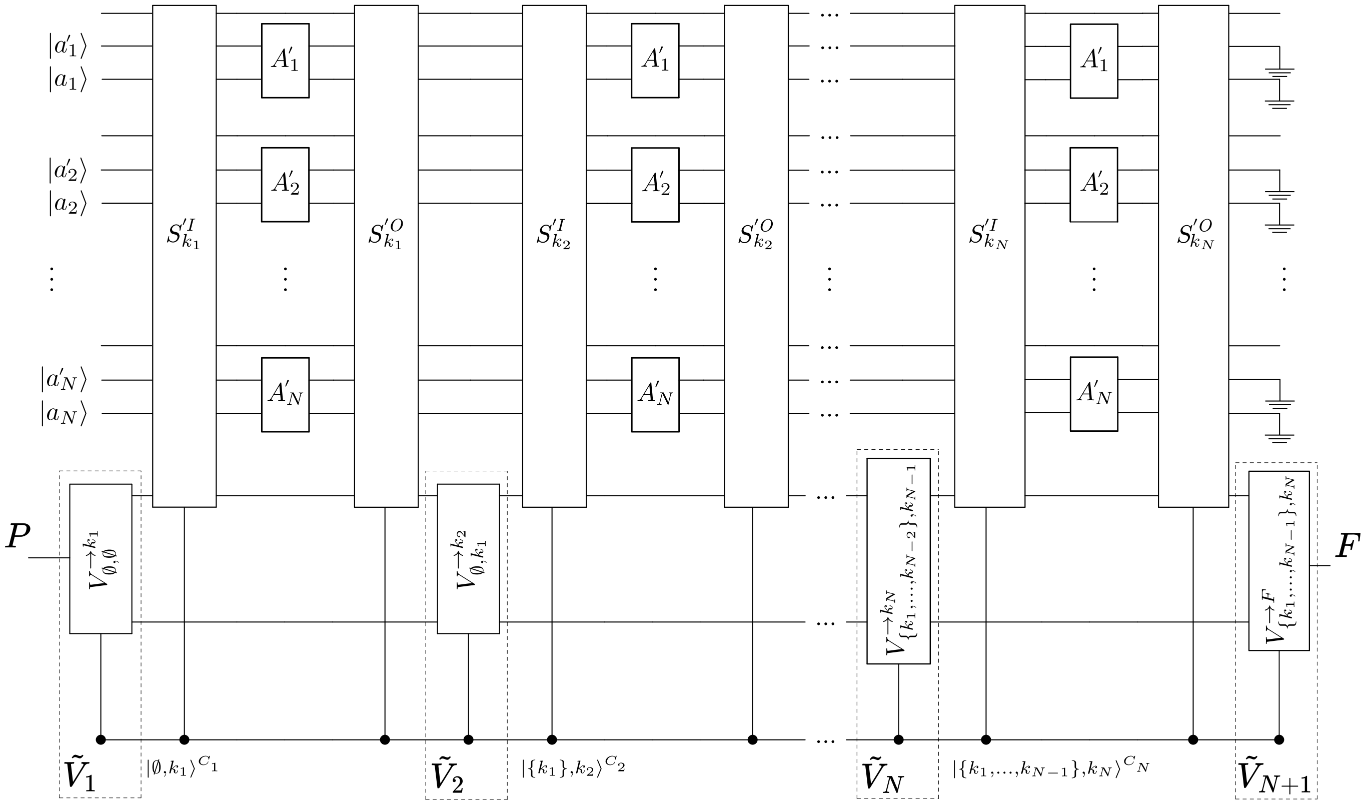

Figure 8. Fixed order simulation of a QC-QC with input extended unitary channels $A'_1, \ldots, A'_N$.

a copy of each quantum channel A_k is applied on the corresponding auxiliary state in $\mathcal{H}^{\mathrm{aux}^1_k}$. Finally the routing is inverted by $\hat{S}^O_{k_n} : \mathcal{H}^{\mathrm{aux}A^I_{k_n}} \to \mathcal{H}^{\mathrm{aux}A^O_{k_n}}$, which swaps the auxiliary state in $\mathcal{H}^{\mathrm{aux}^1_{k_n}}$ with the outgoing target state in $\mathcal{H}^{A^I_{k_n}}$, the latter being eventually output in $\mathcal{H}^{A^O_{k_n}}$ and sent to $V^{\to k_{n+1}}_{\{k_1,\ldots,k_{n-1}\},k_n}$, and which transmits with an identity channel all the other auxiliary systems.

The overall controlled routing operations $\hat{S}^I_n$ and $\hat{S}^O_n$ applied at time step n can thus be described analogously to Eq. (33) and Eq. (34) respectively, with the operations $\hat{S}^I_{k_n}$ and $\hat{S}^O_{k_n}$ just defined. Instead of the operations Q_n defined in Eq. (35), here is applied the operator $\hat{Q}_n$, described in the Choi picture as

$$|\hat{Q}_n\rangle\rangle = |\hat{S}^I_n\rangle\rangle * \Big[(|A_1\rangle\rangle \otimes |\mathbb{1}\rangle\rangle^{\mathrm{aux}^2_1} \otimes \ldots \otimes |\mathbb{1}\rangle\rangle^{\mathrm{aux}^{N-1}_1}) \otimes \ldots \otimes (|A_N\rangle\rangle \otimes |\mathbb{1}\rangle\rangle^{\mathrm{aux}^2_N} \otimes \ldots \otimes |\mathbb{1}\rangle\rangle^{\mathrm{aux}^{N-1}_N}) \otimes |\mathbb{1}\rangle\rangle^{\tilde{A}^I_n})\Big] * |\hat{S}^O_n\rangle\rangle. \tag{45}$$

For any QC-QC, we can compose alternatively its internal operations $\tilde{V}_n$ with the operations $\hat{Q}_n$. For any states $|\psi_P\rangle \in \mathcal{H}^P$, $|a^i_k\rangle \in \mathcal{H}^{\mathrm{aux}^i_k}$ for $k = 1, \ldots, N$ and $i = 1, \ldots, N-1$, we define the output state before tracing α_F as

$$|\hat{\psi}^{\mathrm{sim}}_{F\alpha_F}\rangle = (|\psi_P\rangle \bigotimes_{k=1}^{N} \bigotimes_{i=1}^{N-1} |a^i_k\rangle) * |\tilde{V}_1\rangle\rangle * |\hat{Q}_1\rangle\rangle * \cdots * |\tilde{V}_N\rangle\rangle * |\hat{Q}_N\rangle\rangle * |\tilde{V}_{N+1}\rangle\rangle. \tag{46}$$

