

Essential Unitarity for Higher-Order Quantum Computation

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We develop a semantic framework for higher-order quantum computation based on a boundary-centric presentation of compact closed categories, building on the Kelly–Laplaza construction as reformulated by Abramsky. Morphisms are represented as polarized boundary linkings composed by execution, and a unit-free monoidal sum provides reversible control and branching.

Within this setting, we identify a notion of *essential unitarity* that extends ordinary unitary evolution from first-order quantum processes to higher-order interfaces. At first order, essential unitarity coincides with standard unitarity; for higher-order interfaces, it characterizes when information is preserved relative to the boundary. We prove that essential unitarity is the unique predicate compatible with dagger-monoidal structure, coherence reindexing, and currying, and agreeing with ordinary unitarity at first order, for all morphisms that coherently mix uniform signature blocks—a condition satisfied throughout the quantum core.

We show that this framework is expressively sufficient for higher-order quantum computation. In particular, one-slot, equal-size, purity-preserving supermaps admit coherent realizations entirely within the framework, including a coherent quantum switch.

1 Introduction

Since the foundational work of Abramsky and Coecke [3], Categorical Quantum Mechanics (CQM) has shown that much of the structural backbone of quantum theory—tensorial composition, duality, entanglement, classical interfaces, and measurement—can be understood abstractly within dagger compact closed categories equipped with a symmetric monoidal sum [11]. In this setting, quantum processes are treated independently of any particular representation such as Hilbert spaces, and string diagrams provide a rigorous, representation-free account of their interaction and composition.

Over the past two decades, this structural viewpoint has unified diverse aspects of quantum theory. Diagrammatic calculi such as the ZX-calculus [10] give complete rewrite systems for linear maps on qubits; dagger Frobenius algebras capture observables and classical interfaces [12]; and the CPM construction [21] shows how mixed-state quantum mechanics and completely positive maps arise canonically from dagger compact structure. Together, these developments make clear that core quantum phenomena are deeply encoded in the interaction of monoidal composition, duality, and the dagger.

At the same time, a fundamental question remains unresolved: how should unitarity be understood for open or higher-order quantum processes?

Higher-order quantum processes and open interfaces. At first order, physical realizability of reversible dynamics is captured by unitarity: a closed-system evolution $f : A \rightarrow A$ is reversible precisely when $f^\dagger f = f f^\dagger = \text{id}_A$. But quantum computation extends beyond this setting. Higher-order processes such as the quantum switch [9], which coherently superposes the causal orders $f \circ g$ and $g \circ f$ under quantum control, are physically realizable and reversible, yet do not arise as unitary endomorphisms on

any single state space. This raises a fundamental question: *what is the correct notion of unitarity for higher-order quantum processes—one that captures operational reversibility beyond endomorphisms?*

Boundary semantics. We address this problem by shifting attention from syntactic generators to boundary structure. Our starting point is the free compact closed category of Kelly–Laplaza [17], in the concrete combinatorial presentation given by Abramsky [2] that separates the roles of permutations, loops, and polarities. In this presentation, objects are signed finite sets, morphisms are polarized bijections between boundary ports, and composition is given by path execution with cycle formation. Compact closed structure maps are essentially identities: all semantic content resides in how boundaries are wired and how execution propagates connections. Our development builds directly on this factorization.

This boundary-centric view yields a particularly transparent semantics for unit-free multiplicative linear logic (MLL) [14]. Each proof denotes a polarized wiring with exposed boundary ports, and cut elimination corresponds to execution without the creation of closed components. As a result, the unit-free MLL fragment is intrinsically loop-free: execution never produces scalar effects. In this sense, it isolates a purely structural notion of computation in which information flows only through boundaries.

We take Abramsky’s presentation of the Kelly–Laplaza construction as primitive, and extend it in two directions: first, by lifting from individual signed sets to finite nonempty families (obtaining a semimonoidal $\oplus$ without unit); and second, by passing to a $\mathbb{C}$ -linear completion following Abramsky’s parameterization on prescribed scalar monoids [2, §4.6]. The execution operation coincides with the feedback familiar from the Geometry of Interaction [15, 1, 4], though its role here is secondary: it serves to operationalize composition of normal forms rather than to provide the primary semantic foundation.

Additive structure without classical effects. To model control and branching while preserving reversibility, we extend this multiplicative core with a unit-free monoidal sum $\oplus$. At the structural level, this sum comes with neither injections nor projections: the permutation fragment $\text{Perm}^\pm$ admits no copying, erasure, or measurement [16]. (The ambient $\mathbb{C}$ -linear completion $\text{Perm}(\mathbb{C})$ does contain non-unitary maps, including analogues of injections; the “no classical effects” property is inherited by the quantum core $\mathbb{QC}$, whose every morphism is essentially unitary.) Rather than introducing distributivity via coherence isomorphisms [18], we work in distributed normal form: objects are finite disjoint unions of interfaces, and tensor distributes over sum strictly at the level of objects.

The resulting structural category, denoted $\text{Perm}^\pm$, is a dagger symmetric monoidal category equipped with a symmetric semimonoidal sum; compact closure holds at the summand level (within **KL**), and all higher-order moves are performed summandwise via boundary squaring (§2).

Linearity and controlled quantum evolution. To model quantum amplitudes, we pass to a $\mathbb{C}$ -linear matrix completion in which closed execution cycles contribute scalar factors (§2). This linearity is ambient: it provides a setting for superposition and exponentiation, but does not identify all linear maps as physically meaningful.

As in first-order quantum mechanics—where unitary dynamics is generated by exponentiating self-adjoint involutions such as the Pauli operators [19]—here quantum evolution is obtained by exponentiating involutive structural morphisms arising from MLL proofs. These exponentials are essentially unitary (Proposition 4.7) and generate a distinguished subcategory, the (unitary) *quantum core* $\mathbb{QC}$ (Definition 4.8), whose every morphism is essentially unitary (Theorem 4.12).

Essential unitarity. Given a morphism $f : A \rightarrow B$, the Kelly–Laplaza linking re-expresses f as a boundary operator T_f on the boundary port sets of $A^* \otimes B$. In this squared presentation, every morphism, regardless of type, is evaluated as a matrix on a single pair of port sets. Unitarity thus becomes a uniform boundary condition rather than a type-dependent notion.

We say that f is *essentially unitary* if T_f is unitary (Definition 3.12). In this squared presentation, source and target coexist within one pair of port sets, and the equation $T_f^\dagger T_f = I$ typechecks as a matrix equation independently of the shapes or polarities of A and B . Thus even for higher-order interfaces, information preservation is evaluated at the boundary level.

The construction is insensitive to currying: all presentations of the same morphism yield the same boundary operator, so the higher-order character of the interface is dissolved at the boundary. It is invariant under reindexing, compatible with composition up to scalar factors arising from closed execution cycles, and stable under multiplicative and additive structure (§3).

For first-order endomorphisms—the traditional setting of unitary quantum mechanics—essential unitarity reduces to ordinary unitarity (Remark 3.13). For higher-order morphisms, whose source and target carry mixed polarities and need not coincide, it provides a principled generalization: information is preserved at the boundary even when no global endomorphism interpretation exists.

Indeed, essential unitarity is the *unique* predicate on morphisms between uniform types—including all types generated in $\mathbb{QC}$ —that is reflecting dagger-monoidal, stable under coherence reindexing and currying, and agrees with ordinary unitarity at first order (Theorem 3.16). Uniformity (pairwise isomorphic summands) is exactly the condition under which currying can reduce a morphism to the first-order regime. No morphism in $\mathbb{QC}$ mixes summands of incommensurable boundary size, so the result applies throughout the unitary fragment of interest.

Expressiveness. As a concrete illustration, we show that for any object X , the structural endomorphisms of the sum type $X^{\oplus n}$ —generated by summand permutations, summand-wise phase taggers, and two-level rotations—give rise to a canonical internal copy of the unitary group $U(n)$ (Theorem 5.1). Under the boundary semantics, summand swaps act as self-adjoint involutions; exponentiating them generates all unitary transformations on the n -fold sum. Ordinary finite-dimensional unitary quantum mechanics is therefore not assumed but recovered structurally, uniformly in the underlying type X .

Finally, we show that $\mathbb{QC}$ is expressively complete for one-slot, equal-size, purity-preserving supermaps: every such higher-order operation admits a coherent unitary realization solely from the constructions available in $\mathbb{QC}$ (Corollary 5.4). This expressivity already suffices to realize genuinely higher-order quantum control, including a coherent quantum switch with explicit control wire (§5.3).

1.1 Related Work

Supermaps and higher-order quantum theory. The study of *supermaps*—transformations on quantum operations—originates with [7, 6, 8], which fix an operational interface (slots equipped with a causal order), characterize admissible higher-order maps, and prove representation theorems: every such map factors as conjugation by a unitary on an extended system, followed by partial trace over ancillary degrees of freedom. Subsequent work introduces higher-order processes as primitive typed objects subject to various structural constraints [20, 5, 22, 23, 24]—ranging from complete positivity and normalization to causal consistency or, conversely, to the systematic relaxation of definite causal order. In each case the approach is *from above*: one starts with an interface specification or type discipline and determines which maps are physically admissible.

The present approach. We work *from below*. The starting point is the free compact closed category of Kelly–Laplaza [17] in the concrete presentation of Abramsky [2], extended with a unit-free monoidal additive sum. Higher-order types are already present via compact closure; no separate type hierarchy is introduced. The key mechanism is that compact closure *squares* every morphism: the boundary bijection τ_f internalizes $f : A \rightarrow B$ as a matrix on the boundary port sets of $A^* \otimes B$, dissolving the distinction between first-order and higher-order interfaces at the boundary level. Essential unitarity—the requirement that the internalized operator T_f be unitary—then typechecks uniformly at all types, and is moreover invariant under currying (Remark 3.5).

This leads to a different treatment of higher-order processes. In the representation-theoretic approach, a supermap acts on density operators and its realization requires tracing out ancillary systems; the memory is consumed by partial trace. Here, supermaps are linear contexts in the quantum core $\mathbb{QC}$, and their coherent realizations preserve the memory wire explicitly—the control system remains visible rather than being traced out (Corollary 5.4). The Kelly–Laplaza construction, via compact closure, compiles higher-order structure into first-order boundary interaction; Geometry of Interaction [15, 1, 4] then provides the execution model for the resulting compositions. The present development is purely unitary; extending to measurement, partial trace, and mixed-state semantics is a direction for future work.

Structure of the paper. Section 2 introduces the polarized structural core $\text{Perm}^\pm$ and its $\mathbb{C}$ -linear completion $\text{Perm}(\mathbb{C})$. Section 3 defines essential unitarity. Section 4 constructs the quantum core $\mathbb{QC}$ and proves that every morphism in it is essentially unitary. Section 5 establishes expressiveness: internal unitary groups, causally consistent supermaps, and a coherent quantum switch with explicit control wire.

The full paper with proofs is provided as an appendix.

2 The Polarized Structural Core

We construct $\text{Perm}(\mathbb{k})$, a $\mathbb{k}$ -linear completion of the permutation category, in three stages.

The underlying combinatorics is governed by the rig groupoid **FinBij** of finite sets and bijections, equipped with disjoint union $(+, \emptyset)$ and cartesian product $(\times, \{*\})$, satisfying strict distributivity. As the biinitial rig category, **FinBij** controls the *index-level* structure of families—how objects combine under $\oplus$ and how $\otimes$ distributes—independently of any port-level data.

Polarized port structure, however, requires a generator: the objects of $\text{Perm}(\mathbb{k})$ are built from a single atomic positive interface $(1, 0)$ via duality, tensor, and finite nonempty sums.

Stage 1: The Kelly–Laplaza base. The free dagger compact closed category on one generating object is described concretely by Kelly–Laplaza [2, 17]. The generator produces the atomic positive interface $(1, 0)$; duality gives $(0, 1)$, and tensor builds all signed interfaces

$$(m, n) = (1, 0)^{\otimes m} \otimes (0, 1)^{\otimes n}.$$

Objects are thus signed finite sets $M = (M^+, M^-)$, and morphisms $f : M \rightarrow N$ are polarized bijections

$$f^\sharp : M^+ \sqcup N^- \cong N^+ \sqcup M^-$$

equipped with a loop count $\kappa \in \mathbb{N}$.

Composition is path execution with cycle formation: gluing along the shared boundary decomposes each connected component into either a path contributing to the composite wiring or a closed cycle. If $\ell(g, f)$ denotes the number of new cycles created, then

$$\kappa(g \circ f) = \kappa(f) + \kappa(g) + \ell(g, f).$$

Associativity and correctness of loop accounting follow from Kelly–Laplaza coherence [2, 17]. The dagger is inverse wiring; duality is polarity reversal: $M^* := (M^-, M^+)$.

The tensor unit is $\mathbf{1} := (\emptyset, \emptyset)$, the empty interface, arising as the nullary tensor (zero copies of the generator on each side).

Stage 2: Finite nonempty families (distributed normal form). We lift from individual interfaces to finite nonempty families, obtaining the rig structure of **FinBij** at the index level.

Definition 2.1 (Distributed objects). *An object is a finite nonempty indexed family*

$$A = \bigoplus_{i \in I} M_i, \quad M_i \in \mathbf{KL}.$$

The empty family is forbidden; thus $\oplus$ has no unit.

The additive sum $\oplus$ is concatenation of families (corresponding to $+$ on index sets); the multiplicative tensor distributes bilinearly over $\oplus$ (corresponding to $\times$ on index sets): $A \otimes B = \bigoplus_{(i,j) \in I \times J} (M_i \otimes N_j)$. At the port level, tensor is disjoint union: $(M^+, M^-) \otimes (N^+, N^-) = (M^+ \sqcup N^+, M^- \sqcup N^-)$. Duality extends pointwise to families.

Definition 2.2 (Permutation subcategory). *Let $\text{Perm}^\pm$ be the dagger category whose objects are distributed objects and whose morphisms $f : A \rightarrow B$ (with $A = \bigoplus_{i \in I} M_i$, $B = \bigoplus_{j \in J} N_j$) are permutation matrices: there is a bijection $\pi : I \rightarrow J$ and, for each $i \in I$, a polarized bijection $f_{\pi(i),i} : M_i \rightarrow N_{\pi(i)}$ in $\mathbf{KL}$; all other entries are zero. Composition reduces to entrywise KL execution; the dagger reverses wiring entry by entry and transposes the matrix.*

Remark 2.3 (Currying). A morphism $f : A \otimes B \rightarrow C$ and its curry $\tilde{f} : A \rightarrow B^* \otimes C$ share the same underlying bijection $f^\sharp$; currying merely repartitions boundary ports. This extends linearly to $\text{Perm}(\mathbb{k})$.

Stage 3: $\mathbb{k}$ -linear completion. We introduce a formal indeterminate δ to track closed execution cycles. Let $\mathbb{k} := \mathbb{C}[\delta]$; all scalars generated by the construction below lie in this polynomial ring. Later (§3), we specialize $\delta \mapsto d$ for a chosen real $d > 0$, $d \neq 1$, via the evaluation homomorphism $\mathbb{C}[\delta] \rightarrow \mathbb{C}$.

Definition 2.4 ($\mathbb{k}$ -linear completion). *For distributed objects $A = \bigoplus_{i \in I} M_i$ and $B = \bigoplus_{j \in J} N_j$, define*

$$\text{Perm}(\mathbb{k})(A, B) := \text{Mat}_{J \times I}(\mathbb{k}[\mathbf{KL}(M_i, N_j)]),$$

the $\mathbb{k}$ -vector space of $J \times I$ matrices whose entries are finite $\mathbb{k}$ -linear combinations of KL morphisms. The dagger is conjugate transpose. Composition is defined by the categorical trace: given $f : A \rightarrow B$ and $g : B \rightarrow C$,

$$g \circ f := \text{Tr}^B(f \bowtie g),$$

where $f \bowtie g$ is the stacking of f and g along the shared B -boundary, and Tr^B is the canonical trace in $\mathcal{L} := \mathbb{k}[\mathbf{KL}]$ (Proposition 2.5).

On basis elements (single KL morphisms in one matrix entry), this recovers entrywise Kelly–Laplaza composition with loop counting.

The trace can be computed as a feedback transfer function: partition the ports of $f \bowtie g$ into external (A, C) and internal (B) ; in (E, I) block form $f \bowtie g = \begin{pmatrix} H & X \\ Y & D \end{pmatrix}$ with swap S_B on the internal gluing interface, giving $g \circ f = H + X(I - S_B D)^{-1} S_B Y$. On basis morphisms the internal bounce $S_B D$ is a partial permutation, hence nilpotent on the non-cyclic components, so the inverse is a finite Neumann series; for general $\mathbb{k}$ -linear combinations, the categorical trace (Proposition 2.5) provides an algebraic formulation that is always defined.

