Transversal AND in Quantum Codes*

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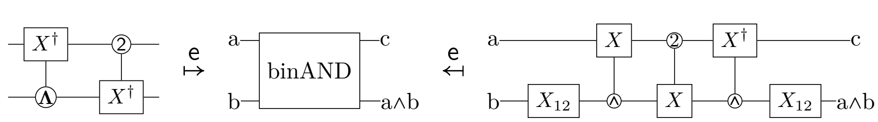
Overview and motivation: The AND gate is not reversible—on qubits. However, it *is* reversible on qutrits, making it a building block for efficient simulation of qubit computation using qutrits. Qutrit computation offers a promising path to more efficient error-correcting codes, with better thresholds and yield rates in magic state distillation [2]. While prior work has explored various constructions of qutrit codes [3], less attention has been given to finding qutrit codes with specific transversal logical operators, especially ones that are both non-Clifford and entangling. In this work, we take a logical operator-centric approach to code construction by reversing the typical procedure for designing stabilizer codes. Instead of deriving the logical operators from the fixed stabilizer set of a code, we start with a chosen logical operator, constructing the code around it by adding stabilizers. This work constitutes one of few efforts on improving qubit computation framed with respect to qutrit stabilizer computation, and to our knowledge is the first to construct qutrit stabilizer error-correcting codes expressedly to this end.

Our approach to constructing qutrit codes with a AND gate logical operation centers on two components. The first is exact circuit synthesis, leveraging the known connection between symmetric T-depth one circuits and error-correcting code constructions [4, 5]. The second is the ZX-calculus [6, 7], by means of which a number of works have investigated new ways to represent and compile for quantum error-correcting codes. However, while more progress has been made on verifying and proving properties of codes and on compiling fault-tolerant circuits, there is exciting unmet potential in the use of ZX-calculus to construct specific new codes and protocols. Thus, this presents an important open question: *How can we construct a non-trivial qutrit QEC code with a transversal qubit logical gate?*

Main contributions: Our main result is the construction of a novel qutrit $[\![6,2,2]\!]$ quantum error-correcting code with a transversal implementation of the AND gate. The key insight in our approach is that a symmetric T-depth one circuit decomposition — composed of a CX circuit, T and T dagger gates, followed by the CX circuit in reverse — of a given unitary can be interpreted as a CSS code. We can increase the code distance by augmenting the code circuit with additional stabilizers while preserving the logical gate. This results in a code with a "built-in" transversal implementation of the original unitary. We observe that there are multiple two-qutrit Clifford+T unitaries that realize the AND gate with T-count 3, and its generalizations to n qubits with T-count $3n-3$. We use the aforementioned methods to construct the qutrit $[\![6,2,2]\!]$ quantum error-correcting code with a transversal implementation of the AND gate, which can be further concatenated to attain a $[\![48,2,4]\!]$ code with the same transversal logical gate. Furthermore, we present several protocols for mixed qubit-qutrit codes which we call Qubit Subspace Codes, and for magic state distillation and injection of the $|0\rangle$-controlled qutrit Z gate and the AND gate.

Theorem 1: Emulation. There exists multiple qutrit Clifford+T circuits which *emulate* a AND gate – the behavior (i.e., truth table rows) of the first two levels $|0\rangle$ and $|1\rangle$ of the qutrit circuit matches that of the qubit AND gate. Additional space afforded by the $|2\rangle$ level allows for AND to be implemented unitarily. We will use the symmetric qutrit circuit on the right of 1 to construct a code.

a	b	c	a∧b
0	0	0	0
0	1	2	0
0	2	1	2
1	0	1	0
1	1	0	1
1	2	2	1
2	0	2	2
2	1	1	1
2	2	0	2



a	b	c	a∧b
0	0	0	0
0	1	2	0
0	2	0	2
1	0	1	0
1	1	1	1
1	2	0	1
2	0	1	2
2	1	2	1
2	2	2	2

(1)