For any unitary channels $A_1, \ldots, A_N$, the output of the simulation is the map $\hat{\mathcal{C}}_{\mathcal{QC}-\mathcal{QC}}(A_1^{\otimes N}, \ldots, A_N^{\otimes N})$, which is defined for any states $|\psi_P\rangle \in \mathcal{H}^P$, $|a^i_k\rangle \in \mathcal{H}^{\mathrm{aux}^i_k}$ for $k = 1, \ldots, N$ and $i = 1, \ldots, N-1$, as

$$\hat{\mathcal{C}}_{\mathcal{QC}-\mathcal{QC}}(A_1^{\otimes N}, \ldots, A_N^{\otimes N})[|\psi_P\rangle\langle\psi_P|, |a^1_1\rangle\langle a^1_1|, \ldots, |a^{N-1}_N\rangle\langle a^{N-1}_N|] = \mathrm{Tr}_{\alpha_F}[|\hat{\psi}^{\mathrm{sim}}_{F\alpha_F}\rangle\langle\hat{\psi}^{\mathrm{sim}}_{F\alpha_F}|]. \tag{47}$$

Theorem 6. *The action of any QC-QC on N local unitary channels $\{A_k : \mathcal{H}^{A^I_k} \to \mathcal{H}^{A^O_k}\}_{k=1}^N$, as described by $\mathcal{QC}-\mathcal{QC}(A_1, \ldots, A_N)$ in Eq. (32), can be catalytically simulated by a quantum circuit that has access to N calls of each quantum channel A_k, i.e., N^2 copies of the channels in total, as described by $\hat{\mathcal{C}}_{\mathcal{QC}-\mathcal{QC}}(A_1^{\otimes N}, \ldots,\ A_N^{\otimes N})$ in*

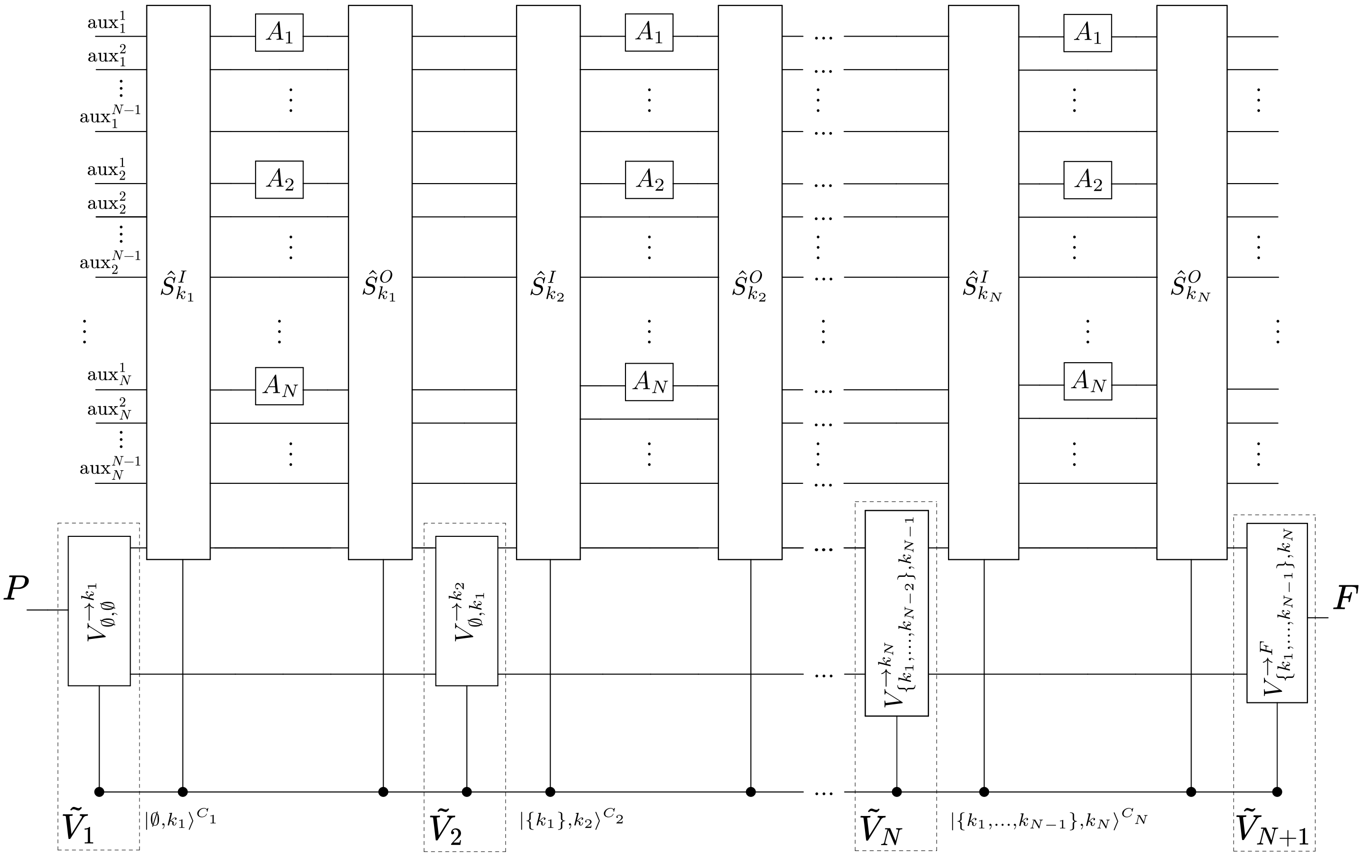

Figure 9. Catalytic fixed order simulation of a QC-QC with input unitary channels $A_1, \ldots, A_N$.

Eq. (47). *This amounts to*

$$\hat{\mathcal{C}}_{\mathcal{QC}-\mathcal{QC}}(A_1^{\otimes N}, \ldots, A_N^{\otimes N}) = \mathcal{QC}-\mathcal{QC}(A_1, \ldots, A_N) \otimes A_1^{\otimes(N-1)} \otimes \ldots \otimes A_N^{\otimes(N-1)},$$

for all input unitary channels $A_1, \ldots, A_N$.

Proof. The proof can be obtained in the same way than the proofs of the last two theorems. It consists in computing iteratively the output state of the simulation scheme presented in Fig. 9 and showing its factorises in the tensor product of two states in $\mathcal{H}^F \otimes \mathcal{H}^{\mathrm{aux}}$, where the state in $\mathcal{H}^F$ matches the output state of the QC-QC as defined in Eq. (32), and the state in $\mathcal{H}^{\mathrm{aux}}$ is of the form $\bigotimes_{k=1}^N \bigotimes_{i=1}^{N-1} A_k |a_k^i\rangle$, which indeed shows the catalytic property of the simulation as defined in Definition 3. □

V. OPTIMALITY OF THE SIMULATIONS

We have provided various simulation schemes for QC-CCs and QC-QCs, showing that they can be efficiently simulated with a quantum circuit and at most a quadratic number of calls to the operations realised by the parties. Conversely, one may wonder whether this quadratic scaling is necessary to simulate all QC-CCs or all QC-QCs.