Proposition 2.5 (Categorical trace and associativity). *The $\mathbb{k}$ -linear envelope $\mathcal{L} := \mathbb{k}[\mathbf{KL}]$ is dagger compact closed and carries a canonical trace. Composition in $\text{Perm}(\mathbb{k})$ is associative.*

The category $\text{Perm}(\mathbb{k})$ is a dagger symmetric semimonoidal category with duality and distributed normal forms; its multiplicative structure is inherited from $\mathbf{KL}$ and its additive indexing is governed by **FinBij**. After specializing $\delta \mapsto d \in \mathbb{R}_{>0} \setminus \{1\}$, we write $\text{Perm}(\mathbb{C})$ for the resulting $\mathbb{C}$ -linear category; by abuse of notation we continue to write δ for the chosen scalar d .

First-order objects.

Definition 2.6 (First-order objects). *An object $A = \bigoplus_{i \in I} M_i$ is first-order if every summand is pure-sign with the same polarity: either $M_i = (m_i, 0)$ for all i , or $M_i = (0, m_i)$ for all i . A morphism is first-order if its domain and codomain are first-order of the same polarity. Write $\mathbf{H}_n := (1, 0)^{\oplus n}$ for the first-order object of size n . In this regime there is no mixed boundary, hence no feedback; composition is purely matrix multiplication.*

3 Boundary Transport and Essential Unitarity

We proceed in three steps: first we isolate the boundary ports of a morphism; then, using the Kelly–Laplaza linking, a morphism $f : A \rightarrow B$ is internalized as an operator on the single cut object $A^* \otimes B$, obtained by retyping its polarized boundary ports; finally, we analyze how this boundary operator behaves under composition and monoidal structure, and define essential unitarity.

3.1 Boundary ports

For a signed interface $M = (M^+, M^-)$, write $\text{Pos}(M) := M^+$ and $\text{Neg}(M) := M^-$. For a distributed object $A = \bigoplus_{i \in I} M_i$, set

$$\text{Pos}(A) := \bigsqcup_{i \in I} \text{Pos}(M_i), \quad \text{Neg}(A) := \bigsqcup_{i \in I} \text{Neg}(M_i).$$

These respect $\otimes$ and $\oplus$ in the evident way, and duality reverses polarity: $\text{Pos}(A^*) = \text{Neg}(A)$ and $\text{Neg}(A^*) = \text{Pos}(A)$. Hence for any morphism $f : A \rightarrow B$,

$$\text{Pos}(A^* \otimes B) = \text{Neg}(A) \sqcup \text{Pos}(B), \quad \text{Neg}(A^* \otimes B) = \text{Pos}(A) \sqcup \text{Neg}(B).$$

Definition 3.1 (Total port set). *For a distributed object $A = \bigoplus_{i \in I} M_i$, the total port set of A is $|A| := \text{Pos}(A) \sqcup \text{Neg}(A)$.*

Definition 3.2 (Boundary port sets). *For objects A, B of $\text{Perm}^\pm$, the boundary port sets are*

$$P_{A,B} := \text{Neg}(A) \sqcup \text{Pos}(B), \quad N_{A,B} := \text{Pos}(A) \sqcup \text{Neg}(B).$$

These are finite sets; the evaluated boundary operator (Definition 3.6) maps them to the full matrix algebra $\mathbb{C}^{N_{A,B} \times P_{A,B}}$.

The port sets $P_{A,B}$ and $N_{A,B}$ coincide with $\text{Pos}(A^* \otimes B)$ and $\text{Neg}(A^* \otimes B)$ respectively. Each port retains its summand-of-origin label (from A or B), which is used by the monoidal decomposition laws (Lemma 3.10).

Remark 3.3 (Notation guide: ports and polarity). Three senses of polarity interact: (a) *interface*: $M = (M^+, M^-)$; (b) *morphism boundary*: $f : A \rightarrow B$ has domain ports $\text{Pos}(A) \sqcup \text{Neg}(B)$ and codomain ports $\text{Neg}(A) \sqcup \text{Pos}(B)$; (c) *boundary labels*: $P_{A,B} = \text{Neg}(A) \sqcup \text{Pos}(B)$ and $N_{A,B} = \text{Pos}(A) \sqcup \text{Neg}(B)$, named by their role in the cut object $A^* \otimes B$.

3.2 Boundary transport

Let $f : A \rightarrow B$ be a morphism of $\text{Perm}^\pm$ with underlying polarized wiring $f^\sharp : \text{Pos}(A) \sqcup \text{Neg}(B) \xrightarrow{\cong} \text{Pos}(B) \sqcup \text{Neg}(A)$.

Definition 3.4 (Boundary bijection). *The boundary bijection of f , $\tau_f : \text{Pos}(A^* \otimes B) \xrightarrow{\cong} \text{Neg}(A^* \otimes B)$, is the bijection*

$$\tau_f := (\text{Neg}(A) \sqcup \text{Pos}(B) \xrightarrow{\cong} \text{Pos}(B) \sqcup \text{Neg}(A) \xrightarrow{(f^\sharp)^{-1}} \text{Pos}(A) \sqcup \text{Neg}(B)),$$

where the unlabeled arrow is the canonical swap.

For distributed objects, τ_f acts blockwise: for $f \in \text{Perm}^\pm(A, B)$ with underlying permutation π , the boundary bijection τ_f maps the ports of M_i and $N_{\pi(i)}$ lying in $P_{A,B}$ (i.e. $\text{Neg}(M_i) \sqcup \text{Pos}(N_{\pi(i)})$) to the corresponding ports in $N_{A,B}$ (i.e. $\text{Pos}(M_i) \sqcup \text{Neg}(N_{\pi(i)})$), via the KL entry $f_{\pi(i),i}$. Its internalization $\underline{\tau}_f$ is the permutation matrix in $\mathbb{C}^{N_{A,B} \times P_{A,B}}$ with $(\underline{\tau}_f)_{\tau_f(p),p} = 1$.

This construction separates the structural (wiring) and the scalar (loop count) content of a morphism and dissolves the higher-order character at the boundary level.

Remark 3.5 (Axiom-link extraction and squaring). A morphism $f : A \rightarrow B$ in Kelly–Laplaza normal form consists of axiom links (the edges of $f^\sharp$) together with a loop count $\kappa(f)$. The boundary bijection τ_f extracts the axiom links, repackaging $f^\sharp$ as a bijection between the two polarities of the single cut object $A^* \otimes B$.

This *squares* the morphism, which is essential at higher types: $\text{eval} : (A^* \otimes B) \otimes A \rightarrow B$ has source and target of different shape, yet τ_{eval} reduces to two bundles of straight wires on $((A^* \otimes B) \otimes A)^* \otimes B \cong A \otimes (B^* \otimes A^*) \otimes B$.

Moreover, τ is invariant under currying: $f : A \otimes B \rightarrow C$ and its curry $\tilde{f} : A \rightarrow B^* \otimes C$ share the same underlying bijection $f^\sharp$ (Remark 2.3) and the same cut object $A^* \otimes B^* \otimes C$, so $\tau_f = \tau_{\tilde{f}}$.

3.3 The evaluated boundary operator

Fix a real scalar $\delta > 0$, $\delta \neq 1$; this ensures that any nonzero loop count contributes a detectable scalar factor.

We extract from each morphism in $\text{Perm}(\mathbb{C})$ a matrix on the boundary port sets.

Definition 3.6 (Evaluated boundary operator). *The evaluated boundary operator assigns to each morphism $f : A \rightarrow B$ in $\text{Perm}(\mathbb{C})$ a matrix $T_f \in \mathbb{C}^{N_{A,B} \times P_{A,B}}$ by $\mathbb{C}$ -linear extension from KL matrix units. A KL matrix unit $e \in \text{Perm}(\mathbb{C})(A, B)$ has a single KL morphism $g : M_i \rightarrow N_j$ in entry (j, i) and zeros elsewhere; define $T_e := \delta^{\kappa(g)} \cdot \underline{\tau}_g$, the partial permutation matrix internalizing the boundary bijection τ_g on the ports of M_i and N_j (and zero elsewhere), scaled by $\delta^{\kappa(g)}$. For $u = \sum_r \alpha_r e_r$, set $T_u := \sum_r \alpha_r T_{e_r}$.*

Remark 3.7 (Computing T in practice). On a permutation matrix $f \in \text{Perm}^\pm(A, B)$, $T_f = \delta^{\kappa(f)} \cdot \underline{\tau}_f$; on a monomial $[\lambda] \circ f$, each block is further scaled by $\lambda_{\pi(i)}$. On $\mathbb{C}$ -linear combinations, T extends by linearity.

The assignment $f \mapsto T_f$ is $\mathbb{C}$ -linear on hom-spaces, compatible with dagger ($T_{f^\dagger} = (T_f)^\dagger$, Lemma 3.11), and respects the monoidal structure ($T_{f \oplus g} = T_f \oplus T_g$, $T_{f \otimes g} = T_f \otimes T_g$, Lemma 3.10). On the loop-free fragment, T preserves the categorical trace (Lemma 4.11).

3.4 Composition law on basis morphisms and the scalar anomaly

On basis morphisms, the feedback transfer function decomposes into a path component and a cycle-counting scalar. This decomposition is useful for the structural fragment (§4) but is not needed for the general closure argument.

Proposition 3.8 (Composition law for τ and κ [2, Proposition 1]). *Let $f : M \rightarrow N$ and $g : N \rightarrow P$ in KL. Then $\tau_{g \circ f}$ is the path component and $\ell(g, f)$ the cycle count of the Kelly–Laplaza execution graph obtained by gluing τ_f and τ_g along the shared N -boundary, and $\kappa(g \circ f) = \kappa(f) + \kappa(g) + \ell(g, f)$.*

Corollary 3.9 (Scalar anomaly on basis morphisms). *For $f : A \rightarrow B$ and $g : B \rightarrow C$ in $\text{Perm}^\pm$,*

$$T_{g \circ f} = \delta^{\ell(g, f)} \cdot (T_g \odot T_f),$$

where $T_g \odot T_f$ is the internalized path component and $\delta^{\ell(g, f)}$ the cycle factor. On the acyclic structural fragment ($\ell = 0$), feedback coincides with $\odot$.

3.5 Dagger (bi)monoidal interaction of the boundary operator

Lemma 3.10 (Monoidal laws for τ , κ , and T). *For $f : A \rightarrow B$ and $g : C \rightarrow D$ in $\text{Perm}^\pm$, the boundary port sets of $f \oplus g$ decompose as $P_{A \oplus C, B \oplus D} = P_{A, B} \sqcup P_{C, D}$ and $N_{A \oplus C, B \oplus D} = N_{A, B} \sqcup N_{C, D}$, and under this identification*

$$\tau_{f \oplus g} = \tau_f \sqcup \tau_g, \quad \kappa(f \oplus g) = \kappa(f) + \kappa(g).$$

Similarly, under $(A \otimes C)^* \otimes (B \otimes D) \cong (A^* \otimes B) \otimes (C^* \otimes D)$,

$$\tau_{f \otimes g} = \tau_f \sqcup \tau_g, \quad \kappa(f \otimes g) = \kappa(f) + \kappa(g).$$

If $\kappa(f) = \kappa(g) = 0$, internalizing gives $T_{f \oplus g} = T_f \oplus T_g$ and $T_{f \otimes g} = T_f \otimes T_g$.

Lemma 3.11 (Dagger). *For any $f : A \rightarrow B$ in $\text{Perm}^\pm$, $\kappa(f^\dagger) = \kappa(f)$, $\tau_{f^\dagger} = \tau_f^{-1}$, and $T_{f^\dagger} = (T_f)^\dagger$.*

3.6 Essential unitarity

We now formulate essential unitarity as an *internal* unitarity condition on the squared boundary operator.

Definition 3.12 (Essential unitarity). *A morphism $f : A \rightarrow B$ in $\text{Perm}(\mathbb{C})$ is essentially unitary (EU) if*

$$T_f^\dagger T_f = I_{P_{A, B}} \quad \text{and} \quad T_f T_f^\dagger = I_{N_{A, B}},$$

where products and dagger (conjugate transpose) are computed in $\mathbb{C}^{N_{A, B} \times P_{A, B}}$.

The definition is δ -independent: the assumption $\delta > 0$, $\delta \neq 1$ enters only in that EU forces loop-freeness on basis morphisms: since $T_{h_r} = \delta^{\kappa(h_r)} \tau_{h_r}$, unitarity of T_f forces $|\delta^\kappa| = 1$, which for real $\delta > 0$, $\delta \neq 1$ requires $\kappa = 0$. (Remark 4.3). Every structural isomorphism is EU (its T_f is a permutation matrix), and EU is insensitive to currying since $T_f = T_{\tilde{f}}$ (Remark 3.5). Cf. Selinger's structural unitaries [21].

Remark 3.13 (EU coincides with unitarity at first order). For a first-order morphism $f : (n, 0) \rightarrow (n, 0)$ (Definition 2.6), the KL linking is a bijection $\{1, \dots, n\} \rightarrow \{1, \dots, n\}$ —an ordinary permutation with no mixed-polarity ports. Since execution of $f^\dagger \circ f$ involves only positive ports, no closed components can form: $\ell(f^\dagger, f) = 0$ and $\kappa(f^\dagger \circ f) = 2\kappa(f)$. EU forces $\kappa(f) = 0$, so $T_f = \tau_f$ is the permutation matrix of f . Hence EU and internal dagger unitarity coincide on first-order morphisms.

3.7 Uniqueness

We now show that any predicate compatible with dagger-monoidal structure, coherence, and currying coincides with essential unitarity.

Ordinary unitarity on first-order types already satisfies (R1) and (R2): tensor, dagger and sum of unitaries are unitary (and conversely), and coherence reindexing is invisible to unitarity. Axiom (R4) anchors any candidate predicate to this known case.

The only genuinely new constraint is (R3). A higher-order process $f : A \otimes B \rightarrow C$ and its curry $\tilde{f} : A \rightarrow B^* \otimes C$ describe the same process at different types (Remark 2.3); a predicate that accepted one but rejected the other would artificially restrict which higher-order processes count as unitary. Currying invariance ensures the lifted notion is as expressive as the type system allows.

Definition 3.14 (Uniform type). A distributed object $A = \bigoplus_{i \in I} M_i$ of $\text{Perm}^\pm$ is uniform if all summands share the same interface, i.e. $A \cong X^{\oplus n}$ for some single KL interface $X = (m^+, m^-)$ and $n \geq 1$.

Definition 3.15 (Interface-decomposable morphism). $u : A \rightarrow B$ in $\text{Perm}(\mathbb{C})$ is interface-decomposable if it admits a decomposition $u \cong u_1 \oplus \dots \oplus u_r$ where each u_k has uniform source and uniform target.

Uniform types include every single interface (the $n = 1$ case), every first-order object of the form $(p, 0)^{\oplus n}$ (in particular $\mathbf{H}_n = (1, 0)^{\oplus n}$), and every object $X^{\oplus n}$ appearing in §5. Interface-decomposability is preserved by currying (which acts on each summand independently), by dagger (which preserves each block), and by composition (the intermediate type's interface decomposition constrains both factors).

The theorem below shows that these constraints leave no freedom: EU is the unique extension of first-order unitarity to interface-decomposable morphisms.

Theorem 3.16 (Uniqueness). Let R be a predicate on morphisms of $\text{Perm}(\mathbb{C})$ satisfying:

- (R1) (Reflecting dagger-monoidal) $R(u \otimes v) \Leftrightarrow R(u) \wedge R(v)$, $R(u \oplus v) \Leftrightarrow R(u) \wedge R(v)$, $R(u^\dagger) \Leftrightarrow R(u)$, and $R(\text{id}_A)$.
 - (R2) (Structural invariance) If s, t are $\otimes/\oplus$ coherence or distributivity isomorphisms, $R(u) \Rightarrow R(s \circ u \circ t)$.
 - (R3) (Currying invariance) $R(u) \Leftrightarrow R(\tilde{u})$ whenever defined.
 - (R4) (First-order unitarity) For first-order u in $\text{Perm}(\mathbb{C})$, $R(u) \Leftrightarrow u^\dagger u = \text{id} \wedge u u^\dagger = \text{id}$.
- Then for every interface-decomposable morphism u , $R(u) = \text{EU}(u)$.