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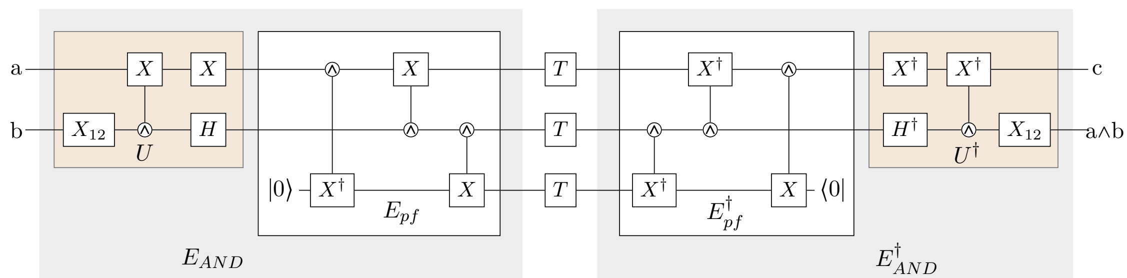
In the paper, we describe how to exactly emulate **all** qubit Clifford gates in qutrit Clifford+T except the qubit H gate (which is impossible to exactly and deterministically emulate with single-qutrit Clifford+T gates [8]). In general, the qubit Clifford operations do not neatly embed in the qutrit Clifford fragment. We show that all binary classical logic circuits can be emulated through exact synthesis in the qutrit Clifford+T gate set. We give qutrit constructions of common qubit gates, summarized in Table 1. These results outperform the best-known constant-ancilla qubit decomposition [9].

Gates	T	R	CX	2q	Gates	T	R	CX	2q
AND/OR	3	0	4	1	n-ary AND [10]	$3n-3$	0	$4n-4$	$n-1$
X (i.e. NOT)	0	0	0	0	Z	0	1	0	0
CX [11]	6	0	8	1	CZ [12]	0	3	3	1
CCX [11]	12	0	15	3	CCZ [12]	0	3	5	3
C^nX, linear depth	$6n$	0	$6n+3$	$2n-1$	C^nZ, linear depth	$6n-12$	3	$6n-7$	$2n-1$
C^nX, log depth [10]	$6n$	0	$8n$	$2n-1$	C^nZ, log depth	$6n-12$	3	$8n-5$	$2n-1$

Table 1: Resource counts for AND qubit emulations unitarily without ancillae by qutrit gates.

Lemma 2: A symmetric T-depth one circuit is a QEC code. We find a symmetric T-depth one qutrit Clifford+T circuit decomposition of the AND gate as follows:

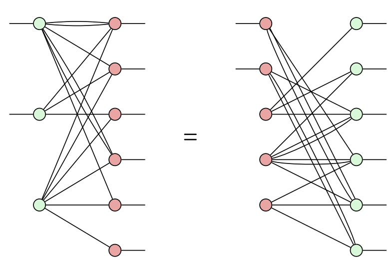


(2)

where the inner CX+T circuit (implementing $|0\rangle$-controlled Z) is from Figure 7 of [11]. The first half of inner CX+T circuit (excluding T gates, labeled E_{pf} in 2) corresponds to a phase-free ZX-diagram and thus a CSS code [13] encoder. It forms a qutrit CSS $[\![3,2,1]\!]$ code with Z distance 1. To increase code distance, we add X-type stabilizers while ensuring that the logical gate stays the same – taking inspiration from an observation about the qubit $[\![8,3,2]\!]$ code construction [4]. Unlike [4], however, naïvely adding an X-type stabilizer to the $[\![3,2,1]\!]$ qutrit code does not preserve the AND gate. We demonstrate in our paper how to modify the circuit to preserve the logical binary AND gate.

Theorem 3: Main code construction. We present a $[\![6,2,2]\!]$ qutrit code implementing a transversal AND. The inner CSS code (which implements the $|0\rangle$-controlled Z) has encoder maps, stabilizer generators, and logical operators given by:

$$\overline{X}_1 = X_1^2 X_2 X_4^2 X_5$$
$$\overline{X}_2 = X_1 X_2 X_3$$
$$S_{X1} = X_1 X_2 X_3 X_4 X_5 X_6$$



$$\overline{Z}_1 = Z_4 Z_5^2$$
$$\overline{Z}_2 = Z_3 Z_6^2$$
$$S_{Z1} = Z_1 Z_2 Z_3$$
$$S_{Z2} = Z_2 Z_3^2 Z_4^2 Z_5$$
$$S_{Z3} = Z_4 Z_5 Z_6 \tag{3}$$