We first note that not all QC-CCs and QC-QCs require N^2 calls to the external channels to be simulated. For instance a quantum comb being compatible with a fixed causal order, it is clear that it can be simulated with only N calls to the external channels. More generally, any QC-CC or QC-QC which does not involve a classical or quantum control of all the $N!$ possible causal orders between N parties can be simulated with strictly less than N^2 copies of the party's operations.

Considering the particular cases of the N-classical switch or the N-quantum switch, which do involve all the $N!$ possible causal orders, it was claimed in [15] that that a quadratic number of queries to the external operations is asymptotically necessary to simulate these processes. The proof is based on a connection between the simulation problem and a combinatorial problem known as the *permutation as substrings* problem [30], which asks the shortest string of $\{1, \dots, N\}$ that contains all permutations on N elements as subsequences. More precisely, it is assumed in [15] that a lower bound on the simulation problem is given by the solution to the combinatorial problem, which is known to be asymptotically N^2 [31, 32]. Here we argue that the step connecting the simulation problem to the combinatorial problem is not justified in [15]. Indeed, that the simulation scheme requires all the $N!$ possible causal orders to be available as sub-branches of the circuit seems like a reasonable guess, but no formal proof is given. The only formal proof of such a claim is provided in [25] under the additional requirement that the simulation circuit is only made of so-called rewiring gates. We leave for further work a rigorous proof of the optimality of the quadratic scaling in the asymptotic limit, for N-classical and quantum switches, as well as for general QC-CCs and QC-QCs. An interesting question is also the optimal query complexity for finite N. It is for instance known that the classical (resp. quantum) switch requires 3 queries to the input quantum channels (resp. unitary channels) to be simulated. Such a result can be straightforwardly generalised to all 2-slots QC-CCs and QC-QCs. However, understanding how the optimal QC-CC or QC-QC simulation behave for any finite N is less clear, we leave this as an open question.

VI. CONCLUSION

We showed that any QC-CC and any QC-QC acting on unitary channels can be simulated by fixed-order quantum circuits with at most a quadratic cost in quantum query complexity, hence, efficiently. We conjecture that our simulations are optimal, in the required number of queries, in the regime of large N. That is, we conjecture that our provably attainable upper bound of $\mathcal{O}(N^2)$ is asymptotically tight.

While we show that the computational power of QC-CCs is limited, at least in terms of quantum query complexity, to at most a quadratic advantage over fixed-order quantum circuits, it remains an open question to understand for which kinds of computational and information-processing tasks this quadratic advantage can be achieved, if any. At the same time, our results do not forbid greater advantages with respect to other notions of cost/performance. For example, when comparing superchannels of a fixed number of slots in a given task, QC-CCs can provide advantages over fixed-order quantum circuits in terms of measures such as probability of success or precision, one such example being reported in Ref. [17]. It would be interesting to find out for which other tasks such advantages are possible and how they scale in, e.g., the dimension of the input quantum channels. Our results imply, nevertheless, that any such advantages can always be matched by fixed-order circuits with at most a quadratic cost in number of queries. Another interesting avenue would be to investigate whether allowing convex mixtures of fixed-order circuits can reduce the simulation cost further.

It also remains an open question whether a universal simulation of QC-QCs is possible using fixed-order quantum circuits.

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APPENDICES

1. Proof of Theorem 1

Proof. Let a QC-CC. It is defined by a collection of internal operations

$$\{\mathcal{M}_{\emptyset}^{\to k_1}, \ldots, \mathcal{M}_{(k_1,\ldots,k_n)}^{\to k_{n+1}}, \ldots, \mathcal{M}_{(k_1,\ldots,k_N)}^{\to F}\}.$$

We are going to show that the simulation scheme proposed in Fig. 3 simulates the action of the QC-CC by showing that the output state of the simulation scheme, as described in Eq. (18), is exactly the output state of the QC-CC for any choice of external operation $\mathcal{A}_k$ and initial state ρ_P.

The output state of the simulation scheme can be computed iteratively, by composing alternatively the operations $\tilde{M}_n$ and $\mathcal{R}_n$. The auxiliary systems are initialised in the pure state $\rho_{\text{aux}} = \bigotimes_{i=1}^N \rho_i \in \mathcal{L}(\mathcal{H}^{\text{aux}})$, with $\rho_i = |a_i\rangle\!\langle a_i|$. Given the initial state $\rho_P \in \mathcal{L}(\mathcal{H}^P)$, the internal states $\varrho'_{(n)}$ before and $\varrho_{(n+1)}$ after each operation $\tilde{\mathcal{M}}_{n+1}$ are

$$\varrho'_{(n)} = (\rho_P \otimes \rho_1 \otimes \ldots \otimes \rho_N) * \tilde{M}_1 * R_1 * \tilde{M}_2 * \cdots * \tilde{M}_n * R_n$$

and

$$\varrho_{(n+1)} = (\rho_P \otimes \rho_1 \otimes \ldots \otimes \rho_N) * \tilde{M}_1 * R_1 * \tilde{M}_2 * \cdots * R_n * \tilde{M}_{n+1}. \tag{48}$$

The state of the simulation scheme at the first time step before the first external operations $\mathcal{A}_k$ are applied

$$\begin{aligned}\varrho_{(1)} &= (\rho_P * \tilde{M}_1) \otimes (\rho_1 \otimes \ldots \otimes \rho_N) \\ &= \left(\sum_{k_1} \rho_P * M_{()}^{\to k_1} \otimes [\![(k_1)]\!]^{C_1}\right) \otimes (\rho_1 \otimes \ldots \otimes \rho_N)\end{aligned} \tag{49}$$