4 The Quantum Core

The *quantum core* $\mathbb{Q}\mathbb{C}$ is built in three stages, moving from pure structure, to reversible control, to coherent dynamics: unit-free MLL (acyclicity gives loop-free permutation matrices, hence trivially EU); addition of $\oplus$ (correctness lifts componentwise); exponentials of structural involutions in the $\mathbb{C}$ -linear

completion (unitarity is purely algebraic). The construction is unit-free by necessity; units cause scalar loops that are fatal to invertibility.

4.1 The structural fragment

Definition 4.1 (The categories $\text{Perm}_{\text{MLL}}^\pm$ and $\text{Perm}_{\text{MLL}(+)}^\pm$). *Let $\text{Perm}_{\text{MLL}}^\pm$ be the subcategory of $\text{Perm}^\pm$ whose objects are signed interfaces (m, n) and whose morphisms are those obtained by interpreting unit-free MLL proofs in the $*$ -autonomous structure of **KL** (every compact closed category is $*$ -autonomous, so this interpretation is canonical). Let $\text{Perm}_{\text{MLL}(+)}^\pm$ be the smallest subcategory of $\text{Perm}^\pm$ containing $\text{Perm}_{\text{MLL}}^\pm$ and closed under $\oplus$ on objects and morphisms, $\otimes$, dual, and composition. The free addition of $\oplus$ preserves acyclicity: every morphism in $\text{Perm}_{\text{MLL}(+)}^\pm$ decomposes blockwise into $\text{Perm}_{\text{MLL}}^\pm$ components and structural reindexings.*

Corollary 4.2 (EU for $\text{Perm}_{\text{MLL}(+)}^\pm$). *Every $f \in \text{Perm}_{\text{MLL}(+)}^\pm$ is essentially unitary.*

Every unit-free MLL proof denotes a linking with $\kappa(f) = 0$: cut elimination preserves acyclicity, so feedback creates no closed components. Hence $T_f = \tau_f$ is a permutation matrix, therefore unitary. Since $\oplus$ -extension and structural reindexings preserve the block-sum structure of T (Lemma 3.10), the result lifts to all of $\text{Perm}_{\text{MLL}(+)}^\pm$.

Remark 4.3 (Homogeneous loop degree in $\mathbb{QC}$). Every generator of $\mathbb{QC}$ is a $\mathbb{C}$ -linear combination of $\text{MLL}(+)$ morphisms, each with $\kappa = 0$. Since $\mathbb{QC}$ is built from these generators by operations that preserve loop-freeness (Lemma 4.4, Proposition 4.7), every $\mathbb{QC}$ morphism has all its $\text{MLL}(+)$ components at loop degree zero. In particular, EU cannot be achieved by cancellation between terms of different loop degree; the scalar δ^κ is uniformly 1 throughout $\mathbb{QC}$.

Lemma 4.4 (No new loops in $\text{MLL}(+)$). *For composable morphisms $f, g \in \text{Perm}_{\text{MLL}(+)}^\pm$, execution creates no new cycle components: $\ell(g, f) = 0$.*

Corollary 4.5 (Execution product on $\text{MLL}(+)$). *For composable $f, g \in \text{Perm}_{\text{MLL}(+)}^\pm$, the scalar anomaly vanishes: $T_{g \circ f} = T_g \odot T_f$. That is, transported composition coincides with the GOI execution product on the structural fragment.*

4.2 Exponentials of involutions

This is the point at which coherent quantum evolution enters.

Definition 4.6 (Exponential generators). *Let $S \in \text{Perm}_{\text{MLL}(+)}^\pm(A, A)$ be an involution ($S \circ S = \text{id}_A$). For $\theta \in \mathbb{R}$, define $\exp_i(\theta, S) := \cos \theta \text{id}_A + i \sin \theta S$ in $\text{Perm}(\mathbb{C})$.*

Any structural involution in the $\text{MLL}(+)$ image qualifies, including higher-order reflections on compound interfaces, so the construction generalizes qubit rotations to the full distributed setting. Since $S^2 = \text{id}$ forces each cross-summand entry to be an isomorphism, and isomorphic KL interfaces have equal port counts, $\exp_i(\theta, S)$ always acts on a uniform type (Definition 3.14).

Proposition 4.7 (Essential unitarity of exponentials). *For every involution $S \in \text{Perm}_{\text{MLL}(+)}^\pm(A, A)$ and angle θ , $\exp_i(\theta, S)$ is essentially unitary.*

4.3 The quantum core

Combining structure, reversible control, and coherent rotations yields the quantum core.

Definition 4.8 (The quantum core $\mathbb{QC}$). *Let $\mathbb{QC}$ be the smallest dagger subcategory of $\text{Perm}(\mathbb{C})$ closed under $\otimes$, $\oplus$, dual, and composition, containing the $\text{MLL}(+)$ image and, for every structural involution S in that image and every $\theta \in \mathbb{R}$, the exponential $\exp_i(\theta, S)$.*

The quantum core is a unit-free dagger SMCC: evaluation $\text{ev}_{B,C}$ and currying are available, but the coevaluation $\eta_A : \mathbf{1} \rightarrow A^* \otimes A$ is excluded.

Theorem 4.9 (Unit-free dagger SMCC structure of $\mathbb{QC}$). *The quantum core $\mathbb{QC}$ is a dagger symmetric monoidal closed category (without tensor unit; that is, a symmetric biclosed category in the sense of [13]). Its internal hom is $B \multimap C := B^* \otimes C$, and evaluation is given by the canonical structural wiring $\text{ev}_{B,C} : (B^* \otimes C) \otimes B \rightarrow C$.*

The boundary operator T does not preserve the categorical trace in general: newly formed execution cycles contribute a scalar δ^ℓ that is visible to T but invisible to the matrix-level feedback (Corollary 3.9). We isolate the hypothesis under which this discrepancy vanishes, satisfied by $\mathbb{QC}$.

Definition 4.10 (Loop-free under B -trace). *Let $h : A \otimes B \rightarrow C \otimes B$ in $\text{Perm}(\mathbb{C})$ with presentation $h = \sum_r \gamma_r h_r$ as a finite $\mathbb{C}$ -linear combination of basis morphisms. h is loop-free under B -trace if $\ell_B(h_r) = 0$ for every r with $\gamma_r \neq 0$, where $\ell_B(h_r)$ counts the new cycles created by tracing h_r over B .*

Lemma 4.11 (Trace preservation of T on loop-free traces). *Let $h : A \otimes B \rightarrow C \otimes B$ be loop-free under B -trace. Then $T_{\text{Tr}^B(h)} = \text{Tr}^{|B|}((\text{id} \oplus S_B) T_h)$, where $\text{Tr}^{|B|}$ is the canonical JSV trace in $\text{Mat}(\mathbb{C})$, and S_B is the polarity reconciliation swap identifying the internal B -port rows ($\text{Neg}(B) \sqcup \text{Pos}(B)$) with the internal B -port columns ($\text{Pos}(B) \sqcup \text{Neg}(B)$).*

Theorem 4.12 (EU closure for $\mathbb{QC}$). *Every morphism of $\mathbb{QC}$ is essentially unitary.*

5 Internal Unitary Actions

We show that $\mathbb{QC}$ is expressively complete for coherent higher-order quantum computation.

5.1 Internal $\text{U}(n)$ on $X^{\oplus n}$

The summand-level endomorphism algebra of $X^{\oplus n}$ is canonically isomorphic to $M_n(\mathbb{C})$, with summand permutations acting as permutation matrices and summandwise phases as diagonal unitaries (each phase $e^{i\theta_k} \text{id}_X = \exp_i(\theta_k, \text{id}_X)$ is in $\mathbb{QC}$; $\oplus$ -closure gives arbitrary diagonal unitaries). Exponentiating the additive swap $S_X : X \oplus X \rightarrow X \oplus X$ generates two-level rotations; by the Givens factorization, these together with diagonal unitaries generate all of $\text{U}(n)$. These standard generators are each realised by morphisms in $\mathbb{QC}$, so the generation fact lifts to an internal unitary action.

Theorem 5.1 (Internal $\text{U}(n)$ on $X^{\oplus n}$). *For $n \geq 1$ and any object X , the endomorphism algebra $\text{End}_{\text{Perm}(\mathbb{C})}(X^{\oplus n})$ contains a canonical internal copy of $\text{U}(n)$, acting unitarily on the n summand copies of X and realised entirely by morphisms of $\mathbb{QC}$.*

In contrast, $X \otimes X$ supports only the abelian $\text{U}(1)$ below.

Twist-generated $U(1)$ on $X \otimes X$. Symmetry $\sigma_{X,X} : X \otimes X \rightarrow X \otimes X$ is a Hermitian involution in $\mathbb{QC}$.

Theorem 5.2 (Twist-generated $U(1)$). *For every object X , the one-parameter family $\theta \mapsto \exp_i(\theta, \sigma_{X,X}) = \cos \theta \text{id} + i \sin \theta \sigma_{X,X}$ defines an internal unitary $U(1)$ representation inside $\text{End}_{\text{Perm}(\mathbb{C})}(X \otimes X)$, realised entirely within $\mathbb{QC}$. This action is abelian; no nonabelian Lie action can be generated from a single Hermitian involution (the generated Lie algebra is one-dimensional).*

$\theta = \frac{\pi}{2n}$ shows that this one-parameter subgroup contains n -th roots of $\sigma_{X,X}$ up to global phase.

5.2 Contexts as supermaps

Because each hole occurs exactly once, alinear context Φ_t cannot duplicate or discard an argument.

Theorem 5.3 (Linear contexts induce supermaps). *Let $\mathcal{U}$ be any class of EU morphisms in $\mathbb{QC}$. An n -hole linear context is a well-typed judgment $\square_1 : A_1 \rightarrow B_1, \dots, \square_n : A_n \rightarrow B_n \vdash t : C \rightarrow D$ with $t ::= \square_i \mid u \mid (t_2 \circ t_1) \mid (t_1 \otimes t_2) \mid (t_1 \oplus t_2)$, $u \in \mathcal{U}$, and each $\square_i$ occurring exactly once. Then t induces a map $\Phi_t : \prod_i \text{Perm}(\mathbb{C})(A_i, B_i) \rightarrow \text{Perm}(\mathbb{C})(C, D)$ that is multilinear (in the ambient $\mathbb{C}$ -linear category $\text{Perm}(\mathbb{C})$) and unitary-preserving: if each argument is EU, so is the result.*

Thus, every $\mathbb{QC}$ -linear context defines a multilinear, EU-preserving, causally consistent supermap—consistency is enforced by linear syntax rather than external axioms. $\mathbb{QC}$ itself is not $\mathbb{C}$ -linear (arbitrary linear combinations of EU morphisms need not be EU); multilinearity refers to the ambient $\text{Perm}(\mathbb{C})$.

5.3 The coherent quantum switch

We exhibit the quantum switch as a coherent higher-order morphism in $\mathbb{QC}$, parametric in a control type. Unlike the extensional formulation of [9], in which the control system is traced out after use, the present construction retains the control wire throughout.

Fix an object A and let $U_A := A^* \otimes A$. For any object X , define the *controlled switch* $\text{QSwitch}_X : (X \oplus X) \otimes U_A \otimes U_A \rightarrow (X \oplus X) \otimes U_A$ summandwise by $\text{QSwitch}_X := (\text{id}_X \otimes \text{comp}_A) \oplus (\text{id}_X \otimes (\text{comp}_A \circ \sigma_{U_A, U_A}))$, where $\text{comp}_A : U_A \otimes U_A \rightarrow U_A$ is internal composition of endomorphisms (currying $\text{ev}_{A,A} \circ (\text{id}_{U_A} \otimes \text{ev}_{A,A})$), and σ_{U_A, U_A} is the symmetry. In the first (resp. second) summand, the inputs are composed as $f \circ g$ (as $g \circ f$). The control wire $X \oplus X$ is preserved coherently.

Every component— comp_A , σ , id_X , and the $\oplus$ -decomposition—is a structural map of $\text{Perm}_{\text{MLL}(+)}^\pm$, so QSwitch_X is EU by Corollary 4.2 and lies in $\mathbb{QC}$.

Quantum behaviour arises from the control wire. By Theorem 5.1, $X \oplus X$ carries an internal $SU(2)$ action; preparing the control in a superposition of summands (via a rotation $\exp_i(\theta, S_X)$) before applying QSwitch_X yields a coherent superposition of the two causal orders $f \circ g$ and $g \circ f$. The switch itself is purely structural wiring; the quantum content resides entirely in the preparation of the control.

Definability of purity-preserving supermaps. The coherent quantum switch exemplifies a general principle: programs in $\mathbb{QC}$ define coherent higher-order operations on their holes, inducing purity-preserving supermaps with explicit memory. The converse holds for equal-size supermaps:

Corollary 5.4 (Definability of equal-size purity-preserving supermaps). *Let $\Phi : (A \rightarrow A) \rightarrow (B \rightarrow B)$ be a purity-preserving supermap in the sense of [8]. Assume that A and B have the same finite size, i.e. the same number of $\oplus$ -summands. Then Φ admits a coherent unitary realization in $\mathbb{QC}$: there exist ancillary systems M, N of equal size and unitaries $V_1 : B \otimes M \rightarrow A \otimes N$ and $V_2 : A \otimes N \rightarrow B \otimes M$ such that $\Phi^\sharp(f) := V_2 \circ (f \otimes \text{id}_N) \circ V_1 : B \otimes M \rightarrow B \otimes M$ is definable in $\mathbb{QC}$ for every $f : A \rightarrow A$.*

Purity-preserving comb realizations are thus interpreted by EU morphisms, and EU is stable under the linear contexts that define supermaps.

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Essential Unitarity for Higher-Order Quantum Computation

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We develop a semantic framework for higher-order quantum computation based on a boundary-centric presentation of compact closed categories, building on the Kelly–Laplaza construction as reformulated by Abramsky. Morphisms are represented as polarized boundary linkings composed by execution, and a unit-free monoidal sum provides reversible control and branching.

Within this setting, we identify a notion of *essential unitarity* that extends ordinary unitary evolution from first-order quantum processes to higher-order interfaces. At first order, essential unitarity coincides with standard unitarity; for higher-order interfaces, it characterizes when information is preserved relative to the boundary. We prove that essential unitarity is the unique predicate compatible with dagger-monoidal structure, coherence reindexing, and currying, and agreeing with ordinary unitarity at first order, for all morphisms that coherently mix uniform signature blocks—a condition satisfied throughout the quantum core.

We show that this framework is expressively sufficient for higher-order quantum computation. In particular, one-slot, equal-size, purity-preserving supermaps admit coherent realizations entirely within the framework, including a coherent quantum switch.

A Introduction

Since the foundational work of Abramsky and Coecke [3], Categorical Quantum Mechanics (CQM) has shown that much of the structural backbone of quantum theory—tensorial composition, duality, entanglement, classical interfaces, and measurement—can be understood abstractly within dagger compact closed categories equipped with a symmetric monoidal sum [12]. In this setting, quantum processes are treated independently of any particular representation such as Hilbert spaces, and string diagrams provide a rigorous, representation-free account of their interaction and composition.

Over the past two decades, this structural viewpoint has unified diverse aspects of quantum theory. Diagrammatic calculi such as the ZX-calculus [11] give complete rewrite systems for linear maps on qubits; dagger Frobenius algebras capture observables and classical interfaces [13]; and the CPM construction [28] shows how mixed-state quantum mechanics and completely positive maps arise canonically from dagger compact structure. Together, these developments make clear that core quantum phenomena are deeply encoded in the interaction of monoidal composition, duality, and the dagger.

At the same time, a fundamental question remains unresolved: how should unitarity be understood for open or higher-order quantum processes?

Higher-order quantum processes and open interfaces. At first order, physical realizability of reversible dynamics is captured by unitarity: a closed-system evolution $f : A \rightarrow A$ is reversible precisely when $f^\dagger f = f f^\dagger = \text{id}_A$. But quantum computation extends beyond this setting. Higher-order processes such as the quantum switch [10], which coherently superposes the causal orders $f \circ g$ and $g \circ f$ under quantum control, are physically realizable and reversible, yet do not arise as unitary endomorphisms on

any single state space. This raises a fundamental question: *what is the correct notion of unitarity for higher-order quantum processes—one that captures operational reversibility beyond endomorphisms?*

Boundary semantics. We address this problem by shifting attention from syntactic generators to boundary structure. Our starting point is the free compact closed category of Kelly–Laplaza [24], in the concrete combinatorial presentation given by Abramsky [2] that separates the roles of permutations, loops, and polarities. In this presentation, objects are signed finite sets, morphisms are polarized bijections between boundary ports, and composition is given by path execution with cycle formation. Compact closed structure maps are essentially identities: all semantic content resides in how boundaries are wired and how execution propagates connections. Our development builds directly on this factorization.