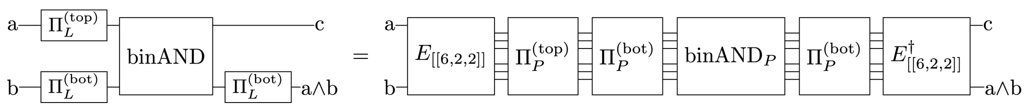
By pushing logical X and Z spiders through the full encoder, we obtain the following logical operators for the "overall" code implementing transversal AND:

$$\overline{X}_1 = X_1^2 X_2 X_4^2 X_5 \qquad \overline{Z}_1 = X_1 X_2 X_3 Z_4 Z_5^2 \tag{4}$$
$$\overline{X}_2 = X_1 X_2^2 Z_3^2 X_4 X_5^2 Z_6 \qquad \overline{Z}_2 = X_1 X_2 X_3 \tag{5}$$

We verify that the code distance is 2 by checking the minimum weight of all possible logical operators (such as by multiplying each logical operator with all possible linear combinations of the stabilizers).

Corollary 4: Code concatenation. We construct a $[\![48, 2, 4]\!]$ qutrit code with transversal AND gate implemented by 24 T and 24 T† gates. The $[\![6, 2, 2]\!]$ qutrit code with a transversal AND gate (implemented by 3 T and 3 T† gates), can be concatenated as the outer code with the inner $[\![8, 1, 2]\!]$ qutrit CSS code with transversal T gate from [14]. As both are qutrit CSS codes and the inner CSS code's X and Z logical operators are implemented by physical X's and Z's respectively, the resulting code distance is the product of that of the inner and outer codes.

Qubit Subspace Projection: We introduce Qubit Subspace Projection as a logical operation in qutrit codes. Depending on how these results are utilized, these can be viewed from the perspectives of gauge fixing of subsystem codes, and as adding additional stabilizers to construct mixed qubit-qutrit codes which we call Qubit Subspace Codes. We show there exists a code with $k = 2$ logical qubits and

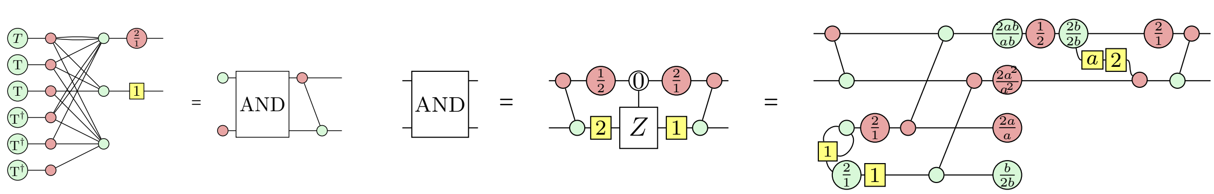


$n = 6$ physical qutrits that has the same stabilizers as that of E_{AND}. The construction relies on the qubit subspace projection (SP) gadget, which projects a logical qutrit into the $|0\rangle$ and $|1\rangle$ subspace.

(6)

It has the above logical and physical implementations, respectively on the left and right sides of Eq. 6 (note this uses the ZH diagram for $\neg\,|0\rangle$-controlled X from [15]). To find the physical implementation of the SP gadget in the $[\![6, 2, 2]\!]$ code, we can push the SP gadget through the full Clifford encoder using PTE [16] (which we show applies to qutrits). We observe that physical gadget, written as a resource state is a qutrit state prepared as the *qubit* $|+\rangle$, an equal superposition of $|0\rangle$ and $|1\rangle$.

Magic state distillation and injection: We prove correctness of magic state distillation and deterministic injection of the AND gate for qutrit CSS codes. By decoding 3 T and 3 T† states (which can be recursively distilled) in the code without the CX in U, we get a magic state for the AND gate up to this CX (see left side of 7). We can then perform deterministic injection (see right side of 7). The resulting correction is a Clifford unitary where all but one gate—the $Z(2b, 2b)$ gate—are Pauli or CX gates, and hence transversally implementable on CSS codes [17].



(7)

Future work: A natural question to investigate through in-depth comparative study would be how theorized qutrit codes perform under different noise models and hardware constraints. This would be a step towards experimental implementation of our code constructions.

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