Applying the first external operations in parallel along with the swap operations we obtain the state at the end of the first time step as

$$\begin{aligned}\varrho'_{(1)} &= \varrho_{(1)} * R_1 = \varrho_{(1)} * S_1^I * (A_1 \otimes \ldots \otimes A_N \otimes |\mathbb{1}\rangle\!\rangle\!\langle\!\langle\mathbb{1}|^{\tilde{A}_1^I}) * S_1^O \\ &= \sum_{k_1} \left[\rho_P * M_{()}^{\to k_1} \otimes (\rho_1 \otimes \ldots \otimes \rho_N)\right] * S_{k_1}^I \otimes [\![(k_1)]\!]^{C_1'} * (A_1 \otimes \ldots \otimes A_N \otimes \mathbb{1}) * S_1^O \\ &= \sum_{k_1} \left[\rho_P * M_{()}^{\to k_1} * A_{k_1} \otimes (\rho_1 * A_1 \otimes \ldots \otimes \rho_{k_1-1} * A_{k_1-1} \otimes \rho_{k_1} \otimes \rho_{k_1+1} * A_{k_1+1} \ldots \otimes \rho_N * A_N)\right] \otimes [\![(k_1)]\!]^{C_1'} * S_1^O \\ &= \sum_{k_1} \left[\rho_P * M_{()}^{\to k_1} * A_{k_1} \otimes (\rho_1 * A_1 \otimes \ldots \otimes \rho_{k_1-1} * A_{k_1-1} \otimes \rho_{k_1} \otimes \rho_{k_1+1} * A_{k_1+1} \ldots \otimes \rho_N * A_N)\right] \otimes [\![(k_1)]\!]^{C_1''}\end{aligned} \tag{50}$$

The state of the simulation scheme at the second time step before the external operations are applied is

$$\begin{aligned}\varrho_{(2)} &= \varrho_{(1)} * R_1 * \tilde{M}_2 \\ &= \sum_{k_1,k_2} \left[\rho_P * M_{()}^{\to k_1} * A_{k_1} * M_{(k_1)}^{\to k_2} \otimes (\rho_1 * A_1 \otimes \ldots \otimes \rho_{k_1-1} * A_{k_1-1} \otimes \rho_{k_1} \otimes \rho_{k_1+1} * A_{k_1+1} \ldots \otimes \rho_N * A_N)\right] \otimes [\![(k_1,k_2)]\!]^{C_2}.\end{aligned} \tag{51}$$

At the end of the second time step, once the external operations and the swap operation have been applied, we note that each external operation $\mathcal{A}_k$ is applied twice on the corresponding auxiliary system in state ρ_k, except for $k = k_1, k_2$ since $\mathcal{A}_{k_1}, \mathcal{A}_{k_2}$ are respectively applied only once on the corresponding auxiliary state since the other copy of this operation was applied on the target system state:

$$\begin{aligned}\varrho'_{(2)} &= \varrho_{(1)} * R_1 * \tilde{M}_2 * R_2 \\ &= \sum_{k_1,k_2} [\rho_P * M_{()}^{\to k_1} * A_{k_1} * M_{(k_1)}^{\to k_2} * A_{k_2} \otimes (\rho_1 * A_1^{*2} \otimes \ldots \otimes \rho_{k_1-1} * A_{k_1-1}^{*2} \otimes \rho_{k_1} * A_{k_1} \otimes \rho_{k_1+1} * A_{k_1+1}^{*2} \otimes \ldots \otimes \ldots \\ &\qquad \ldots \rho_{k_2-1} * A_{k_2-1}^{*2} \otimes \rho_{k_2} * A_{k_2} \otimes \rho_{k_2+1} * A_{k_2+1}^{*2} \otimes \ldots \otimes \rho_N * A_N^{*2})] \otimes [\![(k_1,k_2)]\!]^{C_2''},\end{aligned} \tag{52}$$

where $A_k^{*2} = A_k * A_k$ is the Choi matrix of the channel $\mathcal{A}_k^{\circ 2} = \mathcal{A}_k \circ \mathcal{A}_k$ obtained by composing the channel $\mathcal{A}_k$ with itself. Deriving iteratively the state after each time step, we eventually obtain the state at the last time step N:

$$\begin{aligned}\varrho'_{(N)} &= \varrho_{(1)} * R_1 * \ldots * \tilde{M}_N * R_N \\ &= \sum_{(k_1,\ldots,k_N)} [\rho_P * M_{\emptyset}^{\to k_1} * A_{k_1} * \cdots * M_{(k_1,\ldots,k_{N-1})}^{\to k_N} * A_{k_N} \otimes (\rho_1 * A_1^{*(N-1)} \otimes \ldots \otimes \rho_N * A_N^{*(N-1)})] \otimes [\![(k_1,\ldots,k_N)]\!]^{C''_N}.\end{aligned} \tag{53}$$

Applying the last internal operation $\tilde{\mathcal{M}}_{N+1}$:

$$\begin{aligned}\varrho_F &= \varrho_{(1)} * R_1 * \ldots * \tilde{M}_N * R_N * \tilde{M}_{N+1} \\ &= \sum_{(k_1,\ldots,k_N)} \rho_P * M_{\emptyset}^{\to k_1} * A_{k_1} * \cdots * M_{(k_1,\ldots,k_{N-1})}^{\to k_N} * A_{k_N} * M_{(k_1,\ldots,k_N)}^{\to F} \otimes (\rho_1 * A_1^{*(N-1)} \otimes \ldots \otimes \rho_N * A_N^{*(N-1)}) \\ &= \left[\sum_{(k_1,\ldots,k_N)} \rho_P * M_{\emptyset}^{\to k_1} * A_{k_1} * \cdots * M_{(k_1,\ldots,k_{N-1})}^{\to k_N} * A_{k_N} * M_{(k_1,\ldots,k_N)}^{\to F}\right] \otimes (\rho_1 * A_1^{*(N-1)} \otimes \ldots \otimes \rho_N * A_N^{*(N-1)})\end{aligned} \tag{54}$$

Tracing out the auxiliary systems in $\mathcal{H}^{\text{aux}}$, we indeed retrieve the output state of the QC-CC, as defined in Eq. (14). Note that because the correlations with the auxiliary systems are classical, one recovers the intermediate QC-CC state at each single time step of the simulation by tracing out the auxiliary systems. We observe that $\mathcal{A}_k$ is applied exactly $N-1$ times on each auxiliary system $\rho_k = |a_k\rangle\langle a_k|$, thus the simulation is not catalytic. This already hints at the fact that in order to define catalytic simulation, one needs to introduce extra auxiliary systems and distribute these $N-1$ applications of each $\mathcal{A}_k$ input channel on those auxiliary systems.