This boundary-centric view yields a particularly transparent semantics for unit-free multiplicative linear logic (MLL) [18]. Each proof denotes a polarized wiring with exposed boundary ports, and cut elimination corresponds to execution without the creation of closed components. As a result, the unit-free MLL fragment is intrinsically loop-free: execution never produces scalar effects. In this sense, it isolates a purely structural notion of computation in which information flows only through boundaries.

We take Abramsky’s presentation of the Kelly–Laplaza construction as primitive, and extend it in two directions: first, by lifting from individual signed sets to finite nonempty families (obtaining a semimonoidal $\oplus$ without unit); and second, by passing to a $\mathbb{C}$ -linear completion following Abramsky’s parameterization on prescribed scalar monoids [2, §4.6]. The execution operation coincides with the feedback familiar from the Geometry of Interaction [19, 1, 4], though its role here is secondary: it serves to operationalize composition of normal forms rather than to provide the primary semantic foundation.

Additive structure without classical effects. To model control and branching while preserving reversibility, we extend this multiplicative core with a unit-free monoidal sum $\oplus$. At the structural level, this sum comes with neither injections nor projections: the permutation fragment $\text{Perm}^\pm$ admits no copying, erasure, or measurement [22]. (The ambient $\mathbb{C}$ -linear completion $\text{Perm}(\mathbb{C})$ does contain non-unitary maps, including analogues of injections; the “no classical effects” property is inherited by the quantum core $\mathbb{QC}$, whose every morphism is essentially unitary.) Rather than introducing distributivity via coherence isomorphisms [25], we work in distributed normal form: objects are finite disjoint unions of interfaces, and tensor distributes over sum strictly at the level of objects.

The resulting structural category, denoted $\text{Perm}^\pm$, is a dagger symmetric monoidal category equipped with a symmetric semimonoidal sum; compact closure holds at the summand level (within **KL**), and all higher-order moves are performed summandwise via boundary squaring (§B).

Linearity and controlled quantum evolution. To model quantum amplitudes, we pass to a $\mathbb{C}$ -linear matrix completion in which closed execution cycles contribute scalar factors (§B). This linearity is ambient: it provides a setting for superposition and exponentiation, but does not identify all linear maps as physically meaningful.

As in first-order quantum mechanics—where unitary dynamics is generated by exponentiating self-adjoint involutions such as the Pauli operators [26]—here quantum evolution is obtained by exponentiating involutive structural morphisms arising from MLL proofs. These exponentials are essentially unitary (Proposition D.8) and generate a distinguished subcategory, the (unitary) *quantum core* $\mathbb{QC}$ (Definition D.9), whose every morphism is essentially unitary (Theorem D.14).

Essential unitarity. Given a morphism $f : A \rightarrow B$, the Kelly–Laplaza linking re-expresses f as a boundary operator T_f on the boundary port sets of $A^* \otimes B$. In this squared presentation, every morphism, regardless of type, is evaluated as a matrix on a single pair of port sets. Unitarity thus becomes a uniform boundary condition rather than a type-dependent notion.

We say that f is *essentially unitary* if T_f is unitary (Definition C.13). In this squared presentation, source and target coexist within one pair of port sets, and the equation $T_f^\dagger T_f = I$ typechecks as a matrix equation independently of the shapes or polarities of A and B . Thus even for higher-order interfaces, information preservation is evaluated at the boundary level.

The construction is insensitive to currying: all presentations of the same morphism yield the same boundary operator, so the higher-order character of the interface is dissolved at the boundary. It is invariant under reindexing, compatible with composition up to scalar factors arising from closed execution cycles, and stable under multiplicative and additive structure (§C).

For first-order endomorphisms—the traditional setting of unitary quantum mechanics—essential unitarity reduces to ordinary unitarity (Remark C.14). For higher-order morphisms, whose source and target carry mixed polarities and need not coincide, it provides a principled generalization: information is preserved at the boundary even when no global endomorphism interpretation exists.

Indeed, essential unitarity is the *unique* predicate on morphisms between uniform types—including all types generated in $\mathbb{QC}$ —that is reflecting dagger-monoidal, stable under coherence reindexing and currying, and agrees with ordinary unitarity at first order (Theorem C.17). Uniformity (pairwise isomorphic summands) is exactly the condition under which currying can reduce a morphism to the first-order regime. No morphism in $\mathbb{QC}$ mixes summands of incommensurable boundary size, so the result applies throughout the unitary fragment of interest.

Expressiveness. As a concrete illustration, we show that for any object X , the structural endomorphisms of the sum type $X^{\oplus n}$ —generated by summand permutations, summand-wise phase taggers, and two-level rotations—give rise to a canonical internal copy of the unitary group $U(n)$ (Theorem E.3). Under the boundary semantics, summand swaps act as self-adjoint involutions; exponentiating them generates all unitary transformations on the n -fold sum. Ordinary finite-dimensional unitary quantum mechanics is therefore not assumed but recovered structurally, uniformly in the underlying type X .

Finally, we show that $\mathbb{QC}$ is expressively complete for one-slot, equal-size, purity-preserving supermaps: every such higher-order operation admits a coherent unitary realization solely from the constructions available in $\mathbb{QC}$ (Corollary E.6). This expressivity already suffices to realize genuinely higher-order quantum control, including a coherent quantum switch with explicit control wire (§E.3).

A.1 Related Work

Supermaps and higher-order quantum theory. The study of *supermaps*—transformations on quantum operations—originates with [8, 7, 9], which fix an operational interface (slots equipped with a causal order), characterize admissible higher-order maps, and prove representation theorems: every such map factors as conjugation by a unitary on an extended system, followed by partial trace over ancillary degrees of freedom. Subsequent work introduces higher-order processes as primitive typed objects subject to various structural constraints [27, 5, 29, 30, 31]—ranging from complete positivity and normalization to causal consistency or, conversely, to the systematic relaxation of definite causal order. In each case the approach is *from above*: one starts with an interface specification or type discipline and determines which maps are physically admissible.

The present approach. We work *from below*. The starting point is the free compact closed category of Kelly–Laplaza [24] in the concrete presentation of Abramsky [2], extended with a unit-free monoidal additive sum. Higher-order types are already present via compact closure; no separate type hierarchy is introduced. The key mechanism is that compact closure *squares* every morphism: the boundary bijection τ_f internalizes $f : A \rightarrow B$ as a matrix on the boundary port sets of $A^* \otimes B$, dissolving the distinction between first-order and higher-order interfaces at the boundary level. Essential unitarity—the requirement that the internalized operator T_f be unitary—then typechecks uniformly at all types, and is moreover invariant under currying (Remark C.5).

This leads to a different treatment of higher-order processes. In the representation-theoretic approach, a supermap acts on density operators and its realization requires tracing out ancillary systems; the memory is consumed by partial trace. Here, supermaps are linear contexts in the quantum core $\mathbb{QC}$, and their coherent realizations preserve the memory wire explicitly—the control system remains visible rather than being traced out (Corollary E.6). The Kelly–Laplaza construction, via compact closure, compiles higher-order structure into first-order boundary interaction; Geometry of Interaction [19, 1, 4] then provides the execution model for the resulting compositions. The present development is purely unitary; extending to measurement, partial trace, and mixed-state semantics is a direction for future work.

Structure of the paper. Section B introduces the polarized structural core $\text{Perm}^\pm$ and its $\mathbb{C}$ -linear completion $\text{Perm}(\mathbb{C})$. Section C defines essential unitarity. Section D constructs the quantum core $\mathbb{QC}$ and proves that every morphism in it is essentially unitary. Section E establishes expressiveness: internal unitary groups, causally consistent supermaps, and a coherent quantum switch with explicit control wire.

B The Polarized Structural Core

We construct $\text{Perm}(\mathbb{k})$, a $\mathbb{k}$ -linear completion of the permutation category, in three stages.

The underlying combinatorics is governed by the rig groupoid **FinBij** of finite sets and bijections, equipped with disjoint union $(+, \emptyset)$ and cartesian product $(\times, \{*\})$, satisfying strict distributivity. As the biinitial rig category, **FinBij** controls the *index-level* structure of families—how objects combine under $\oplus$ and how $\otimes$ distributes—independently of any port-level data.

Polarized port structure, however, requires a generator: the objects of $\text{Perm}(\mathbb{k})$ are built from a single atomic positive interface $(1, 0)$ via duality, tensor, and finite nonempty sums.

Stage 1: The Kelly–Laplaza base. The free dagger compact closed category on one generating object is described concretely by Kelly–Laplaza [2, 24]. The generator produces the atomic positive interface $(1, 0)$; duality gives $(0, 1)$, and tensor builds all signed interfaces

$$(m, n) = (1, 0)^{\otimes m} \otimes (0, 1)^{\otimes n}.$$

Objects are thus signed finite sets $M = (M^+, M^-)$, and morphisms $f : M \rightarrow N$ are polarized bijections

$$f^\sharp : M^+ \sqcup N^- \cong N^+ \sqcup M^-$$

equipped with a loop count $\kappa \in \mathbb{N}$.

Composition is path execution with cycle formation: gluing along the shared boundary decomposes each connected component into either a path contributing to the composite wiring or a closed cycle. If $\ell(g, f)$ denotes the number of new cycles created, then

$$\kappa(g \circ f) = \kappa(f) + \kappa(g) + \ell(g, f).$$

Associativity and correctness of loop accounting follow from Kelly–Laplaza coherence [2, 24]. The dagger is inverse wiring; duality is polarity reversal: $M^* := (M^-, M^+)$.

The tensor unit is $\mathbf{1} := (\emptyset, \emptyset)$, the empty interface, arising as the nullary tensor (zero copies of the generator on each side).

Stage 2: Finite nonempty families (distributed normal form). We lift from individual interfaces to finite nonempty families, obtaining the rig structure of **FinBij** at the index level.

Definition B.1 (Distributed objects). *An object is a finite nonempty indexed family*

$$A = \bigoplus_{i \in I} M_i, \quad M_i \in \mathbf{KL}.$$

The empty family is forbidden; thus $\oplus$ has no unit.

The additive sum $\oplus$ is concatenation of families (corresponding to $+$ on index sets); the multiplicative tensor distributes bilinearly over $\oplus$ (corresponding to $\times$ on index sets): $A \otimes B = \bigoplus_{(i,j) \in I \times J} (M_i \otimes N_j)$. At the port level, tensor is disjoint union: $(M^+, M^-) \otimes (N^+, N^-) = (M^+ \sqcup N^+, M^- \sqcup N^-)$. Duality extends pointwise to families.

Definition B.2 (Permutation subcategory). *Let $\text{Perm}^\pm$ be the dagger category whose objects are distributed objects and whose morphisms $f : A \rightarrow B$ (with $A = \bigoplus_{i \in I} M_i$, $B = \bigoplus_{j \in J} N_j$) are permutation matrices: there is a bijection $\pi : I \rightarrow J$ and, for each $i \in I$, a polarized bijection $f_{\pi(i),i} : M_i \rightarrow N_{\pi(i)}$ in $\mathbf{KL}$; all other entries are zero. Composition reduces to entrywise KL execution; the dagger reverses wiring entry by entry and transposes the matrix.*

Remark B.3 (Currying). A morphism $f : A \otimes B \rightarrow C$ and its curry $\tilde{f} : A \rightarrow B^* \otimes C$ share the same underlying bijection $f^\sharp$; currying merely repartitions boundary ports. This extends linearly to $\text{Perm}(\mathbb{k})$.

Stage 3: $\mathbb{k}$ -linear completion. We introduce a formal indeterminate δ to track closed execution cycles. Let $\mathbb{k} := \mathbb{C}[\delta]$; all scalars generated by the construction below lie in this polynomial ring. Later (§C), we specialize $\delta \mapsto d$ for a chosen real $d > 0$, $d \neq 1$, via the evaluation homomorphism $\mathbb{C}[\delta] \rightarrow \mathbb{C}$.

Definition B.4 ($\mathbb{k}$ -linear completion). *For distributed objects $A = \bigoplus_{i \in I} M_i$ and $B = \bigoplus_{j \in J} N_j$, define*

$$\text{Perm}(\mathbb{k})(A, B) := \text{Mat}_{J \times I}(\mathbb{k}[\mathbf{KL}(M_i, N_j)]),$$

the $\mathbb{k}$ -vector space of $J \times I$ matrices whose entries are finite $\mathbb{k}$ -linear combinations of KL morphisms. The dagger is conjugate transpose. Composition is defined by the categorical trace: given $f : A \rightarrow B$ and $g : B \rightarrow C$,

$$g \circ f := \text{Tr}^B(f \bowtie g),$$

where $f \bowtie g$ is the stacking of f and g along the shared B -boundary, and Tr^B is the canonical trace in $\mathcal{L} := \mathbb{k}[\mathbf{KL}]$ (Proposition B.7).

On basis elements (single KL morphisms in one matrix entry), this recovers entrywise Kelly–Laplaza composition with loop counting.

The trace can be computed as a feedback transfer function: partition the ports of $f \bowtie g$ into external (A, C) and internal (B, I) block form $f \bowtie g = \begin{pmatrix} H & X \\ Y & D \end{pmatrix}$ with swap S_B on the internal gluing interface, giving $g \circ f = H + X(I - S_B D)^{-1} S_B Y$. On basis morphisms the internal bounce $S_B D$ is a partial permutation, hence nilpotent on the non-cyclic components, so the inverse is a finite Neumann series; for general $\mathbb{k}$ -linear combinations, the categorical trace (Proposition B.7) provides an algebraic formulation that is always defined.

Remark B.5 (Monomials and taggers). A *monomial matrix* $[\lambda] \circ \sigma$ is a permutation $\sigma \in \text{Perm}^\pm(A, B)$ composed with per-summand nonzero scalars $\lambda_j \in \mathbb{C}^\times$. A *tagger* $[\lambda]_A = \text{diag}(\lambda_i \cdot \text{id}_{M_i})$ is the identity-permutation case; it is unitary when $|\lambda_i| = 1$ for all i . Monomials span each hom-space of $\text{Perm}(\mathbb{k})$ under $\mathbb{k}$ -linear combination.

Remark B.6 (Finite star on basis morphisms). For basis morphisms (KL matrix units), gluing along the cut object B induces a *partial permutation* on the internal B -ports. Its directed graph decomposes into disjoint components, each either a directed cycle or a chain that eventually exits to the external boundary. Let $\ell(g, f)$ denote the number of cyclic components (the Kelly–Laplaza cycle count), and let D_0 be the restriction of the bounce operator to the *transient* ports (those on chains, not cycles). Then D_0 is acyclic and therefore nilpotent: $D_0^N = 0$ for $N \leq |\text{Pos}(B) \sqcup \text{Neg}(B)|$. Hence the Kleene star is a finite sum $D_0^* := \sum_{n=0}^{N-1} D_0^n$, and the composite is $\delta^{\ell(g, f)} \cdot (H_{g, f} + X D_0^* Y)$, recovering ordinary Kelly–Laplaza execution with each closed cycle contributing one factor of δ .

Proposition B.7 (Categorical trace and associativity). *The $\mathbb{k}$ -linear envelope $\mathcal{L} := \mathbb{k}[\mathbf{KL}]$ is dagger compact closed and carries a canonical trace. Composition in $\text{Perm}(\mathbb{k})$ is associative.*

Proof. Since $\mathbf{KL}$ is dagger compact closed, so is its $\mathbb{k}$ -linear envelope $\mathcal{L}$. A compact closed category carries a canonical trace in the sense of Joyal–Street–Verity [23]: for any morphism $h : A \otimes X \rightarrow C \otimes X$, the trace $\text{Tr}^X(h) : A \rightarrow C$ is defined by *bending the X -wire around* using the cup $\eta_X : \mathbf{1} \rightarrow X^* \otimes X$ and cap $\varepsilon_X : X \otimes X^* \rightarrow \mathbf{1}$, i.e.

$$\text{Tr}^X(h) = (\text{id}_C \otimes \varepsilon_X) \circ (\text{id}_C \otimes \sigma_{X^*, X}) \circ (h \otimes \text{id}_{X^*}) \circ (\text{id}_A \otimes \eta_X),$$

where σ is the symmetry. This is always defined—it is an algebraic composition of morphisms in $\mathcal{L}$, not a limiting process—and satisfies the vanishing, nesting, and naturality axioms of a traced monoidal category.

