

Talk proposal: Quantum statistical functions

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1. The problem: The missing quantum counterpart.—In classical probability theory and statistical mechanics, statistical functions—such as the moment-generating function (MGF), characteristic function (CF), and their logarithmic counterparts—are cornerstone tools. They compactly encode all statistical moments, govern the addition of independent random variables, and form the basis of the generating functional approach in field theory [1, 2].

However, formulating a rigorous, universally applicable quantum counterpart to these functions has remained an outstanding open problem. The fundamental hurdle is twofold: the noncommutativity of quantum observables and the intrinsic contextuality of quantum measurements. Previous attempts to define quantum generating functions have largely been ad-hoc, often tailored to specific applications like full counting statistics [3], or restricted to symmetric operator orderings that fail to capture the rich structure of sequential quantum measurements [4]. The lack of a unified mathematical framework has obscured the deep connections between standard statistical measures, quasiprobability distributions, and post-selected phenomena like weak values. In this work, we address this fundamental gap by proposing a comprehensive and rigorous hierarchy of *quantum statistical functions*.

2. The approach: Canonical purification and generalized operator ordering.—To transcend the limitations of classical analogies, our approach shifts the foundational stage from the mixed-state density operator $\rho \in \mathcal{L}(\mathcal{H})$ to its canonical purification $|\Psi\rangle \in \mathcal{H} \otimes \mathcal{H}^*$ [5]. We define quantum statistical functions directly as inner products within this purified space. For a single observable A , the quantum moment-generating function (QMGF) is unambiguously defined as $\mathcal{M}_A(\theta, \rho) := \langle \Psi | (e^{\theta A} \otimes \mathbb{1}_{\mathcal{H}^*}) | \Psi \rangle = \text{Tr}(e^{\theta A} \rho)$. Differentiating this function perfectly reproduces the standard quantum expectation value and variance.

The true challenge, however, lies in the multivariable case where $[A_j, A_k] \neq 0$. To solve this, we introduce the *generalized operator ordering function*:

$$f_{\mathbf{A}}^{(N,w)}(\boldsymbol{\theta}) := \left[\sum_{\sigma \in S_n} w(\sigma) \left(\prod_{j=1}^n e^{\frac{\theta_{\sigma(j)}}{N} A_{\sigma(j)}} \right) \right]^N. \quad (1)$$

Here, the integer $N \in \mathbb{N}$ controls the Trotterization layer (interpolating between algebraic sums and group-theoretic products), and $w(\sigma)$ is a normalized weight function over the permutation group S_n . The multivariable QMGF is thus defined as $\mathcal{M}_{\mathbf{A}}(\boldsymbol{\theta}, \rho) := \langle \Psi | f_{\mathbf{A}}^{(N,w)}(\boldsymbol{\theta}) | \Psi \rangle$. This parameterized

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function serves as a “Rosetta Stone,” allowing us to systematically traverse the landscape of non-commutative statistics.

3. Solution and unification: Quasiprobabilities.—By systematically varying N and w , our framework rigorously derives and unifies historically disparate quasiprobability distributions:

- **Kirkwood–Dirac (KD) limit:** Setting $N = 1$ and choosing an identity weight $w(\sigma) = \delta_{\sigma, \text{id}}$ preserves the sequential operator algebra, generating the KD distribution [6, 7], which is central to studying quantum contextuality.
- **Margenau–Hill (MH) limit:** Using $N = 1$ with a symmetrized weight yields the MH distribution [8], naturally providing symmetrized covariances.
- **Wigner limit:** Taking $N \rightarrow \infty$ fuses the operators into a single exponential $\exp(\sum \theta_j A_j)$. By extending the parameters to imaginary arguments to form the quantum characteristic function (QCF), the symplectic Fourier transform rigorously recovers the Wigner function [9].

4. Foundational contribution: The extended Bochner’s theorem.—A central highlight of this work is the rigorous formalization of non-classicality via an *extended Bochner’s theorem*. In classical theory, Bochner’s theorem states that a continuous function is the Fourier transform of a finite positive measure if and only if it is positive definite [10].

By formulating the multivariable QCF within the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$, we prove that a unique joint weak distribution $\text{Pr}_{\mathbf{A}}(\mathbf{x}, \rho)$ always exists. Crucially, we establish that this distribution becomes a classical (non-negative) probability measure *if and only if* the multivariable QCF satisfies the positive definiteness condition. This provides a definitive, mathematically rigorous demarcation line: the emergence of quantum non-classicality (i.e., complex or negative values in quasiprobabilities) is explicitly driven by the breakdown of positive definiteness in the characteristic domain due to non-trivial operator orderings.

