The category of quantum graphs is closed

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The phrase "quantum graph homomorphism" can be interpreted in two different ways according to how the words are associated. Remarkably, both "(quantum graph) homomorphisms" and "quantum (graph homomorphisms)" play significant roles in quantum information theory. The two notions are united in the category qGph of quantum graphs that we introduce in [9].

Quantum graphs arise naturally in quantum error correction [2], where they encode the confusability structure of quantum channels and thus provide a simple definition of quantum channel capacity. In that setting, quantum graph structure is defined on von Neumann algebras of the form $M_n(\mathbb{C})$, but it can also be defined on arbitrary von Neumann algebras [16]. The objects of our category qGph are in effect hereditarily atomic von Neumann algebras with such structure. These quantum graphs are the canonical quantum generalization of graphs in noncommutative geometry because hereditarily atomic von Neumann algebras generalize discrete spaces [13], i.e., sets [5].

The morphisms of qGph are the "(quantum graph) homomorphisms." From the perspective of quantum confusability graphs, a morphism of qGph is a quantum channel that is completely entropy-nonincreasing [8] and that maps confusable pairs of states to confusable pairs of states [9, Rem. 6.8]. Another compelling interpretation of these morphisms was given earlier by Verdon in terms of encoding channels, but in that case, it is more appropriate to work with a larger class of morphisms [15, sec. 1.1].

In contrast, "quantum (graph homomorphisms)" is a manner of speaking about quantum nonlocality. Let G and H be finite simple graphs. The (G,H)-homomorphism game is a nonlocal game whose rules are such that a winning strategy exists iff there exists a homomorphism $G \to H$ [11]. The (G,H)-homomorphism game may have a winning quantum strategy even when no winning strategy exists [3]. In this case, it is said that "there exists a quantum homomorphism $G \to H$." Musto, Reutter, and Verdon observed that one may interpret this statement literally in a 2-category of quantum graphs [12]. The first author then observed that, in the framework of noncommutative geometry, one may interpret this statement literally as the existence of a nonempty quantum set of homomorphisms $G \to H$ [5]. In this paper, we construct the quantum set of all homomorphisms $G \to H$.

More generally, we prove the following theorems.

Theorem 1. *The category* qGph *is symmetric monoidal closed.*

Theorem 2. *The category* Gph *of graphs is a* Gph*-enriched coreflective subcategory of* qGph.

Theorem 3. *Let G and H be finite simple graphs. Then, the following are equivalent:*

1. *there exists a winning quantum strategy for the (G,H)-homomorphism game,*
2. *the inner hom object $[G,H]$ in* qGph *is nonempty, i.e., not isomorphic to the empty graph.*

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Both "(quantum graph) homomorphisms" and "quantum (graph homomorphisms)" occur together in the 2-category of quantum graphs of Musto, Reutter, and Verdon [12]. Thus, the primary contribution of this submission is providing a category of quantum graphs that

1. includes infinite quantum graphs,
2. includes quantum graphs of homomorphisms in the sense of both
   (a) category theory,
   (b) quantum information theory,
3. can be motivated entirely from noncommutative geometry.

The quantum graph $[G,H]$ is the quantum graph of all homomorphisms $G \to H$ in the sense of category theory because it is the internal hom object in qGph and in the sense of quantum information theory according to Theorem 3. Similarly, the graph of homomorphisms $G \to H$ is the internal hom object in Gph and satisfies the analogue of Theorem 3 for classical strategies.

The definition of the symmetric monoidal category qGph can be motivated entirely from the basic principles of noncommutative geometry without any appeals to quantum information theory. In short, qGph can be defined by internalizing the obvious definition of Gph within Rel to the category qRel of quantum sets and relations [5]. The definition of qRel can, in turn, be motivated entirely from the duality between C^*-algebras and quantum locally compact Hausdorff spaces [6], which holds by fiat [18].

1 Definition of qGph

The dagger symmetric monoidal category qRel can be given several equivalent definitions. Concisely, it is the category of hereditarily atomic von Neumann algebras and quantum relations in the sense of Weaver [16]. It also has an elementary definition that only assumes familiarity with FinHilb, the dagger compact category of finite-dimensional Hilbert spaces over $\mathbb{C}$ [1, 14]. We recall this definition now.

A *quantum set* X is simply an indexed family of nonzero finite-dimensional Hilbert spaces $\{X_\alpha\}$. The index set $\mathrm{At}(X)$ may be empty, finite, or infinite and the indices $\alpha \in \mathrm{At}(X)$ are said to be the atoms of X. (In [5], quantum sets are self-indexed.) A relation R from X to Y is then a choice of subspaces $R_{\alpha,\beta} \leq L(X_\alpha, Y_\beta)$ for $\alpha \in \mathrm{At}(X)$ and $\beta \in \mathrm{At}(Y)$, where $L(H,K)$ denotes the set of operators from H to K. If we view such relations as matrices of operator spaces, the composition of qRel becomes obvious. The dagger compact structure of qRel is then obtained from that of FinHilb in the obvious way.