Distributed objects form the finite matrix completion of $\mathcal{L}$: $\text{Perm}(\mathbb{k})(A, B) \cong \text{Mat}_{J \times I}(\mathcal{L}(M_i, N_j))$. The matrix completion inherits a canonical traced monoidal structure from $\mathcal{L}$ via blockwise application of the trace [21]. Composition in $\text{Perm}(\mathbb{k})$ is the categorical trace Tr^B applied to the glued morphism. The Haghverdi–Scott correspondence [21] identifies this trace with the feedback transfer function. Associativity follows from the nesting axiom: tracing first over B and then over C coincides with tracing once over $B \otimes C$. $\square$

The category $\text{Perm}(\mathbb{k})$ is a dagger symmetric semimonoidal category with duality and distributed normal forms; its multiplicative structure is inherited from $\mathbf{KL}$ and its additive indexing is governed by **FinBij**. After specializing $\delta \mapsto d \in \mathbb{R}_{>0} \setminus \{1\}$, we write $\text{Perm}(\mathbb{C})$ for the resulting $\mathbb{C}$ -linear category; by abuse of notation we continue to write δ for the chosen scalar d .

First-order objects.

Definition B.8 (First-order objects). *An object $A = \bigoplus_{i \in I} M_i$ is first-order if every summand is pure-sign with the same polarity: either $M_i = (m_i, 0)$ for all i , or $M_i = (0, m_i)$ for all i . A morphism is first-order if its domain and codomain are first-order of the same polarity. Write $\mathbf{H}_n := (1, 0)^{\oplus n}$ for the first-order object of size n . In this regime there is no mixed boundary, hence no feedback; composition is purely matrix multiplication.*

C Boundary Transport and Essential Unitarity

We proceed in three steps: first we isolate the boundary ports of a morphism; then, using the Kelly–Laplaza linking, a morphism $f : A \rightarrow B$ is internalized as an operator on the single cut object $A^* \otimes B$, obtained by retyping its polarized boundary ports; finally, we analyze how this boundary operator behaves under composition and monoidal structure, and define essential unitarity.

C.1 Boundary ports

For a signed interface $M = (M^+, M^-)$, write $\text{Pos}(M) := M^+$ and $\text{Neg}(M) := M^-$. For a distributed object $A = \bigoplus_{i \in I} M_i$, set

$$\text{Pos}(A) := \bigsqcup_{i \in I} \text{Pos}(M_i), \quad \text{Neg}(A) := \bigsqcup_{i \in I} \text{Neg}(M_i).$$

These respect $\otimes$ and $\oplus$ in the evident way, and duality reverses polarity: $\text{Pos}(A^*) = \text{Neg}(A)$ and $\text{Neg}(A^*) = \text{Pos}(A)$. Hence for any morphism $f : A \rightarrow B$,

$$\text{Pos}(A^* \otimes B) = \text{Neg}(A) \sqcup \text{Pos}(B), \quad \text{Neg}(A^* \otimes B) = \text{Pos}(A) \sqcup \text{Neg}(B).$$

Definition C.1 (Total port set). *For a distributed object $A = \bigoplus_{i \in I} M_i$, the total port set of A is $|A| := \text{Pos}(A) \sqcup \text{Neg}(A)$.*

Definition C.2 (Boundary port sets). *For objects A, B of $\text{Perm}^\pm$, the boundary port sets are*

$$P_{A,B} := \text{Neg}(A) \sqcup \text{Pos}(B), \quad N_{A,B} := \text{Pos}(A) \sqcup \text{Neg}(B).$$

These are finite sets; the evaluated boundary operator (Definition C.6) maps them to the full matrix algebra $\mathbb{C}^{N_{A,B} \times P_{A,B}}$.

The port sets $P_{A,B}$ and $N_{A,B}$ coincide with $\text{Pos}(A^* \otimes B)$ and $\text{Neg}(A^* \otimes B)$ respectively. Each port retains its summand-of-origin label (from A or B), which is used by the monoidal decomposition laws (Lemma C.11).

Remark C.3 (Notation guide: ports and polarity). Three senses of polarity interact: (a) *interface*: $M = (M^+, M^-)$; (b) *morphism boundary*: $f : A \rightarrow B$ has domain ports $\text{Pos}(A) \sqcup \text{Neg}(B)$ and codomain ports $\text{Neg}(A) \sqcup \text{Pos}(B)$; (c) *boundary labels*: $P_{A,B} = \text{Neg}(A) \sqcup \text{Pos}(B)$ and $N_{A,B} = \text{Pos}(A) \sqcup \text{Neg}(B)$, named by their role in the cut object $A^* \otimes B$.

C.2 Boundary transport

Let $f : A \rightarrow B$ be a morphism of $\text{Perm}^\pm$ with underlying polarized wiring $f^\# : \text{Pos}(A) \sqcup \text{Neg}(B) \xrightarrow{\cong} \text{Pos}(B) \sqcup \text{Neg}(A)$.

Definition C.4 (Boundary bijection). *The boundary bijection of f , $\tau_f : \text{Pos}(A^* \otimes B) \xrightarrow{\cong} \text{Neg}(A^* \otimes B)$, is the bijection*

$$\tau_f := (\text{Neg}(A) \sqcup \text{Pos}(B) \xrightarrow{\cong} \text{Pos}(B) \sqcup \text{Neg}(A) \xrightarrow{(f^\#)^{-1}} \text{Pos}(A) \sqcup \text{Neg}(B)),$$

where the unlabeled arrow is the canonical swap.

For distributed objects, τ_f acts blockwise: for $f \in \text{Perm}^\pm(A, B)$ with underlying permutation π , the boundary bijection τ_f maps the ports of M_i and $N_{\pi(i)}$ lying in $P_{A,B}$ (i.e. $\text{Neg}(M_i) \sqcup \text{Pos}(N_{\pi(i)})$) to the corresponding ports in $N_{A,B}$ (i.e. $\text{Pos}(M_i) \sqcup \text{Neg}(N_{\pi(i)})$), via the KL entry $f_{\pi(i),i}$. Its internalization $\underline{\tau}_f$ is the permutation matrix in $\mathbb{C}^{N_{A,B} \times P_{A,B}}$ with $(\underline{\tau}_f)_{\tau_f(p),p} = 1$.

This construction separates the structural (wiring) and the scalar (loop count) content of a morphism and dissolves the higher-order character at the boundary level.

Remark C.5 (Axiom-link extraction and squaring). A morphism $f : A \rightarrow B$ in Kelly–Laplaza normal form consists of axiom links (the edges of $f^\sharp$) together with a loop count $\kappa(f)$. The boundary bijection τ_f extracts the axiom links, repackaging $f^\sharp$ as a bijection between the two polarities of the single cut object $A^* \otimes B$.

This *squares* the morphism, which is essential at higher types: $\text{eval} : (A^* \otimes B) \otimes A \rightarrow B$ has source and target of different shape, yet τ_{eval} reduces to two bundles of straight wires on $((A^* \otimes B) \otimes A)^* \otimes B \cong A \otimes (B^* \otimes A^*) \otimes B$.

Moreover, τ is invariant under currying: $f : A \otimes B \rightarrow C$ and its curry $\tilde{f} : A \rightarrow B^* \otimes C$ share the same underlying bijection $f^\sharp$ (Remark B.3) and the same cut object $A^* \otimes B^* \otimes C$, so $\tau_f = \tau_{\tilde{f}}$.

C.3 The evaluated boundary operator

Fix a real scalar $\delta > 0$, $\delta \neq 1$; this ensures that any nonzero loop count contributes a detectable scalar factor.

We extract from each morphism in $\text{Perm}(\mathbb{C})$ a matrix on the boundary port sets.

Definition C.6 (Evaluated boundary operator). *The evaluated boundary operator assigns to each morphism $f : A \rightarrow B$ in $\text{Perm}(\mathbb{C})$ a matrix $T_f \in \mathbb{C}^{N_{A,B} \times P_{A,B}}$ by $\mathbb{C}$ -linear extension from KL matrix units. A KL matrix unit $e \in \text{Perm}(\mathbb{C})(A, B)$ has a single KL morphism $g : M_i \rightarrow N_j$ in entry (j, i) and zeros elsewhere; define $T_e := \delta^{\kappa(g)} \cdot \tau_g$, the partial permutation matrix internalizing the boundary bijection τ_g on the ports of M_i and N_j (and zero elsewhere), scaled by $\delta^{\kappa(g)}$. For $u = \sum_r \alpha_r e_r$, set $T_u := \sum_r \alpha_r T_{e_r}$.*

Remark C.7 (Computing T in practice). On a permutation matrix $f \in \text{Perm}^\pm(A, B)$, $T_f = \delta^{\kappa(f)} \cdot \tau_f$; on a monomial $[\lambda] \circ f$, each block is further scaled by $\lambda_{\pi(i)}$. On $\mathbb{C}$ -linear combinations, T extends by linearity.

The assignment $f \mapsto T_f$ is $\mathbb{C}$ -linear on hom-spaces, compatible with dagger ($T_{f^\dagger} = (T_f)^\dagger$, Lemma C.12), and respects the monoidal structure ($T_{f \oplus g} = T_f \oplus T_g$, $T_{f \otimes g} = T_f \otimes T_g$, Lemma C.11). On the loop-free fragment, T preserves the categorical trace (Lemma D.13).

Remark C.8 (Feedback computation of $T_{g \circ f}$). Partition the ports of $T_f \oplus T_g$ into external $E = P_{A,C} \sqcup N_{A,C}$ and internal $I = \text{Pos}(B) \sqcup \text{Neg}(B)$; in (E, I) block form $T_f \oplus T_g = \begin{pmatrix} H & X \\ Y & D \end{pmatrix}$ with swap $S_B : \text{Neg}(B) \sqcup \text{Pos}(B) \rightarrow \text{Pos}(B) \sqcup \text{Neg}(B)$. The matrix-level feedback external map $\Phi(x) := Hx + Xy$, where y solves $(I - S_B D)y = S_B Yx$, captures the path component of the execution. On basis morphisms, $T_{g \circ f} = \delta^{\ell(g,f)} \Phi$ (Corollary C.10); when $\ell(g, f) = 0$, feedback agrees with transported composition. When $(I - S_B D)$ is invertible,

$$\Phi = H + X(I - S_B D)^{-1} S_B Y. \quad (1)$$

C.4 Composition law on basis morphisms and the scalar anomaly

On basis morphisms, the feedback transfer function decomposes into a path component and a cycle-counting scalar. This decomposition is useful for the structural fragment (§D) but is not needed for the general closure argument.

Proposition C.9 (Composition law for τ and k [2, Proposition 1]). *Let $f : M \rightarrow N$ and $g : N \rightarrow P$ in **KL**. Then $\tau_{g \circ f}$ is the path component and $\ell(g, f)$ the cycle count of the Kelly–Laplaza execution graph obtained by gluing τ_f and τ_g along the shared N -boundary, and $\kappa(g \circ f) = \kappa(f) + \kappa(g) + \ell(g, f)$.*

Proof. The boundary bijection τ_f on $M^* \otimes N$ is the Kelly–Laplaza linking of the cut object, with $\text{Pos}(M^* \otimes N) = \text{Neg}(M) \sqcup \text{Pos}(N)$ and $\text{Neg}(M^* \otimes N) = \text{Pos}(M) \sqcup \text{Neg}(N)$. Gluing identifies $\text{Pos}(N)$ in the codomain of τ_f with $\text{Pos}(N)$ in the domain of τ_g (and likewise $\text{Neg}(N)$), reproducing the Kelly–Laplaza execution graph. The paths-and-cycles decomposition is [2, Proposition 1]. $\square$

Corollary C.10 (Scalar anomaly on basis morphisms). *For $f : A \rightarrow B$ and $g : B \rightarrow C$ in $\text{Perm}^\pm$,*

$$T_{g \circ f} = \delta^{\ell(g, f)} \cdot (T_g \odot T_f),$$

where $T_g \odot T_f$ is the internalized path component and $\delta^{\ell(g, f)}$ the cycle factor. On the acyclic structural fragment ($\ell = 0$), feedback coincides with $\odot$.

Proof. Immediate from Proposition C.9 and $T_f = \delta^{\kappa(f)} \cdot \tau_f$. $\square$

C.5 Dagger (bi)monoidal interaction of the boundary operator

Lemma C.11 (Monoidal laws for τ , κ , and T). *For $f : A \rightarrow B$ and $g : C \rightarrow D$ in $\text{Perm}^\pm$, the boundary port sets of $f \oplus g$ decompose as $P_{A \oplus C, B \oplus D} = P_{A, B} \sqcup P_{C, D}$ and $N_{A \oplus C, B \oplus D} = N_{A, B} \sqcup N_{C, D}$, and under this identification*

$$\tau_{f \oplus g} = \tau_f \sqcup \tau_g, \quad \kappa(f \oplus g) = \kappa(f) + \kappa(g).$$

Similarly, under $(A \otimes C)^ \otimes (B \otimes D) \cong (A^* \otimes B) \otimes (C^* \otimes D)$,*

$$\tau_{f \otimes g} = \tau_f \sqcup \tau_g, \quad \kappa(f \otimes g) = \kappa(f) + \kappa(g).$$

If $\kappa(f) = \kappa(g) = 0$, internalizing gives $T_{f \oplus g} = T_f \oplus T_g$ and $T_{f \otimes g} = T_f \otimes T_g$.

Proof. For $\oplus$: the boundary port sets are defined by Pos and Neg of the source and target (Definition C.2). Since $\text{Neg}(A \oplus C) = \text{Neg}(A) \sqcup \text{Neg}(C)$ and $\text{Pos}(B \oplus D) = \text{Pos}(B) \sqcup \text{Pos}(D)$, $P_{A \oplus C, B \oplus D} = \text{Neg}(A) \sqcup \text{Neg}(C) \sqcup \text{Pos}(B) \sqcup \text{Pos}(D) = P_{A, B} \sqcup P_{C, D}$, and likewise for N . The linking of $f \oplus g$ is the disjoint union of the linkings of f and g , so $\tau_{f \oplus g} = \tau_f \sqcup \tau_g$. The loop counter is additive since execution in the two components is disjoint.

For $\otimes$: in the Kelly–Laplaza construction, the tensor of objects is disjoint union of ports, and the linking of a tensor composite is the disjoint union of the linkings. Under the identification $(A \otimes C)^* \otimes (B \otimes D) \cong (A^* \otimes B) \otimes (C^* \otimes D)$, the boundary linking of $f \otimes g$ therefore decomposes as $\tau_{f \otimes g} = \tau_f \sqcup \tau_g$. The loop counter is additive since execution in the two factors is disjoint. The T laws follow by internalization. $\square$

Lemma C.12 (Dagger). *For any $f : A \rightarrow B$ in $\text{Perm}^\pm$, $\kappa(f^\dagger) = \kappa(f)$, $\tau_{f^\dagger} = \tau_f^{-1}$, and $T_{f^\dagger} = (T_f)^\dagger$.*

Proof. Dagger inverts $f^\sharp$ and preserves the loop counter, so $\kappa(f^\dagger) = \kappa(f)$ and $\tau_{f^\dagger} = \tau_f^{-1}$. Internalization sends inversion to dagger, so $\tau_{f^\dagger} = (\tau_f)^\dagger$. Since δ is real,

$$T_{f^\dagger} = \delta^{\kappa(f)} (\tau_f)^\dagger = (\delta^{\kappa(f)} \tau_f)^\dagger = (T_f)^\dagger. \quad \square$$

C.6 Essential unitarity

We now formulate essential unitarity as an *internal* unitarity condition on the squared boundary operator.

Definition C.13 (Essential unitarity). *A morphism $f : A \rightarrow B$ in $\text{Perm}(\mathbb{C})$ is essentially unitary (EU) if*

$$T_f^\dagger T_f = I_{P_{A,B}} \quad \text{and} \quad T_f T_f^\dagger = I_{N_{A,B}},$$

where products and dagger (conjugate transpose) are computed in $\mathbb{C}^{N_{A,B} \times P_{A,B}}$.

The definition is δ -independent: it is a condition on T_f alone, with no δ appearing. The assumption $\delta > 0$, $\delta \neq 1$ enters only in the implication that EU forces loop-freeness on basis morphisms: since $T_{h_r} = \delta^{\kappa(h_r)} \tau_{h_r}$ with τ_{h_r} a permutation matrix, unitarity of T_f forces $|\delta^\kappa| = 1$, which for real $\delta > 0$, $\delta \neq 1$ requires $\kappa = 0$; in $\mathbb{Q}\mathbb{C}$ all components have $\kappa = 0$ (Remark D.3). Every structural isomorphism is EU (its T_f is a permutation matrix), and the condition is insensitive to currying since $T_f = T_{\tilde{f}}$ (Remark C.5). Compare Selinger’s structural unitaries [28].