□

2. Proof of Theorem 2

Proof. Consider the pure auxiliary state $\rho_{\text{aux}} = \bigotimes_{i=1}^N \rho_i \otimes \rho'_i \in \mathcal{L}(\mathcal{H}^{\text{aux}})$, with $\rho_i \otimes \rho'_i = |a_i\rangle\langle a_i| \otimes |a'_i\rangle\langle a'_i|$. For any states $\rho_P \in \mathcal{L}(\mathcal{H}^P)$, $\rho_{A_1} \in \mathcal{L}(\mathcal{H}^{A_1^{I'}}), \ldots, \rho_{A_N} \in \mathcal{L}(\mathcal{H}^{A_N^{I'}})$, the state of the simulation scheme at the first time step before the first external operations $\mathcal{A}'_k$ are applied is

$$\begin{aligned}\varrho_{(1)} &= (\rho_P * \tilde{M}_1) \otimes (\rho_{A_1} \otimes \rho_1 \otimes \rho'_1 \otimes \ldots \otimes \rho_{A_N} \otimes \rho_N \otimes \rho'_N) \\ &= \left(\sum_{k_1} \rho_P * M_{()}^{\to k_1} \otimes [\![(k_1)]\!]^{C_1}\right) \otimes (\rho_{A_1} \otimes \rho_1 \otimes \rho'_1 \otimes \ldots \otimes \rho_{A_N} \otimes \rho_N \otimes \rho'_N)\end{aligned} \tag{55}$$

Applying the first external operations in parallel along with the swap operations we obtain the state at the end of the first time step as

$$\begin{aligned}\varrho'_{(1)} &= \varrho_{(1)} * R'_1 = \varrho_{(1)} * S_1^{\prime I} * \left[(|\mathbb{1}\rangle\!\rangle\langle\!\langle\mathbb{1}|^{A_1^{I'}} \otimes A'_1) \otimes \ldots \otimes (|\mathbb{1}\rangle\!\rangle\langle\!\langle\mathbb{1}|^{A_N^{I'}} \otimes A'_N) \otimes |\mathbb{1}\rangle\!\rangle\langle\!\langle\mathbb{1}|^{\tilde{A}_1^{I'}}\right] * S_1^{\prime O} \\ &= \sum_{k_1} \left[(\rho_P * M_{()}^{\to k_1}) \bigotimes_{i=1}^N (\rho_{A_i} \otimes \rho_i \otimes \rho'_i)\right] * S_{k_1}^I \otimes [\![(k_1)]\!]^{C'_1} * \left[\bigotimes_{j=1}^N (|\mathbb{1}\rangle\!\rangle\langle\!\langle\mathbb{1}|^{A_j^{I'}} \otimes A'_j) \otimes |\mathbb{1}\rangle\!\rangle\langle\!\langle\mathbb{1}|^{\tilde{A}_1^{I'}}\right] * S_1^{\prime O} \\ &= \sum_{k_1} \left[((\rho_P * M_{()}^{\to k_1} \otimes \rho_{A_{k_1}}) * A'_{k_1}) \bigotimes_{\substack{i=1,\\ i\neq k_1}}^N (\rho_{A_i} \otimes ((\rho_i \otimes \rho'_i) * A'_i)) \otimes (\rho_{k_1} \otimes \rho'_{k_1})\right] \otimes [\![(k_1)]\!]^{C''_1} \\ &= \sum_{k_1} \left[((\rho_P \otimes \rho_{A_{k_1}}) * M_{()}^{\to k_1} * A'_{k_1}) \bigotimes_{\substack{i=1,\\ i\neq k_1}}^N (\rho_{A_i} \otimes ((\rho_i \otimes \rho'_i) * A'_i)) \otimes (\rho_{k_1} \otimes \rho'_{k_1})\right] \otimes [\![(k_1)]\!]^{C''_1}.\end{aligned} \tag{56}$$

The state of the simulation scheme at the second time step before the external operations are applied is

$$\begin{aligned}\varrho_{(2)} &= \varrho_{(1)} * R'_1 * \tilde{M}_2 \\ &= \sum_{k_1,k_2} \left[((\rho_P \otimes \rho_{A_{k_1}}) * M_{()}^{\to k_1} * A'_{k_1} * M_{(k_1)}^{\to k_2}) \bigotimes_{i=1,i\neq k_1}^N (\rho_{A_i} \otimes ((\rho_i \otimes \rho'_i) * A'_i)) \otimes (\rho_{k_1} \otimes \rho'_{k_1})\right] \otimes [\![(k_1,k_2)]\!]^{C_2}.\end{aligned} \tag{57}$$

At the end of the second time step, once the external operations and the swap operation have been applied, we note that each external operation $\mathcal{A}'_k$ is applied twice on the corresponding auxiliary system in state ρ_k, except for $k = k_1, k_2$ since $\mathcal{A}'_{k_1}, \mathcal{A}'_{k_2}$ are respectively applied only once on the corresponding auxiliary state since the other copy of this operation was applied on the target system state:

$$\begin{aligned}
\varrho'_{(2)} &= \varrho_{(1)} * R'_1 * \tilde{M}_2 * R'_2 \\
&= \sum_{k_1,k_2} \Big[((((\rho_P \otimes \rho_{A_{k_1}}) * M_{()}^{\to k_1} * A'_{k_1} * M_{(k_1)}^{\to k_2}) \otimes \rho_{A_{k_2}}) * A'_{k_2}) \bigotimes_{\substack{i=1,\\ i\neq k_1,k_2}}^{N} (\rho_{A_i} \otimes ((\rho_i \otimes \rho'_i) * A_i'^{*2})) \bigotimes_{j=k_1,k_2} ((\rho_j \otimes \rho'_j) * A'_j)\Big] \otimes [\![(k_1,k_2)]\!]^{C_2} \\
&= \sum_{k_1,k_2} \Big[((\rho_P \otimes \rho_{A_{k_1}} \otimes \rho_{A_{k_2}}) * M_{()}^{\to k_1} * A'_{k_1} * M_{(k_1)}^{\to k_2} * A'_{k_2}) \bigotimes_{\substack{i=1,\\ i\neq k_1,k_2}}^{N} (\rho_{A_i} \otimes ((\rho_i \otimes \rho'_i) * A_i'^{*2})) \bigotimes_{j=k_1,k_2} ((\rho_j \otimes \rho'_j) * A'_j)\Big] \otimes [\![(k_1,k_2)]\!]^{C_2}.
\end{aligned} \tag{58}$$