The category Rel of sets and relations may be viewed as a subcategory of qRel by identifying each set A with the quantum set X such that $\mathrm{At}(X) = A$ and $X_\alpha = \mathbb{C}$ for all $\alpha \in A$ and doing similarly for relations. Both Rel and qRel are dagger compact quantaloids [4], and this structure may be used to internalize the definitions of many kinds of objects. For example, internalizing a definition of groups in Rel yields the class of all groups, and internalizing that definition in qRel yields the class of all (discrete) quantum groups [7]. We apply the same approach to the category Gph of graphs.

Definition 4. We define the category Gph of quantum graphs and homomorphisms.

1. A *graph* G is a pair (V_G, e_G), where V_G is a set and e_G is a relation $V_G \to V_G$ such that $e_G^\dagger = e_G$.
2. A *homomorphism* $\phi \colon G \to H$ is a relation $V_G \to V_H$ such that
   (a) $\phi^\dagger \circ \phi \geq \mathrm{id}_G$,
   (b) $\phi \circ \phi^\dagger \leq \mathrm{id}_H$,
   (c) $\phi \circ e_G \leq e_H \circ \phi$.

Thus, Gph is the category of graphs such that loops are allowed but multiple edges are forbidden, and a map $\phi\colon V_G \to V_H$ is a homomorphism iff $\phi(g_1) \sim \phi(g_2)$ whenever $g_1 \sim g_2$ are vertices of G.

Definition 5. We define the symmetric monoidal structure on Gph.

1. The monoidal unit is the graph $K_1 = (\{*\}, \varnothing)$.
2. The monoidal product of G and H is the graph $G \square H = (V_G \times V_H, e_G \times \mathrm{id}_H \vee \mathrm{id}_G \times e_H)$.

Evidently, $G \square H$ is the *box product* of G and H. It is distinct from the *tensor product* $G \times H = (V_G \times V_H, e_G \times e_H)$, which is the categorical product in Gph.

Proposition 6. *The symmetric monoidal category* Gph *is closed in the sense that the functor* $G \mapsto G \square H$ *has a right adjoint* $G \mapsto [H,G]$ *for every graph* H.

It is straightforward to verify that the vertices of the graph $[G,H]$ are homomorphisms $G \to H$ with $\phi_1 \sim \phi_2$ iff $\phi_1(g) \sim \phi_2(g)$ for all $g \in V_G$. However, having defined the graph $[G,H]$ internally to Rel, we can now make the same definition in qRel.

Definition 7. We define the symmetric monoidal category qGph by internalizing Definitions 4 and 5 in qGph rather than in Gph.

We note that in any dagger compact quantaloid, condition (c) of Definition 4(2) is equivalent to both $\phi \circ e_G \circ \phi^\dagger \leq e_H$ and $e_G \leq \phi^\dagger \circ e_H \circ \phi$ modulo conditions (a) and (b), which implies that for entropy-nonincreasing quantum channels [8], Weaver's two notions of a CP morphism [17] coincide.

2 Proofs of the theorems

We now prove Theorem 1 by categorifying the proof that Gph is symmetric monoidal closed, as in [10]. Theorem 2 is established by working carefully with the "inclusion" functor Inc: Rel $\to$ qRel, which is full and faithful and preserves the structure of dagger compact quantaloids. Theorem 3 is established by a straightforward calculation that applies these categorical results, after proving the following lemma.

Lemma 8. *Let G be a finite simple graph, and let Q_n be the edgeless finite quantum graph such that $\ell(V_{Q_n}) = M_n(\mathbb{C})$. Then, the graph* $\mathsf{qGph}(Q_n, G)$ *is isomorphic to the graph $M(G,n)$ of [11, Def. 2.8].*

Proof of Theorem 3. For each positive integer n, we calculate that

$$\begin{aligned}
\mathsf{qGph}(Q_n,[G,H]) &\cong \mathsf{qGph}(Q_n \square G, H)\\
&\cong \mathsf{qGph}(G \square Q_n, H)\\
&\cong \mathsf{qGph}(G,[Q_n,H])\\
&\cong \mathsf{Gph}(G, \mathsf{qGph}(K_1,[Q_n,H]))\\
&\cong \mathsf{Gph}(G, \mathsf{qGph}(K_1 \square Q_n, H))\\
&\cong \mathsf{Gph}(G, \mathsf{qGph}(Q_n, H))\\
&\cong \mathsf{Gph}(G, M(H,n)).
\end{aligned}$$

All but the last of these isomorphisms follow by Theorems 1 and 2. The last isomorphism follows by Lemma 8. By [11, Thm. 2.9], condition 1 is equivalent to the condition that $\mathsf{Gph}(G,M(H,n))$ is nonempty for some n. On the other hand, it is elementary that $\mathsf{qGph}(Q_n,[G,H])$ is nonempty for some n iff $[G,H]$ is nonempty, i.e., iff condition 2 holds. $\square$

We also include a short proof that every finite reflexive quantum graph is the confusability graph of a quantum channel, which was previously established via a more general argument by Verdon [15].

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