Remark C.14 (EU coincides with unitarity at first order). For a first-order morphism $f : (n, 0) \rightarrow (n, 0)$ (Definition B.8), the KL linking is a bijection $\{1, \dots, n\} \rightarrow \{1, \dots, n\}$ —an ordinary permutation with no mixed-polarity ports. Since execution of $f^\dagger \circ f$ involves only positive ports, no closed components can form: $\ell(f^\dagger, f) = 0$ and $\kappa(f^\dagger \circ f) = 2\kappa(f)$. EU forces $\kappa(f) = 0$, so $T_f = \tau_f$ is the permutation matrix of f . Hence EU and internal dagger unitarity coincide on first-order morphisms. This extends to distributed first-order objects $A = (p, 0)^{\oplus n}$: since all ports are positive, composition in $\text{Perm}(\mathbb{C})(A, A)$ is ordinary matrix multiplication with no feedback, so T preserves composition strictly ($T_{g \circ f} = T_g T_f$) and is injective on this hom-space (distinct basis morphisms map to distinct matrices, using $\delta > 0$, $\delta \neq 1$). Hence $u^\dagger u = \text{id}$ iff $T_u^\dagger T_u = I$, and likewise for $u u^\dagger$.

C.7 Uniqueness

We now show that any predicate compatible with dagger-monoidal structure, coherence, and currying coincides with essential unitarity.

Ordinary unitarity on first-order types already satisfies (R1) and (R2): tensor, dagger and sum of unitaries are unitary (and conversely), and coherence reindexing is invisible to unitarity. Axiom (R4) anchors any candidate predicate to this known case.

The only genuinely new constraint is (R3). A higher-order process $f : A \otimes B \rightarrow C$ and its curry $\tilde{f} : A \rightarrow B^* \otimes C$ describe the same process at different types (Remark B.3); a predicate that accepted one but rejected the other would artificially restrict which higher-order processes count as unitary. Currying invariance ensures the lifted notion is as expressive as the type system allows.

Definition C.15 (Uniform type). *A distributed object $A = \bigoplus_{i \in I} M_i$ of $\text{Perm}^\pm$ is uniform if all summands share the same interface, i.e. $A \cong X^{\oplus n}$ for some single KL interface $X = (m^+, m^-)$ and $n \geq 1$.*

Definition C.16 (Interface-decomposable morphism). *A morphism $u : A \rightarrow B$ in $\text{Perm}(\mathbb{C})$ is interface-decomposable if it admits a decomposition $u \cong u_1 \oplus \dots \oplus u_r$ where each u_k has uniform source and uniform target.*

Uniform types include every single interface (the $n = 1$ case), every first-order object of the form $(p, 0)^{\oplus n}$ (in particular $\mathbf{H}_n = (1, 0)^{\oplus n}$), and every object $X^{\oplus n}$ appearing in §E. Interface-decomposability is preserved by currying (which acts on each summand independently), by dagger (which preserves each block), and by composition (the intermediate type’s interface decomposition constrains both factors).

The theorem below shows that these constraints leave no freedom: EU is the unique extension of first-order unitarity to interface-decomposable morphisms.

Theorem C.17 (Uniqueness). *Let R be a predicate on morphisms of $\text{Perm}(\mathbb{C})$ satisfying:*

(R1) (Reflecting dagger-monoidal) $R(u \otimes v) \Leftrightarrow R(u) \wedge R(v)$, $R(u \oplus v) \Leftrightarrow R(u) \wedge R(v)$, $R(u^\dagger) \Leftrightarrow R(u)$, and $R(\text{id}_A)$.

(R2) (Structural invariance) *If s, t are $\otimes/\oplus$ coherence or distributivity isomorphisms, $R(u) \Rightarrow R(s \circ u \circ t)$.*

(R3) (Currying invariance) $R(u) \Leftrightarrow R(\tilde{u})$ *whenever defined.*

(R4) (First-order unitarity) *For first-order u in $\text{Perm}(\mathbb{C})$, $R(u) \Leftrightarrow u^\dagger u = \text{id} \wedge u u^\dagger = \text{id}$.*

Then for every interface-decomposable morphism u , $R(u) = \text{EU}(u)$.

Proof. By (R1), $R(u_1 \oplus \dots \oplus u_r) \Leftrightarrow R(u_1) \wedge \dots \wedge R(u_r)$. Since $T_{u \oplus v} = T_u \oplus T_v$ (Lemma C.11), a block-diagonal matrix is unitary iff each block is, so $\text{EU}(u) \Leftrightarrow \text{EU}(u_1) \wedge \dots \wedge \text{EU}(u_r)$. It therefore suffices to prove $R(u_k) = \text{EU}(u_k)$ for each uniform-to-uniform block u_k .

Write the source as $X^{\oplus n}$ and the target as $Y^{\oplus m}$ with $X = (a, b)$, $Y = (c, d)$.

Step 1 (Curry to first order). Distributivity gives $X^{\oplus n} \cong (a, 0)^{\oplus n} \otimes (0, b)$ and $Y^{\oplus m} \cong (c, 0)^{\oplus m} \otimes (0, d)$. By (R2), reindexing by these distributivity isomorphisms preserves R . Currying out the $(0, b)$ factor and uncurrying the $(0, d)$ factor (each step by (R3), with associativity and commutativity reindexings by (R2)) yields a morphism

$$\hat{u} : (a+d, 0)^{\oplus n} \longrightarrow (b+c, 0)^{\oplus m}$$

with $R(\hat{u}) \Leftrightarrow R(u_k)$. This is first-order. Since currying and coherence reindexings leave the boundary linking untouched (Remark C.5), the port sets $P_{A,B}$ and $P_{\hat{A},\hat{B}}$ are canonically identified and $T_{\hat{u}} = T_{u_k}$ as matrices on these port sets.

Step 2 (Apply first-order unitarity). Since $u_k : (a, b) \rightarrow (c, d)$ exists in **KL**, the port-count condition for a polarized bijection gives $a+d = b+c =: p$. Hence $\hat{u} : (p, 0)^{\oplus n} \rightarrow (p, 0)^{\oplus m}$.

If $n \neq m$ (and $p > 0$), then $\hat{u}$ maps between first-order spaces of different total size, so $\hat{u}$ cannot satisfy both $\hat{u}^\dagger \hat{u} = \text{id}$ and $\hat{u} \hat{u}^\dagger = \text{id}$ simultaneously (one requires at least as many columns as rows, the other the reverse); hence $R(\hat{u})$ is false by (R4). Meanwhile $T_{\hat{u}}$ is a non-square matrix ($|P_{A,B}| \neq |N_{A,B}|$), so $\text{EU}(u_k)$ is false. Both sides are false and the conclusion holds. (When $p = 0$ every summand is the tensor unit $(0, 0) = \mathbf{1}$, which lies outside the unit-free fragment; types constructible from the generator $(1, 0)$ via $\otimes$, duality, and nonempty $\oplus$ always have $p \geq 1$, so this case does not arise.)

If $n = m$, write $\hat{u} : (p, 0)^{\oplus n} \rightarrow (p, 0)^{\oplus n}$. By (R4), $R(\hat{u}) \Leftrightarrow \hat{u}^\dagger \hat{u} = \text{id} \wedge \hat{u} \hat{u}^\dagger = \text{id}$. By Remark C.14, this is equivalent to $T_{\hat{u}}$ being unitary. Since $T_{\hat{u}} = T_{u_k}$, we conclude $R(u_k) \Leftrightarrow \text{EU}(u_k)$. $\square$

Lemma C.18 (Loop-freeness is forced). *If R satisfies (R1)–(R4) and $R(f)$ for some basis morphism $f \in \text{Perm}^\pm$ that is interface-decomposable, then $\kappa(f) = 0$.*

Proof. The proof of Theorem C.17 yields T_f unitary. Since f is a basis morphism, $T_f = \delta^{\kappa(f)} \underline{\tau}_f$ where $\underline{\tau}_f$ is a permutation matrix. Unitarity forces $|\delta^{\kappa(f)}| = 1$; since $\delta > 0$ and $\delta \neq 1$, this gives $\kappa(f) = 0$. $\square$

Remark C.19 (Scope of uniqueness). The only morphisms not covered by Theorem C.17 are those that *entangle across interface types*: given a non-uniform object $A = (3, 1)^{\oplus 2} \oplus (5, 3)$, a morphism $u : A \rightarrow A$ whose matrix entries connect a $(3, 1)$ -summand to a $(5, 3)$ -summand is not interface-decomposable. After currying, these summands become $(4, 0)$ and $(8, 0)$ respectively—incommensurable boundary sizes that cannot be related by any first-order unitary. Axioms (R1)–(R4) are therefore insufficient to determine R on such morphisms.

This gap is immaterial for the quantum core: $\mathbb{QC}$ is generated by identity-based operations—structural maps and exponentials of structural involutions on blocks $X^{\oplus n}$ —which never introduce cross-interface entries. Closure under $\otimes$ and duality preserves uniformity ($X^{\oplus n} \otimes Y^{\oplus m} \cong (X \otimes Y)^{\oplus nm}$; $(X^{\oplus n})^* = (X^*)^{\oplus n}$).

Closure under $\oplus$ may produce non-uniform types, but since generators act within uniform blocks, every $\mathbb{Q}\mathbb{C}$ morphism is interface-decomposable.

D The Quantum Core

The *quantum core* $\mathbb{Q}\mathbb{C}$ is built in three stages, moving from pure structure, to reversible control, to coherent dynamics: unit-free MLL (acyclicity gives loop-free permutation matrices, hence trivially EU); addition of $\oplus$ (correctness lifts componentwise); exponentials of structural involutions in the $\mathbb{C}$ -linear completion (unitarity is purely algebraic). The construction is unit-free by necessity; units cause scalar loops that are fatal to invertibility.

D.1 The structural fragment

Definition D.1 (The categories $\text{Perm}_{\text{MLL}}^{\pm}$ and $\text{Perm}_{\text{MLL}(+)}^{\pm}$). *Let $\text{Perm}_{\text{MLL}}^{\pm}$ be the subcategory of $\text{Perm}^{\pm}$ whose objects are signed interfaces (m, n) and whose morphisms are those obtained by interpreting unit-free MLL proofs in the $*$ -autonomous structure of $\mathbf{KL}$ (every compact closed category is $*$ -autonomous, so this interpretation is canonical). Let $\text{Perm}_{\text{MLL}(+)}^{\pm}$ be the smallest subcategory of $\text{Perm}^{\pm}$ containing $\text{Perm}_{\text{MLL}}^{\pm}$ and closed under $\oplus$ on objects and morphisms, $\otimes$, dual, and composition. The free addition of $\oplus$ preserves acyclicity: every morphism in $\text{Perm}_{\text{MLL}(+)}^{\pm}$ decomposes blockwise into $\text{Perm}_{\text{MLL}}^{\pm}$ components and structural reindexings.*

Corollary D.2 (EU for $\text{Perm}_{\text{MLL}(+)}^{\pm}$). *Every $f \in \text{Perm}_{\text{MLL}(+)}^{\pm}$ is essentially unitary.*

Every unit-free MLL proof denotes a linking with $\kappa(f) = 0$: cut elimination preserves acyclicity, so feedback creates no closed components. Hence $T_f = \tau_f$ is a permutation matrix, therefore unitary. Since $\oplus$ -extension and structural reindexings preserve the block-sum structure of T (Lemma C.11), the result lifts to all of $\text{Perm}_{\text{MLL}(+)}^{\pm}$.

Remark D.3 (Homogeneous loop degree in $\mathbb{Q}\mathbb{C}$). Every generator of $\mathbb{Q}\mathbb{C}$ is a $\mathbb{C}$ -linear combination of $\text{MLL}(+)$ morphisms, each with $\kappa = 0$. Since $\mathbb{Q}\mathbb{C}$ is built from these generators by operations that preserve loop-freeness (Lemma D.4, Proposition D.8), every $\mathbb{Q}\mathbb{C}$ morphism has all its $\text{MLL}(+)$ components at loop degree zero. In particular, EU cannot be achieved by cancellation between terms of different loop degree; the scalar δ^{κ} is uniformly 1 throughout $\mathbb{Q}\mathbb{C}$.

Lemma D.4 (No new loops in $\text{MLL}(+)$). *For composable morphisms $f, g \in \text{Perm}_{\text{MLL}(+)}^{\pm}$, execution creates no new cycle components: $\ell(g, f) = 0$.*

Proof. Let $f, g \in \text{Perm}_{\text{MLL}(+)}^{\pm}$ be composable, say $f : A \rightarrow B$ and $g : B \rightarrow C$. By definition of $\text{Perm}_{\text{MLL}(+)}^{\pm}$, both f and g arise as interpretations of unit-free $\text{MLL}(+)$ proof nets. Their composite $g \circ f$ is interpreted by cut elimination of the composite proof structure obtained by cutting the output B -conclusion of f against the input B -assumption of g . Unit-free MLL correctness is acyclicity of the switching graphs [15, Theorem 1], and this property is preserved by cut elimination. Acyclicity implies that every connected component of the execution graph reaches at least one external port (a closed component would give a switching cycle), so no cyclic components can form: $\ell(g, f) = 0$. The $\oplus$ -reindexings in $\text{MLL}(+)$ are structural isomorphisms and do not introduce axiom links, so the argument extends to all of $\text{Perm}_{\text{MLL}(+)}^{\pm}$. $\square$

Corollary D.5 (Execution product on $\text{MLL}(+)$). *For composable $f, g \in \text{Perm}_{\text{MLL}(+)}^\pm$, the scalar anomaly vanishes: $T_{g \circ f} = T_g \odot T_f$. That is, transported composition coincides with the GOI execution product on the structural fragment.*

Proof. By Corollary C.10, $T_{g \circ f} = \delta^{\ell(g,f)}(T_g \odot T_f)$. By Lemma D.4, $\ell(g, f) = 0$. $\square$

D.2 Exponentials of involutions

This is the point at which coherent quantum evolution enters.

We first record the structural fact underlying the uniform-type condition.

Lemma D.6 (Involutions preserve interface size). *Let $A = \bigoplus_{i \in I} M_i$ and let $S \in \text{Perm}^\pm(A, A)$ satisfy $S \circ S = \text{id}_A$. Then S permutes the summands of A , and every summand in the same orbit has the same interface: if S maps M_i to M_j then $|M_i^+| = |M_j^+|$ and $|M_i^-| = |M_j^-|$. In particular, if S acts transitively on the summands, then A is uniform.*

Proof. Since $S \in \text{Perm}^\pm(A, A)$, it is a permutation matrix: there is an involution $\sigma : I \rightarrow I$ and, for each i , a loop-free polarized bijection $S_{\sigma(i), i} : M_i \rightarrow M_{\sigma(i)}$. The condition $S^2 = \text{id}$ gives $S_{\sigma(\sigma(i)), \sigma(i)} \circ S_{\sigma(i), i} = \text{id}_{M_i}$, so $S_{\sigma(i), i}$ is an isomorphism in **KL** with inverse $S_{i, \sigma(i)}$. It remains to show that an isomorphism $M \cong N$ in **KL** forces $|M^+| = |N^+|$ and $|M^-| = |N^-|$. Write $M = (a, b)$ and $N = (c, d)$ with $a + d = c + b$ (the port-count condition for a morphism to exist). An isomorphism $f : M \rightarrow N$ with inverse $g : N \rightarrow M$ requires $(g \circ f)^\# = \text{id}$ on $a + b$ elements with zero loops. The execution of $g \circ f$ glues along $c + d$ internal ports; if $a + b \neq c + d$ then $a > c$ or $b > d$ (since $a - b = c - d$), forcing $f^\#$ to route at least one source-positive port directly to a source-negative port. Such cross-polarity paths are incompatible with the identity bijection. Hence $a + b = c + d$, and combined with $a - b = c - d$ this gives $a = c$ and $b = d$. $\square$

Definition D.7 (Exponential generators). *Let $S \in \text{Perm}_{\text{MLL}(+)}^\pm(A, A)$ be an involution ($S \circ S = \text{id}_A$). For $\theta \in \mathbb{R}$, define $\exp_i(\theta, S) := \cos \theta \text{id}_A + i \sin \theta S$ in $\text{Perm}(\mathbb{C})$.*

Any structural involution in the $\text{MLL}(+)$ image qualifies, including higher-order reflections on compound interfaces, so the construction generalizes qubit rotations to the full distributed setting. By Lemma D.6, every such involution permutes summands of equal interface size, so $\exp_i(\theta, S)$ always acts on a uniform type (Definition C.15); in particular Theorem C.17 applies to every such exponential.