Deriving iteratively the state after each time step, we eventually obtain the state at the last time step N:

$$\begin{aligned}
&\varrho'_{(N)} = \varrho_{(1)} * R'_1 * \ldots * \tilde{M}_N * R'_N \\
&= \sum_{(k_1,\ldots,k_N)} [((\rho_P \otimes \rho_{A_{k_1}} \otimes \ldots \otimes \rho_{A_{k_N}}) * M_{\emptyset}^{\to k_1} * A'_{k_1} * \cdots * M_{(k_1,\ldots,k_{N-1})}^{\to k_N} * A'_{k_N}) \bigotimes_{j=1}^{N} ((\rho_j \otimes \rho'_j) * A_j'^{*(N-1)})] \otimes [\![(k_1,\ldots,k_N)]\!]^{C''_N} \\
&= \left[\sum_{(k_1,\ldots,k_N)} ((\rho_P \otimes \rho_{A_{k_1}} \otimes \ldots \otimes \rho_{A_{k_N}}) * M_{\emptyset}^{\to k_1} * A'_{k_1} * \cdots * M_{(k_1,\ldots,k_{N-1})}^{\to k_N} * A'_{k_N}) \otimes [\![(k_1,\ldots,k_N)]\!]^{C''_N}\right] \bigotimes_{j=1}^{N} ((\rho_j \otimes \rho'_j) * A_j'^{*(N-1)}).
\end{aligned} \tag{59}$$

Applying the last internal operation $\tilde{\mathcal{M}}_{N+1}$:

$$\begin{aligned}
\varrho_F &= \varrho_{(1)} * R'_1 * \ldots * \tilde{M}_N * R'_N * \tilde{M}_{N+1} \\
&= \sum_{(k_1,\ldots,k_N)} \Bigg[\Big((\rho_P \otimes \rho_{A_{k_1}} \otimes \cdots \otimes \rho_{A_{k_N}}) * M_{\emptyset}^{\to k_1} * A'_{k_1} * \cdots \\
&\qquad * M_{(k_1,\ldots,k_{N-1})}^{\to k_N} * A'_{k_N} * M_{(k_1,\ldots,k_N)}^{\to F}\Big) \otimes [\![(k_1,\ldots,k_N)]\!]^{C''_N}\Bigg] \bigotimes_{j=1}^{N} \big((\rho_j \otimes \rho'_j) * A_j'^{N-1}\big).
\end{aligned} \tag{60}$$

Then tracing out the auxiliary systems we recover the output state of the QC-CC as defined in Eq. (19). □

3. Proof of Theorem 4

Proof. Let a QC-QC. It is defined by a collection of internal operations $\{V_{\emptyset,\emptyset}^{\to k_1}, \ldots, V_{\{k_1,\ldots,k_{n-1}\},k_n}^{\to k_{n+1}}, \ldots, V_{\{k_1,\ldots,k_{N-1}\},k_N}^{\to F}\}$. We are going to show that the simulation scheme proposed in Fig. 7 simulates the action of the QC-QC by showing that the output state of the simulation scheme, as described in Eq. (37), is exactly the output state of the QC-QC for any choice of external unitary operation A_k and initial state ρ_P. The output state of the simulation scheme can be computed iteratively, by composing alternatively the operations $\tilde{V}_n$ and Q_n. The auxiliary systems are initialised in the pure state $|\psi_{\text{aux}}\rangle = \bigotimes_{k=1}^{N} |a_k\rangle \in \mathcal{H}^{\text{aux}}$. Given the initial state $|\psi_P\rangle \in \mathcal{H}^P$, for $n = 1, \ldots, N$, the states $|\varphi'_{(n)}\rangle \in \mathcal{H}^{\tilde{A}_n^O \alpha_n C'_n \text{aux}}$ and $|\varphi_{(n+1)}\rangle \in \mathcal{H}^{\tilde{A}_{n+1}^I \alpha_{n+1} C_{n+1} \text{aux}}$ of the global system going through the circuit right before and right after the application of $\tilde{V}_{n+1}$, respectively, are obtained recursively as

$$|\varphi'_{(n)}\rangle = (|\psi_P\rangle \otimes |\psi_{\text{aux}}\rangle) * |\tilde{V}_1\rangle\!\rangle * |Q_1\rangle\!\rangle * |\tilde{V}_2\rangle\!\rangle * |Q_2\rangle\!\rangle * \cdots * |\tilde{V}_n\rangle\!\rangle * |Q_n\rangle\!\rangle, \tag{61}$$

and

$$|\varphi_{(n+1)}\rangle = (|\psi_P\rangle \otimes |\psi_{\text{aux}}\rangle) * |\tilde{V}_1\rangle\!\rangle * |Q_1\rangle\!\rangle * \cdots * |\tilde{V}_n\rangle\!\rangle * |Q_n\rangle\!\rangle * |\tilde{V}_{n+1}\rangle\!\rangle. \tag{62}$$

At the first time step, the internal operation $\tilde{V}_1$ coherently selects an index k_1 and writes it into the control register C_1:

$$
\begin{aligned}
|\varphi_{(1)}\rangle &= (|\psi_P\rangle * |\tilde{V}_1\rangle\rangle) \otimes (|a_1\rangle \otimes \ldots \otimes |a_N\rangle) \\
&= \left(\sum_{k_1} |\psi_P\rangle * |\tilde{V}_{\emptyset,\emptyset}^{\to k_1}\rangle\rangle \otimes |\emptyset, k_1\rangle^{C_1}\right) \otimes (|a_1\rangle \otimes \ldots \otimes |a_N\rangle).
\end{aligned} \tag{63}
$$

The auxiliary systems remain in the product state $\bigotimes_{i=1}^{N} |a_i\rangle$, and no external unitary has yet acted. After applying the routing operation Q_1, the selected unitary A_{k_1} acts on the target system, while all other unitaries A_i act once on their corresponding auxiliary states $|a_i\rangle$. The swap ensures that the correct subsystem is targeted and we obtain

$$
\begin{aligned}
|\varphi'_{(1)}\rangle &= |\varphi_{(1)}\rangle * |Q_1\rangle\rangle = |\varphi_{(1)}\rangle * S_1^I * (|A_1\rangle\rangle \otimes \ldots \otimes |A_N\rangle\rangle \otimes |\mathbb{1}\rangle\rangle) * S_1^O \\
&= \sum_{k_1} \left[|\psi_P\rangle * |\tilde{V}_{\emptyset,\emptyset}^{\to k_1}\rangle\rangle * |A_{k_1}\rangle\rangle \bigotimes_{\substack{i=1,\\ i\neq k_1}}^{N} (|a_i\rangle * |A_i\rangle\rangle) \otimes |a_{k_1}\rangle \right] \otimes |\emptyset, k_1\rangle^{C_1}.
\end{aligned} \tag{64}
$$