5. Resolving anomalies: Conditional statistics and weak values.—By incorporating a positive operator-valued measure (POVM) element Π_m for post-selection, we define the *conditional QMGF*:

$$\mathcal{M}_A(\theta | \Pi_m, \rho) := \frac{\langle \Psi | (\Pi_m e^{\theta A} \otimes \mathbb{1}_{\mathcal{H}^*}) | \Psi \rangle}{\langle \Psi | (\Pi_m \otimes \mathbb{1}_{\mathcal{H}^*}) | \Psi \rangle}. \quad (2)$$

Since their inception, weak values [11] have been debated regarding their physical interpretation and “anomalous” nature. Our framework definitively settles this: the first derivative of the conditional QMGF evaluated at $\theta = 0$ exactly yields the weak value $\text{Ex}_\rho(A | \Pi_m)$, and the second derivative yields the weak variance [12]. In our preprint, we show how any general n -point correlation function can be structurally decomposed into a chain of these weak values. Weak values are thus not physical anomalies, but the rigorous quantum analogues of conditional statistical expectations [13, 14].

6. Impact and application: Quantum econometrics.—Beyond foundational insights, this framework provides a powerful new toolkit for practical quantum metrology. Taking inspiration from classical econometrics [15], we introduce the *quantum method of moments (QMM)* and the *quantum generalized method of moments (QGMM)*.

Current parameter estimation methods (like maximum likelihood) often require full quantum state tomography, which is computationally intractable for large systems. By matching theoretical

moments generated by our multivariable QMGF to empirical data, the QMM allows for efficient parameter estimation. For over-identified systems (where measured observables exceed unknown parameters), the QGMM minimizes statistical error by using an optimal weighting matrix. Remarkably, this optimal matrix is exactly the quantum covariance matrix, seamlessly derived analytically from our MH-type multivariable QMGF. In our full paper, we demonstrate this efficiency using an analytical solution of the 1D transverse-field Ising model, showcasing significant variance reduction.

7. Conclusion and future outlook.—To conclude, this work establish that the generalized operator ordering function evaluated in the canonical purification space is the long-sought master key for quantum statistics. It bridges the gap between operator algebra, information geometry, and operational quasiprobabilities. Moving forward, this framework lays the essential groundwork for developing quantum full counting statistics in non-equilibrium thermodynamics and extending statistical generating functionals to finite-temperature quantum field theory.

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- [1] W. Feller, *An introduction to probability theory and its applications*, Vol. 2 (John Wiley & Sons, 1991).
 - [2] J. Schwinger, [Proc. Natl. Acad. Sci. **37**, 452 \(1951\)](#).
 - [3] M. Esposito, U. Harbola, and S. Mukamel, [Rev. Mod. Phys. **81**, 1665 \(2009\)](#).
 - [4] B. Hetényi and P. Lévy, [Phys. Rev. A **108**, 032218 \(2023\)](#).
 - [5] J. von Neumann, *Mathematical foundations of quantum mechanics*, Princeton Landmarks in Mathematics and Physics (Princeton university press, Princeton, New Jersey, 1955).
 - [6] J. G. Kirkwood, [Phys. Rev. **44**, 31 \(1933\)](#).
 - [7] P. A. M. Dirac, [Rev. Mod. Phys. **17**, 195 \(1945\)](#).
 - [8] H. Margenau and R. N. Hill, [Prog. Theor. Phys. **26**, 722 \(1961\)](#).
 - [9] E. P. Wigner, [Phys. Rev. **40**, 749 \(1932\)](#).
 - [10] S. Bochner, *Harmonic Analysis and the Theory of Probability*, 1st ed. (Publisher, University of California Press, 1955) p. 184.
 - [11] Y. Aharonov, D. Z. Albert, and L. Vaidman, [Phys. Rev. Lett. **60**, 1351 \(1988\)](#).
 - [12] J. Dressel, [Phys. Rev. A **91**, 032116 \(2015\)](#).
 - [13] A. M. Steinberg, [Phys. Rev. A **52**, 32 \(1995\)](#).
 - [14] A. M. Steinberg, [Phys. Rev. Lett. **74**, 2405 \(1995\)](#).
 - [15] L. P. Hansen, [Econometrica **50**, 1029 \(1982\)](#).