Proposition D.8 (Essential unitarity of exponentials). *For every involution $S \in \text{Perm}_{\text{MLL}(+)}^\pm(A, A)$ and angle θ , $\exp_i(\theta, S)$ is essentially unitary.*

Proof. Write $U = \exp_i(\theta, S)$. Since $\kappa(S) = 0$ we have $T_S = \tau_S$ and $T_{\text{id}_A} = I$. By Lemma C.12 and $S^\dagger = S$, $(T_S)^\dagger = T_S$. Since τ_S is an involution, $(T_S)^2 = I$ as an ordinary matrix product. By $\mathbb{C}$ -linearity of T on the hom-space,

$$T_U = \cos \theta \cdot I + i \sin \theta \cdot T_S, \quad T_U^\dagger = \cos \theta \cdot I - i \sin \theta \cdot T_S.$$

Then $T_U^\dagger T_U = (\cos^2 \theta + \sin^2 \theta) I = I$, and similarly $T_U T_U^\dagger = I$. $\square$

D.3 The quantum core

Combining structure, reversible control, and coherent rotations yields the quantum core.

Definition D.9 (The quantum core $\mathbb{QC}$). *Let $\mathbb{QC}$ be the smallest dagger subcategory of $\text{Perm}(\mathbb{C})$ closed under $\otimes$, $\oplus$, dual, and composition, containing the $\text{MLL}(+)$ image and, for every structural involution S in that image and every $\theta \in \mathbb{R}$, the exponential $\exp_i(\theta, S)$.*

The quantum core is a unit-free dagger SMCC: evaluation $\text{ev}_{B,C}$ and currying are available, but the coevaluation $\eta_A : \mathbf{1} \rightarrow A^* \otimes A$ is excluded. because composing with ε_A yields a non-trivial scalar loop for general A .

Theorem D.10 (Unit-free dagger SMCC structure of $\mathbb{QC}$). *The quantum core $\mathbb{QC}$ is a dagger symmetric monoidal closed category (without tensor unit; that is, a symmetric biclosed category in the sense of [16]). Its internal hom is $B \multimap C := B^* \otimes C$, and evaluation is given by the canonical structural wiring $\text{ev}_{B,C} : (B^* \otimes C) \otimes B \rightarrow C$.*

Proof. We work inside $\text{Perm}(\mathbb{C})$ and use that $\mathbb{QC}$ is a dagger subcategory of it, closed under $\otimes$, dual, and composition.

(1) Dagger symmetric monoidal structure. By Definition D.9, $\mathbb{QC}$ is closed under $\otimes$ and contains the $\text{MLL}(+)$ image; in particular it contains the symmetry and associativity structure maps for $\otimes$ (and likewise for $\oplus$, though $\oplus$ is not used for closedness). Since $\mathbb{QC}$ is a dagger subcategory, it is closed under $f \mapsto f^\dagger$ and contains identities. Hence $\mathbb{QC}$ is a dagger symmetric monoidal category (unit-free).

(2) Evaluation lies in $\mathbb{QC}$. For any objects B, C , consider the canonical *yanking* permutation wiring

$$\text{ev}_{B,C} : (B^* \otimes C) \otimes B \rightarrow C$$

which connects the B^* -ports to the B -ports and leaves the C -boundary unchanged. This map is purely structural, hence it lies in the $\text{MLL}(+)$ image, and therefore $\text{ev}_{B,C} \in \mathbb{QC}$.

(3) Currying/uncurrying and the universal property. A basis morphism in $\text{Perm}^\pm$ is determined by its boundary linking τ together with its loop count κ . Under the canonical rebracketing identification of boundaries

$$(A \otimes B)^* \otimes C \cong A^* \otimes (B^* \otimes C),$$

every linking τ with external boundary $A^* \otimes B^* \otimes C$ can be read equally well as a morphism $A \otimes B \rightarrow C$ or as a morphism $A \rightarrow B^* \otimes C$, with the same loop count. This re-cutting defines a bijection on basis morphisms, which extends $\mathbb{C}$ -linearly to a natural isomorphism

$$\text{curry}_{A,B,C} : \mathbb{QC}(A \otimes B, C) \cong \mathbb{QC}(A, B^* \otimes C),$$

with inverse $\text{uncurry}_{A,B,C}$. Moreover, by construction, $\text{uncurry}(g)$ is characterized by

$$\text{uncurry}(g) = \text{ev}_{B,C} \circ (g \otimes \text{id}_B),$$

and since $\text{ev}_{B,C} \in \mathbb{QC}$, $g \in \mathbb{QC}$, and $\mathbb{QC}$ is closed under $\otimes$ and composition, $\text{uncurry}(g) \in \mathbb{QC}$ whenever $g \in \mathbb{QC}$. Dually, currying of f is obtained by the inverse re-cutting, hence lies in $\mathbb{QC}$ as well. Thus $(- \otimes B) \dashv (B^* \otimes -)$ holds in $\mathbb{QC}$ with counit $\text{ev}_{B,C}$, i.e. $\mathbb{QC}$ is symmetric monoidal closed (unit-free) with internal hom $B \multimap C := B^* \otimes C$. Together with (1), $\mathbb{QC}$ is a dagger SMCC. $\square$

Structure, control, and coherent dynamics now close under essential unitarity. The key tools are a general lemma on feedback systems built from unitary matrices (a standard result in quantum input-output network theory [14]), and a trace-preservation result that bridges categorical composition in $\text{Perm}(\mathbb{C})$ with the canonical JSV trace in $\text{Mat}(\mathbb{C})$; see [6, 17].

Lemma D.11 (Well-posed unitary feedback). *Let U be a unitary matrix written in 2×2 block form*

$$U = \begin{pmatrix} H & X \\ Y & D \end{pmatrix} : E_{\text{in}} \oplus I_{\text{in}} \rightarrow E_{\text{out}} \oplus I_{\text{out}},$$

and let $S : I_{\text{out}} \rightarrow I_{\text{in}}$ be unitary. Consider the feedback equation $(I - SD)y = SYx$ for $x \in E_{\text{in}}, y \in I_{\text{in}}$. Then:

- (i) For every x , the equation has at least one solution y .
- (ii) If y_1, y_2 are solutions for the same x , then $Xy_1 = Xy_2$. Hence the external map $\Phi(x) := Hx + Xy$ (for any solution y) is well-defined. Since the feedback equation is linear in (x, y) , Φ is linear.
- (iii) Φ is an isometry. If $\dim E_{\text{in}} = \dim E_{\text{out}}$, then Φ is unitary.

Moreover, Φ coincides with the Joyal–Street–Verity trace in the matrix category: $\Phi = \text{Tr}^{I_{\text{out}}}((\text{id} \oplus S)U)$.

Proof. Define $\tilde{U} := (I_E \oplus S)U = \begin{pmatrix} H & X \\ SY & SD \end{pmatrix}$, which is unitary since S and U are.

Independence (ii). Let $y \in \ker(I - SD)$. Then $\tilde{U} \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} Xy \\ y \end{pmatrix}$, and unitarity of $\tilde{U}$ gives $\|y\|^2 = \|Xy\|^2 + \|y\|^2$, so $Xy = 0$. Hence changing a solution by a kernel element does not change Xy .

Existence (i). We must show $\text{Im}(SY) \subseteq \text{Im}(I - SD)$, equivalently $\ker(I - D^\dagger S^\dagger) \subseteq \ker(Y^\dagger S^\dagger)$ (a vector lies in $\text{Im}(L)$ iff it is orthogonal to $\ker(L^\dagger)$). Let $v \in \ker(I - D^\dagger S^\dagger)$, so $v = D^\dagger S^\dagger v$. Then $\|v\| = \|D^\dagger S^\dagger v\| \leq \|S^\dagger v\| = \|v\|$, giving equality throughout. From $UU^\dagger = I$ we get $DD^\dagger + YY^\dagger = I_{I_{\text{out}}}$; applying to $S^\dagger v$ yields $\|D^\dagger S^\dagger v\|^2 + \|Y^\dagger S^\dagger v\|^2 = \|S^\dagger v\|^2$. Since $\|D^\dagger S^\dagger v\| = \|v\| = \|S^\dagger v\|$, we conclude $Y^\dagger S^\dagger v = 0$.

Isometry (iii). Let y be any solution of $(I - SD)y = SYx$. Then $\tilde{U} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \Phi x \\ y \end{pmatrix}$ and unitarity gives $\|x\|^2 + \|y\|^2 = \|\Phi x\|^2 + \|y\|^2$, hence $\|\Phi x\| = \|x\|$. A square isometry is unitary (its columns are orthonormal and span the codomain).

Trace identification. Unwinding the cup–cap definition of $\text{Tr}^{I_{\text{out}}}$ on basis vectors shows that the JSV trace of $\tilde{U}$ enforces exactly the constraint $y = SYx + SDy$ (i.e. the feedback equation) and returns $Hx + Xy$; by independence of the choice of y , this equals Φ . $\square$

D.4 Trace preservation on the loop-free fragment

The boundary operator T does not preserve the categorical trace in general: newly formed execution cycles contribute a scalar δ^ℓ that is visible to T but invisible to the matrix-level feedback (Corollary C.10). We isolate the hypothesis under which this discrepancy vanishes, satisfied by $\mathbb{Q}\mathbb{C}$.

Definition D.12 (Loop-free under B -trace). *Let $h : A \otimes B \rightarrow C \otimes B$ in $\text{Perm}(\mathbb{C})$ with presentation $h = \sum_r \gamma_r h_r$ as a finite $\mathbb{C}$ -linear combination of basis morphisms. h is loop-free under B -trace if $\ell_B(h_r) = 0$ for every r with $\gamma_r \neq 0$, where $\ell_B(h_r)$ counts the new cycles created by tracing h_r over B .*

Lemma D.13 (Trace preservation of T on loop-free traces). *Let $h : A \otimes B \rightarrow C \otimes B$ be loop-free under B -trace. Then $T_{\text{Tr}^B(h)} = \text{Tr}^{|B|}((\text{id} \oplus S_B)T_h)$, where $\text{Tr}^{|B|}$ is the canonical JSV trace in $\text{Mat}(\mathbb{C})$, and S_B is the polarity reconciliation swap identifying the internal B -port rows ($\text{Neg}(B) \sqcup \text{Pos}(B)$) with the internal B -port columns ($\text{Pos}(B) \sqcup \text{Neg}(B)$).*

Proof. Write $h = \sum_r \gamma_r h_r$ as in Definition D.12. Since the categorical trace Tr^B in $\text{Perm}(\mathbb{C})$ is built from $\otimes$ and composition in a $\mathbb{C}$ -linear category, it is $\mathbb{C}$ -linear; hence $\text{Tr}^B(h) = \sum_r \gamma_r \text{Tr}^B(h_r)$. Applying T (which is $\mathbb{C}$ -linear on hom-spaces) gives $T_{\text{Tr}^B(h)} = \sum_r \gamma_r T_{\text{Tr}^B(h_r)}$.

For each basis term h_r , the only discrepancy between $T_{\text{Tr}^B(h_r)}$ and $\text{Tr}^{|B|}((\text{id} \oplus S_B) T_{h_r})$ is the scalar $\delta^{\ell_B(h_r)}$ from newly formed cycles. By loop-freeness, $\ell_B(h_r) = 0$ whenever $\gamma_r \neq 0$, so $T_{\text{Tr}^B(h_r)} = \text{Tr}^{|B|}((\text{id} \oplus S_B) T_{h_r})$ for all contributing r . Therefore

$$T_{\text{Tr}^B(h)} = \sum_r \gamma_r \text{Tr}^{|B|}((\text{id} \oplus S_B) T_{h_r}) = \text{Tr}^{|B|}((\text{id} \oplus S_B) \sum_r \gamma_r T_{h_r}) = \text{Tr}^{|B|}((\text{id} \oplus S_B) T_h),$$

using $\mathbb{C}$ -linearity of T and of $\text{Tr}^{|B|}$, and the fact that S_B is a fixed permutation matrix (so $(\text{id} \oplus S_B)$ is $\mathbb{C}$ -linear). $\square$

D.5 EU closure

Theorem D.14 (EU closure for $\mathbb{QC}$). *Every morphism of $\mathbb{QC}$ is essentially unitary.*

Proof. By induction on the construction of $\mathbb{QC}$ (Definition D.9).

Generators. $\text{MLL}(+)$ morphisms are EU by Corollary D.2; exponentials by Proposition D.8.

Composition. Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be EU. Form the glued morphism $h := f \bowtie g : A \otimes B \rightarrow C \otimes B$; by Definition B.4, $g \circ f = \text{Tr}^B(h)$, hence $T_{g \circ f} = T_{\text{Tr}^B(h)}$.

Every $\mathbb{QC}$ morphism is a $\mathbb{C}$ -linear combination of $\text{MLL}(+)$ morphisms, each of which has $\kappa = 0$ (Corollary D.2). The $\text{MLL}(+)$ components of $h = f \bowtie g$ are pairs (f_i, g_j) , and Lemma D.4 gives $\ell_B(f_i \bowtie g_j) = 0$ for all such pairs. Hence h is loop-free under B -trace, and Lemma D.13 gives

$$T_{g \circ f} = \text{Tr}^{|B|}((\text{id} \oplus S_B) T_h),$$

where S_B is the gluing swap on internal B -ports (Definition B.4).

It remains to show this is unitary. The boundary operator on the glue is $T_h = T_f \oplus T_g$ (the standard GOI stacking: f and g act on disjoint port blocks, so T_h is block-diagonal); since f and g are EU, T_f and T_g are unitary, hence T_h is unitary. The swap S_B is a permutation matrix, hence unitary, so $(\text{id} \oplus S_B) T_h$ is unitary. By Lemma D.11, $\text{Tr}^{|B|}((\text{id} \oplus S_B) T_h)$ is an isometry. EU of f and g individually forces $|P_{A,B}| = |N_{A,B}|$ and $|P_{B,C}| = |N_{B,C}|$, hence $|P_{A,C}| = |N_{A,C}|$ (the external port sets have equal size), so isometry gives unitarity.

Tensor and sum. $T_{f \otimes g} = T_f \otimes T_g$ and $T_{f \oplus g} = T_f \oplus T_g$ (Lemma C.11); block-diagonal and Kronecker products of unitaries are unitary.

Dagger. $T_{f^\dagger} = (T_f)^\dagger$ (Lemma C.12); the adjoint of a unitary is unitary. $\square$

Because EU excludes scalar loops, discarding, probability extraction, and partial trace are not internal to $\mathbb{QC}$: the quantum core models coherent dynamics, not the classical interface.

E Internal Unitary Actions

We show that $\mathbb{QC}$ is expressively complete for coherent higher-order quantum computation. $X^{\oplus n}$ carries a canonical $\text{U}(n)$, $X \otimes X$ carries only $\text{U}(1)$, and linear contexts over $\mathbb{QC}$ define purity-preserving supermaps including a coherent quantum switch with explicit control.

E.1 Internal $U(n)$ on $X^{\oplus n}$

We begin with a classical Lie-theoretic fact.

Fact E.1 (Two-level generation of $U(n)$). *The unitary group $U(n)$ is generated by diagonal unitaries together with two-level unitaries: matrices that act as an element of $U(2)$ on a chosen pair of coordinates and as the identity elsewhere. Permutation matrices conjugate any two-level unitary to one acting on an arbitrary pair. This is the basis of Givens factorization; see [20, Chapter 5] and [26, Corollary 4.2] for the quantum computing formulation.*

We now internalise this directly on $X^{\oplus n}$.

The summand-level subalgebra. For any object X and $n \geq 1$, the endomorphism algebra $\text{End}_{\text{Perm}(\mathbb{C})}(X^{\oplus n})$ contains a distinguished subalgebra of *summand-level* endomorphisms, acting on each summand as a scalar multiple of id_X . An $n \times n$ scalar matrix (α_{jk}) determines an endomorphism whose (j, k) -component is $\alpha_{jk} \cdot \text{id}_X$. This subalgebra is canonically isomorphic, as a $*$ -algebra, to $M_n(\mathbb{C})$.

Summand permutations. A permutation $p \in S_n$ induces a summand permutation $p_X : X^{\oplus n} \rightarrow X^{\oplus n}$ sending the k th copy of X to the $p(k)$ th copy. This is a structural isomorphism with $p_X^{-1} = p_X^\dagger$, corresponding within the summand-level subalgebra to the permutation matrix of p .

Summand-wise phase taggers. For $(\lambda_1, \dots, \lambda_n) \in U(1)^n$, the endomorphism $D_X(\lambda_1, \dots, \lambda_n) : X^{\oplus n} \rightarrow X^{\oplus n}$ acts by $\lambda_k \cdot \text{id}_X$ on the k th summand. These are unitary, form a commuting subgroup isomorphic to $U(1)^n$, and satisfy $D_X(\lambda_1, \dots, \lambda_n) \circ p_X = p_X \circ D_X(\lambda_{p^{-1}(1)}, \dots, \lambda_{p^{-1}(n)})$.