The second internal operation $\tilde{V}_2$ now acts conditionally on the previously selected index k_1, producing a coherent superposition over k_2. Importantly, the action of $\tilde{V}_2$ depends only on the ordered set $\{k_1\}$, as required by the definition of a QC-QC:

$$
\begin{aligned}
|\varphi_{(2)}\rangle &= |\varphi_{(1)}\rangle * |Q_1\rangle\rangle * |\tilde{V}_2\rangle\rangle \\
&= \sum_{k_1,k_2} \left[|\psi_P\rangle * |\tilde{V}_{\emptyset,\emptyset}^{\to k_1}\rangle\rangle * |A_{k_1}\rangle\rangle * |\tilde{V}_{\emptyset,k_1}^{\to k_2}\rangle\rangle \bigotimes_{\substack{i=1,\\ i\neq k_1}}^{N} (|a_i\rangle * |A_i\rangle\rangle) \otimes |a_{k_1}\rangle \right] \otimes |\{k_1\}, k_2\rangle^{C_2}.
\end{aligned} \tag{65}
$$

Applying Q_2 results in the second round of external unitaries. The unitary A_{k_2} now acts on the target system, while all other unitaries act again on their respective auxiliary systems. Consequently, each auxiliary system associated with an index $i \notin \{k_1, k_2\}$ has undergone two applications of A_i, whereas the auxiliary systems corresponding to k_1 and k_2 have each experienced a single application. The intermediate simulation state is then

$$
\begin{aligned}
|\varphi'_{(2)}\rangle &= |\varphi_{(1)}\rangle * |Q_1\rangle\rangle * |\tilde{V}_2\rangle\rangle * |Q_2\rangle\rangle \\
&= \sum_{k_1,k_2} \left[|\psi_P\rangle * |\tilde{V}_{\emptyset,\emptyset}^{\to k_1}\rangle\rangle * |A_{k_1}\rangle\rangle * |\tilde{V}_{\emptyset,k_1}^{\to k_2}\rangle\rangle * |A_{k_2}\rangle\rangle \bigotimes_{\substack{i=1,\\ i\neq k_1,k_2}}^{N} (|a_i\rangle * |A_i\rangle\rangle^{*2}) \bigotimes_{j=k_1,k_2} (|a_j\rangle * |A_j\rangle\rangle) \right] \otimes |\{k_1\}, k_2\rangle^{C_2}.
\end{aligned} \tag{66}
$$

Iterating this reasoning up to time step N, we observe that the target system undergoes the ordered sequence of controlled unitaries interleaved with the selected external unitaries $A_{k_1}, \ldots, A_{k_N}$. Meanwhile, each auxiliary system $|a_j\rangle$ accumulates $N-1$ applications of the corresponding unitary A_j, independently of the specific sequence $(k_1, \ldots, k_N)$. The dependence on the sequence is entirely registered in the coherent control register $|\{k_1, \ldots, k_{N-1}\}, k_N\rangle^{C_N}$.

$$
\begin{aligned}
|\varphi'_{(N)}\rangle &= |\varphi_{(1)}\rangle * |Q_1\rangle\rangle * \cdots * |\tilde{V}_N\rangle\rangle * |Q_N\rangle\rangle \\
&= \sum_{(k_1,\ldots,k_N)} \left[|\psi_P\rangle * |V_{\emptyset,\emptyset}^{\to k_1}\rangle\rangle * |A_{k_1}\rangle\rangle * \ldots * |V_{\{k_1,\ldots,k_{N-2}\},k_{N-1}}^{\to k_N}\rangle\rangle * |A_{k_N}\rangle\rangle \bigotimes_{j=1}^{N} (|a_j\rangle * |A_j\rangle\rangle^{*N-1}) \right] \otimes |\{k_1, \ldots, k_{N-1}\}, k_N\rangle^{C_N} \\
&= \sum_{(k_1,\ldots,k_N)} \left[|\psi_P\rangle * |V_{\emptyset,\emptyset}^{\to k_1}\rangle\rangle * |A_{k_1}\rangle\rangle * \ldots * |V_{\{k_1,\ldots,k_{N-2}\},k_{N-1}}^{\to k_N}\rangle\rangle * |A_{k_N}\rangle\rangle \right] \otimes |\{k_1, \ldots, k_{N-1}\}, k_N\rangle^{C_N} \left(\bigotimes_{j=1}^{N} |a_j\rangle * |A_j\rangle\rangle^{*N-1} \right)
\end{aligned} \tag{67}
$$

Finally, the last internal operation $\tilde{V}_{N+1}$ coherently maps the control register and target system to the final output space defined as

$$|\varphi_{(F)}\rangle = |\varphi_{(1)}\rangle * |Q_1\rangle\rangle * \cdots * |\tilde{V}_N\rangle\rangle * |Q_N\rangle\rangle * |\tilde{V}_{N+1}\rangle\rangle$$
$$= \sum_{(k_1,\dots,k_N)} \Big[|\psi_P\rangle * |V_{\emptyset,\emptyset}^{\to k_1}\rangle\rangle * |A_{k_1}\rangle\rangle * \dots * |V_{\{k_1,\dots,k_{N-2}\},k_{N-1}}^{\to k_N}\rangle\rangle * |A_{k_N}\rangle\rangle * |V_{\{k_1,\dots,k_{N-1}\},k_N}^{\to F}\rangle\rangle \Big] \left(\bigotimes_{j=1}^{N} |a_j\rangle * |A_j\rangle\rangle^{*N-1} \right) \tag{68}$$

The global state factorises into the desired QC-QC output state in the global future Hilbert space $\mathcal{H}^F$, tensored with a product state of the auxiliary systems that have each undergone $N-1$ applications of their respective unitaries. Tracing out the auxiliary systems therefore yields exactly the output state defined in Eq. (31), which proves that the simulation scheme reproduces the action of the QC-QC.

As in the QC-CC case, we observe that each external unitary A_j is applied $N-1$ times to its auxiliary system. Hence, the simulation is not catalytic, since the auxiliary systems are not returned to their initial states.

□

4. Proof of Theorem 5

Proof. Let a QC-QC be defined by the collection of internal operations

$$\{V_{\emptyset,\emptyset}^{\to k_1}, \dots, V_{\{k_1,\dots,k_{n-1}\},k_n}^{\to k_{n+1}}, \dots, V_{\{k_1,\dots,k_{N-1}\},k_N}^{\to F}\}.$$

We show that the simulation scheme proposed in Fig. 8 reproduces the action of the QC-QC when extended unitary channels $A'_k : \mathcal{H}^{A_k^I A_k^{I'}} \to \mathcal{H}^{A_k^O A_k^{O'}}$ are applied.

The auxiliary systems are initialised in the pure state

$$|\psi_{\text{aux}}\rangle = \bigotimes_{k=1}^{N} (|a_k\rangle \otimes |a'_k\rangle) \in \mathcal{H}^{\text{aux}}.$$

Given initial states $|\psi_P\rangle \in \mathcal{H}^P$ and $|\psi_{A_k}\rangle \in \mathcal{H}^{A_k^{I'}}$, the output state of the simulation scheme before tracing α_F is

$$|\psi'^{\text{sim}}_{F\alpha_F}\rangle = (|\psi_P\rangle \bigotimes_{k=1}^{N} |\psi_{A_k}\rangle \otimes |\psi_{\text{aux}}\rangle) * |\tilde{V}_1\rangle\rangle * |Q'_1\rangle\rangle * \cdots * |\tilde{V}_N\rangle\rangle * |Q'_N\rangle\rangle * |\tilde{V}_{N+1}\rangle\rangle.$$

As in the local-unitary case, the proof proceeds iteratively by composing alternately the internal operations $\tilde{V}_n$ and the routing operations Q'_n.

The first internal operation $\tilde{V}_1$ coherently selects an index k_1:

$$|\varphi_{(1)}\rangle = \left(\sum_{k_1} |\psi_P\rangle * |\tilde{V}_{\emptyset,\emptyset}^{\to k_1}\rangle\rangle \otimes |\emptyset, k_1\rangle^{C_1} \right) \bigotimes_{k=1}^{N} (|\psi_{A_k}\rangle \otimes |a_k\rangle \otimes |a'_k\rangle) . \tag{69}$$

Applying Q'_1 performs the double routing: the target system is swapped with $|a_{k_1}\rangle$, and the primed input system $|\psi_{A_{k_1}}\rangle$ is swapped with $|a'_{k_1}\rangle$. Then each A'_k acts on its corresponding auxiliary pair. Finally the swaps are inverted. We obtain

$$|\varphi'_{(1)}\rangle = \sum_{k_1} \left[(|\psi_P\rangle \otimes |\psi_{A_{k_1}}\rangle) * |\tilde{V}_{\emptyset,\emptyset}^{\to k_1}\rangle\rangle * |A'_{k_1}\rangle\rangle \bigotimes_{i\neq k_1} |\psi_{A_i}\rangle \otimes ((|a_i\rangle \otimes |a'_i\rangle) * |A'_i\rangle\rangle) \, (|a_{k_1}\rangle \otimes |a'_{k_1}\rangle) \right] \otimes |\emptyset, k_1\rangle^{C_1}. \tag{70}$$

Then, the operation $\tilde{V}_2$ acts conditionally on k_1:

$$|\varphi_{(2)}\rangle = \sum_{k_1,k_2} \left[(|\psi_P\rangle \otimes |\psi_{A_{k_1}}\rangle) * |\tilde{V}_{\emptyset,\emptyset}^{\to k_1}\rangle\rangle * |A'_{k_1}\rangle\rangle * |\tilde{V}_{\emptyset,k_1}^{\to k_2}\rangle\rangle \bigotimes_{i\neq k_1} ((|a_i\rangle \otimes |a'_i\rangle) * |A'_i\rangle\rangle) \otimes (|a_{k_1}\rangle \otimes |a'_{k_1}\rangle) \right] \otimes |\{k_1\}, k_2\rangle^{C_2}. \tag{71}$$

After applying Q'_2, a second round of extended unitaries acts. The unitary A'_{k_2} acts on the target and its primed system, while all other unitaries act again on their auxiliary pairs.

$$|\varphi'_{(2)}\rangle = \sum_{k_1,k_2} \Bigg[(|\psi_P\rangle \otimes |\psi_{A_{k_1}}\rangle \otimes |\psi_{A_{k_2}}\rangle) * |\tilde{V}^{\to k_1}_{\emptyset,\emptyset}\rangle\!\rangle * |A'_{k_1}\rangle\!\rangle * |\tilde{V}^{\to k_2}_{\emptyset,k_1}\rangle\!\rangle * |A'_{k_2}\rangle\!\rangle \Bigg] \cdots$$
$$\cdots \bigotimes_{i\notin\{k_1,k_2\}} \Big(|\psi_{A_i}\rangle \otimes (|a_i\rangle \otimes |a'_i\rangle) * |A'_i\rangle\!\rangle^{*\,2} \Big) \bigotimes_{j\in\{k_1,k_2\}} \Big((|a_j\rangle \otimes |a'_j\rangle) * |A'_j\rangle\!\rangle^{*\,2} \Big) \Bigg] \otimes |\{k_1\},k_2\rangle^{C_2}. \tag{72}$$

Iterating the argument up to time step N, we obtain

$$|\varphi'_{(N)}\rangle = \sum_{(k_1,\ldots,k_N)} \Bigg[(|\psi_P\rangle \otimes |\psi_{A_{k_1}}\rangle \otimes \ldots \otimes |\psi_{A_{k_N}}\rangle) * |V^{\to k_1}_{\emptyset,\emptyset}\rangle\!\rangle * |A'_{k_1}\rangle\!\rangle * \cdots * |V^{\to k_N}_{\{k_1,\ldots,k_{N-2}\},k_{N-1}}\rangle\!\rangle * |A'_{k_N}\rangle\!\rangle \Bigg] \cdots$$
$$\cdots \otimes |\{k_1,\ldots,k_{N-1}\},k_N\rangle^{C_N} \bigotimes_{i=1}^{N} \Big((|a_i\rangle \otimes |a'_i\rangle) * |A'_i\rangle\!\rangle^{*\,(N-1)} \Big). \tag{73}$$

Finally applying $\tilde{V}_{N+1}$ yields

$$|\varphi_{(F)}\rangle = \sum_{(k_1,\ldots,k_N)} \Bigg[(|\psi_P\rangle \otimes |\psi_{A_{k_1}}\rangle \otimes \ldots \otimes |\psi_{A_{k_N}}\rangle) * |V^{\to k_1}_{\emptyset,\emptyset}\rangle\!\rangle * |A'_{k_1}\rangle\!\rangle * \cdots * |V^{\to k_N}_{\{k_1,\ldots,k_{N-2}\},k_{N-1}}\rangle\!\rangle * |A'_{k_N}\rangle\!\rangle * |V^{\to F}_{\{k_1,\ldots,k_{N-1}\},k_N}\rangle\!\rangle \Bigg] \cdots$$
$$\cdots \bigotimes_{i=1}^{N} \Big((|a_i\rangle \otimes |a'_i\rangle) * |A'_i\rangle\!\rangle^{*\,(N-1)} \Big). \tag{74}$$

The global state factorises into the desired QC-QC output state defined in Eq. (38), i.e. a product state of the auxiliary systems, each having undergone $N-1$ applications of its corresponding unitary. Tracing out α_F and all auxiliary systems in $\mathcal{H}^{\text{aux}}$ therefore yields exactly the output state of the QC-QC. □