Unitary taggers lie in $\mathbb{QC}$: for each summand M_k , the identity id_{M_k} is an involution in $\text{Perm}_{\text{MLL}(+)}^\pm$, so $\exp_i(\theta_k, \text{id}_{M_k}) = e^{i\theta_k} \text{id}_{M_k}$ belongs to $\mathbb{QC}$ by Definition D.9. Writing $\lambda_k = e^{i\theta_k}$, the tagger is $D_X(\lambda_1, \dots, \lambda_n) = \bigoplus_{k=1}^n \exp_i(\theta_k, \text{id}_{M_k})$, which lies in $\mathbb{QC}$ by $\oplus$ -closure. In particular, the Pauli-Z involution $[1, -1]_{X \oplus X} = \text{id}_X \oplus \exp_i(\pi, \text{id}_X)$ belongs to $\mathbb{QC}$.

Structural rotations. The additive symmetry $S_X : X \oplus X \rightarrow X \oplus X$, swapping the two copies of X , is a structural involution in $\mathbb{QC}$ ($S_X^\dagger = S_X$, $S_X^2 = \text{id}$). By closure under exponentials of involutions, the one-parameter family

$$\exp_i(\theta, S_X) := \cos \theta \cdot \text{id}_{X \oplus X} + i \sin \theta \cdot S_X$$

also lies in $\mathbb{QC}$. Within $\text{End}(X \oplus X) \cong M_2(\mathbb{C})$, S_X corresponds to the Pauli matrix $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Lemma E.2 (Generation of $U(2)$). *Let $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The subgroup generated by $\exp(i\theta X)$ and $\exp(i\phi Z)$ for $\theta, \phi \in \mathbb{R}$ is $SU(2)$; adjoining global phases yields $U(2)$. This is the Euler decomposition [26, Section 4.2].*

Proof. Every $U \in SU(2)$ can be written as $e^{iaZ} e^{ibX} e^{icZ}$ for suitable $a, b, c \in \mathbb{R}$. □

For $n \geq 2$, embedding $X \oplus X$ into the first two summands and extending by the identity yields the two-level rotation

$$R_X(\theta) := \exp_i(\theta, S_X) \oplus \text{id}_{X^{\oplus(n-2)}} : X^{\oplus n} \rightarrow X^{\oplus n}.$$

Theorem E.3 (Internal $U(n)$ on $X^{\oplus n}$). *For $n \geq 1$ and any object X , the endomorphism algebra $\text{End}_{\text{Perm}(\mathbb{C})}(X^{\oplus n})$ contains a canonical internal copy of $U(n)$, acting unitarily on the n summand copies of X and realised entirely by morphisms of $\mathbb{QC}$.*

Proof. The summand-level subalgebra of $\text{End}_{\text{Perm}(\mathbb{C})}(X^{\oplus n})$ is canonically isomorphic, as a $*$ -algebra, to $M_n(\mathbb{C})$ (each matrix entry α_{jk} acts as $\alpha_{jk} \cdot \text{id}_X$). Under this identification: the summand permutation p_X corresponds to the permutation matrix of p ; the phase tagger $D_X(\lambda_1, \dots, \lambda_n)$ corresponds to $\text{diag}(\lambda_1, \dots, \lambda_n)$; and the swap S_X corresponds to the Pauli matrix X .

By Lemma E.2, the diagonal taggers together with the rotation $\exp_i(\theta, S_X)$ generate the embedded $U(2)$ acting on the first two summands. Conjugation by summand permutations moves this $U(2)$ to any pair of summands, yielding all two-level unitaries. By Fact E.1, two-level unitaries and diagonal unitaries generate $U(n)$.

All generators belong to $\mathbb{QC}$: summand permutations lie in the $\text{MLL}(+)$ image; summand-wise phase taggers are constructed from $\oplus$ -closure of exponentials of the identity involution (as shown above); and the rotations $\exp_i(\theta, S_X)$ are exponentials of a structural involution. Since $\mathbb{QC}$ is closed under composition and dagger, the entire $U(n)$ action on $X^{\oplus n}$ is realised inside the unitary core. $\square$

The $U(n)$ action lives in the summand-level subalgebra $M_n(\mathbb{C}) \cdot \text{id}_X$, which is typically a proper subalgebra of $\text{End}(X^{\oplus n}) \cong M_n(\text{End}(X))$; the internal structure of X is left untouched.

Multiplicative versus additive boundaries. $X^{\oplus 2}$ supports a nonabelian $SU(2)$ action, while the tensor square $X \otimes X$ supports only an abelian $U(1)$ generated by the twist (Theorem E.4 below). This separation clarifies the roles of the two monoidal structures: $\otimes$ expresses physical connectivity, while $\oplus$ carries coherent superposition and control. The recovery of $U(n)$ on $X^{\oplus n}$ therefore suffices for the standard finite-dimensional Hilbert space formalism, without conflating dynamical symmetry with structural composition.

Twist-generated $U(1)$ on $X \otimes X$. Symmetry $\sigma_{X,X} : X \otimes X \rightarrow X \otimes X$ is a Hermitian involution in $\mathbb{QC}$.

Theorem E.4 (Twist-generated $U(1)$). *For every object X , the one-parameter family $\theta \mapsto \exp_i(\theta, \sigma_{X,X}) = \cos \theta \text{id} + i \sin \theta \sigma_{X,X}$ defines an internal unitary $U(1)$ representation inside $\text{End}_{\text{Perm}(\mathbb{C})}(X \otimes X)$, realised entirely within $\mathbb{QC}$. This action is abelian; no nonabelian Lie action can be generated from a single Hermitian involution (the generated Lie algebra is one-dimensional).*

Proof. Since $\sigma_{X,X}$ is a Hermitian involution, the essential unitarity of exponentials (Proposition D.8) applies. Separating even and odd powers using $\sigma_{X,X}^2 = \text{id}$ yields the closed form $\exp_i(\theta, \sigma_{X,X}) = \cos \theta \text{id} + i \sin \theta \sigma_{X,X}$. Unitarity follows from the same computation as in the proof of essential unitarity of exponentials. Each $\exp_i(\theta, \sigma_{X,X})$ lies in the $\mathbb{C}$ -linear span of id and $\sigma_{X,X}$, so the resulting one-parameter subgroup is abelian. All morphisms involved belong to $\mathbb{QC}$ by Definition D.9. $\square$

Choosing $\theta = \frac{\pi}{2n}$ shows that the one-parameter subgroup generated by $\sigma_{X,X}$ contains n -th roots of $\sigma_{X,X}$ up to global phase.

E.2 Contexts as supermaps

We now make precise when a syntactic context defines a well-behaved higher-order operation.

Theorem E.5 (Linear contexts induce supermaps). *Let $\mathcal{U}$ be any class of EU morphisms in $\mathbb{QC}$. An n -hole linear context is a well-typed judgment $\square_1 : A_1 \rightarrow B_1, \dots, \square_n : A_n \rightarrow B_n \vdash t : C \rightarrow D$ with $t ::= \square_i \mid u \mid (t_2 \circ t_1) \mid (t_1 \otimes t_2) \mid (t_1 \oplus t_2)$, $u \in \mathcal{U}$, and each $\square_i$ occurring exactly once. Then t induces a map $\Phi_t : \prod_i \text{Perm}(\mathbb{C})(A_i, B_i) \rightarrow \text{Perm}(\mathbb{C})(C, D)$ that is multilinear (in the ambient $\mathbb{C}$ -linear category $\text{Perm}(\mathbb{C})$) and unitary-preserving: if each argument is EU, so is the result.*

Proof. Substitution into a well-typed linear context yields a well-typed morphism in $\text{Perm}(\mathbb{C})$; the single-use condition guarantees each f_i appears exactly once.

Multilinearity. This holds in the ambient category $\text{Perm}(\mathbb{C})$, whose composition and tensor are $\mathbb{C}$ -bilinear. Fix a slot k and fix all arguments except f_k . Proceed by structural induction on t . The base cases $t = \square_i$ and $t = u$ are immediate. If $t = t_2 \circ t_1$, then exactly one of t_1, t_2 contains $\square_k$; thus Φ_t is pre- or post-composition by a fixed morphism, hence linear by bilinearity of $\circ$. If $t = t_1 \otimes t_2$, then $\square_k$ occurs in exactly one factor, so Φ_t is tensoring with a fixed morphism, hence linear by bilinearity of $\otimes$. The case $t = t_1 \oplus t_2$ is analogous: $\oplus$ is $\mathbb{C}$ -bilinear on hom-spaces. Since k was arbitrary, Φ_t is multilinear.

Unitary preservation. Assume each f_i is EU. Induct on t . If $t = \square_i$, then $\Phi_t(\vec{f}) = f_i$ is EU. If $t = u$, then $\Phi_t(\vec{f}) = \llbracket u \rrbracket$ is EU since $u \in \mathcal{U}$. If $t = t_2 \circ t_1$ or $t = t_1 \otimes t_2$ or $t = t_1 \oplus t_2$, then $\Phi_t(\vec{f})$ is a composition, tensor, or sum of EU morphisms by the induction hypothesis, hence EU by Theorem D.14. $\square$

Because each hole occurs exactly once, Φ_t cannot duplicate or discard an argument. Taking $\mathcal{U} = \mathbb{QC}$, every linear context defines a multilinear, EU-preserving, causally consistent supermap—consistency is enforced by linear syntax rather than external axioms. Note that $\mathbb{QC}$ itself is not $\mathbb{C}$ -linear (arbitrary linear combinations of EU morphisms need not be EU); multilinearity refers to the ambient category $\text{Perm}(\mathbb{C})$.

E.3 The coherent quantum switch

We exhibit the quantum switch as a coherent higher-order morphism in $\mathbb{QC}$, parametric in a control type. Unlike the extensional formulation of [10], in which the control system is traced out after use, the present construction retains the control wire throughout.

Fix an object A and let $U_A := A^* \otimes A$. For any object X , define the *controlled switch* $\text{QSwitch}_X : (X \oplus X) \otimes U_A \otimes U_A \rightarrow (X \oplus X) \otimes U_A$ summandwise by $\text{QSwitch}_X := (\text{id}_X \otimes \text{comp}_A) \oplus (\text{id}_X \otimes (\text{comp}_A \circ \sigma_{U_A, U_A}))$, where $\text{comp}_A : U_A \otimes U_A \rightarrow U_A$ is internal composition of endomorphisms (currying $\text{ev}_{A,A} \circ (\text{id}_{U_A} \otimes \text{ev}_{A,A})$), and σ_{U_A, U_A} is the symmetry. In the first (resp. second) summand, the inputs are composed as $f \circ g$ (as $g \circ f$). The control wire $X \oplus X$ is preserved coherently.

Every component— comp_A , σ , id_X , and the $\oplus$ -decomposition—is a structural map of $\text{Perm}_{\text{MLL}(+)}^\pm$, so QSwitch_X is EU by Corollary D.2 and lies in $\mathbb{QC}$.

Quantum behaviour arises from the control wire. By Theorem E.3, $X \oplus X$ carries an internal $\text{SU}(2)$ action; preparing the control in a superposition of summands (via a rotation $\exp_i(\theta, S_X)$) before applying QSwitch_X yields a coherent superposition of the two causal orders $f \circ g$ and $g \circ f$. The switch itself is purely structural wiring; the quantum content resides entirely in the preparation of the control.

Definability of purity-preserving supermaps. The coherent quantum switch exemplifies a general principle: programs in $\mathbb{QC}$ define coherent higher-order operations on their holes, inducing purity-preserving supermaps with explicit memory. The converse holds for equal-size supermaps:

A purity-preserving supermap $\Phi : (A \rightarrow A) \rightarrow (B \rightarrow B)$ admits a pure comb realization (Chiribella [9]): there exist ancillary systems M, N and isometries $V_1 : B \otimes M \rightarrow A \otimes N$ and $V_2 : A \otimes N \rightarrow B \otimes M$ such that $\Phi^\sharp(f) = V_2 \circ (f \otimes \text{id}_N) \circ V_1$. Choosing the canonical first-order encodings $A \cong \mathbf{H}_a$, $B \cong \mathbf{H}_b$, $M \cong \mathbf{H}_m$, $N \cong \mathbf{H}_n$ (Definition B.8), the isometry conditions force $bm = an$; when $a = b$, this gives $m = n$, so both V_1 and V_2 are unitary. Under the encoding, $B \otimes M \cong X^{\oplus bm}$ with $X = (1, 0) \otimes (1, 0)$; Chiribella's unitaries act on summand indices, hence lie in the summand-level $\mathcal{U}(bm)$ provided by Theorem E.3.

Corollary E.6 (Definability of equal-size purity-preserving supermaps). *Let $\Phi : (A \rightarrow A) \rightarrow (B \rightarrow B)$ be a purity-preserving supermap in the sense of [9]. Assume that A and B have the same finite size, i.e. the same number of $\oplus$ -summands. Then Φ admits a coherent unitary realization in $\mathbb{QC}$: there exist*

ancillary systems M, N of equal size and unitaries $V_1 : B \otimes M \rightarrow A \otimes N$ and $V_2 : A \otimes N \rightarrow B \otimes M$ such that $\Phi^\sharp(f) := V_2 \circ (f \otimes \text{id}_N) \circ V_1 : B \otimes M \rightarrow B \otimes M$ is definable in $\mathbb{QC}$ for every $f : A \rightarrow A$.

Proof. We work under the canonical first-order encoding $\mathbf{H}_n := (1, 0)^{\oplus n}$ (Definition B.8), so that size coincides with the number of $\oplus$ -summands and hence with Hilbert-space dimension in Chiribella's framework. Write $A \cong \mathbf{H}_a, B \cong \mathbf{H}_b$.

By the quantum comb realization theorem in the purity-preserving case [9, Theorem 6, specialized to the 1-slot case], the supermap Φ admits a coherent realization by a pure 1-comb: there exist ancillary systems $M \cong \mathbf{H}_m, N \cong \mathbf{H}_n$ and linear isometries

$$V_1 : B \otimes M \longrightarrow A \otimes N, \quad V_2 : A \otimes N \longrightarrow B \otimes M,$$

such that for every $f : A \rightarrow A$,

$$\Phi^\sharp(f) = V_2 \circ (f \otimes \text{id}_N) \circ V_1.$$

Since V_1 and V_2 are isometries and size is multiplicative under $\otimes$ (by strict distributivity, $\text{size}(A \otimes B) = \text{size}(A) \cdot \text{size}(B)$), we have $\text{size}(B) \text{size}(M) \leq \text{size}(A) \text{size}(N)$ and $\text{size}(A) \text{size}(N) \leq \text{size}(B) \text{size}(M)$, hence $\text{size}(B) \text{size}(M) = \text{size}(A) \text{size}(N)$. Thus V_1 and V_2 are isometries between spaces of equal size and therefore are unitary. Tensor distributes bilinearly over $\oplus$, hence $B \otimes M \cong (1, 0)^{\oplus b} \otimes (1, 0)^{\oplus m} \cong ((1, 0) \otimes (1, 0))^{\oplus bm}$, and similarly $A \otimes N \cong ((1, 0) \otimes (1, 0))^{\oplus an}$. Since $\text{size}(A) = \text{size}(B)$, we have $a = b$, and from the equality above it follows that $bm = an$. Hence both tensor products are of the homogeneous form $X^{\oplus k}$ with $X = (1, 0) \otimes (1, 0)$ and $k = bm = an$. By Theorem E.3, for every object X , $\mathbb{QC}$ contains a canonical internal copy of $U(k)$ acting on $X^{\oplus k}$ via the summand-level subalgebra $M_k(\mathbb{C}) \cdot \text{id}_X$. It remains to observe that V_1 and V_2 lie in this subalgebra. In Chiribella's Hilbert-space framework, $V_1 \in U(k)$ acts on a k -dimensional space whose basis vectors correspond to the summands of $X^{\oplus k}$; each basis vector represents one copy of X , and V_1 mixes summand indices while leaving the internal structure of X untouched. Under our encoding this is precisely a summand-level morphism $(V_1)_{jl} \cdot \text{id}_X$, which Theorem E.3 provides. Applying this with $X = (1, 0) \otimes (1, 0)$ and $k = bm = an$, both V_1 and V_2 are definable in $\mathbb{QC}$. Since $\mathbb{QC}$ is closed under tensor, composition, and contains all structural maps, the coherent realization $\Phi^\sharp(f)$ is definable in $\mathbb{QC}$ for every $f : A \rightarrow A$. $\square$

Purity-preserving comb realizations are thus interpreted by EU morphisms, and EU is stable under the linear contexts that define supermaps.

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