ZX Calculus Tutorial

Part 2

Presented by **Razin Shaikh**

15 August 2026

Overview

- Part 1 of the tutorial covered:
  - Basics of ZX calculus
  - Circuit Optimisation
  - MBQC

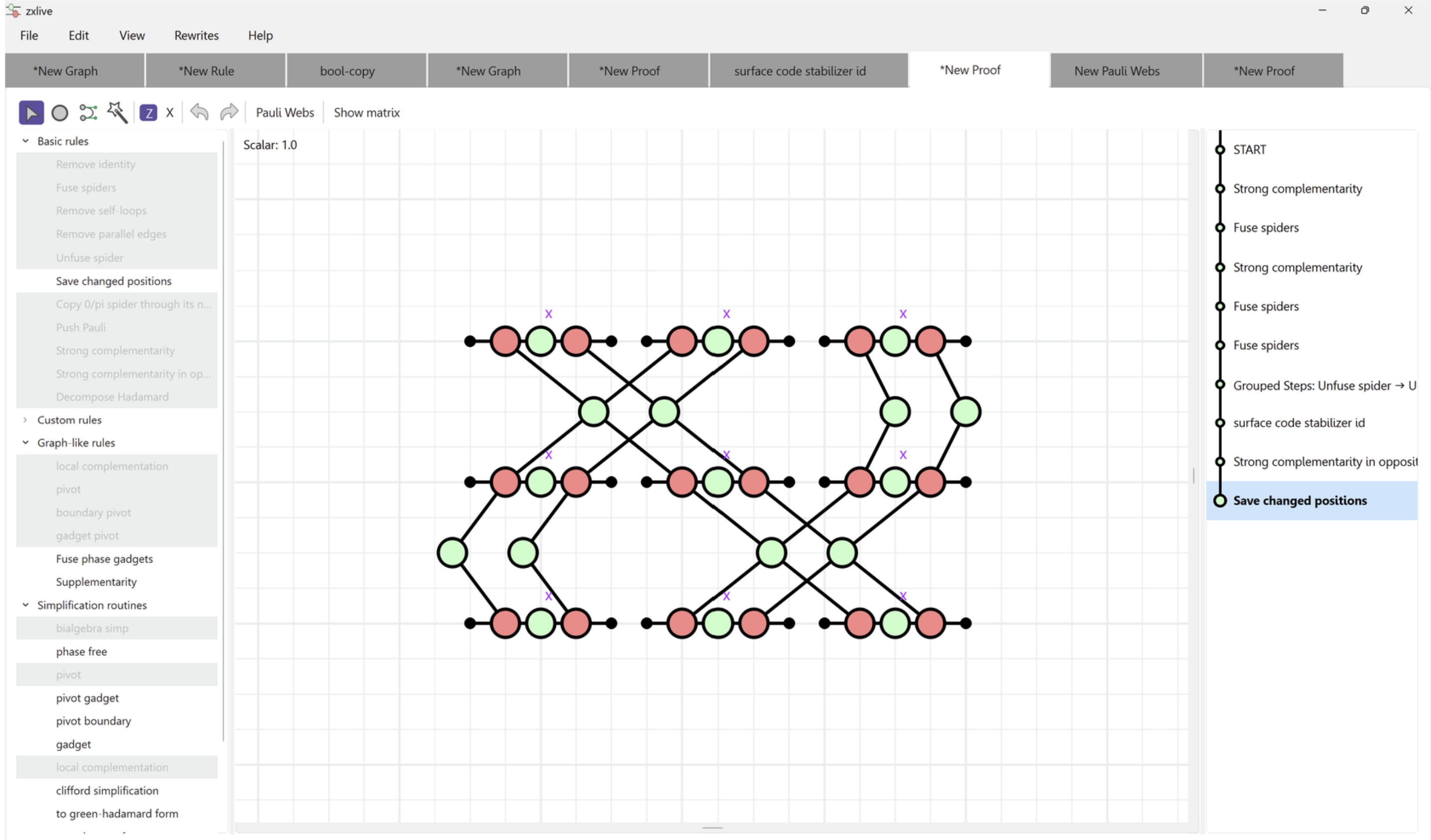
- Part 2 will cover:
  - Quantum Error Correction
  - Higher dimensional variants of ZX calculus

ZXLive tutorial: Monday 17:30 – 18:30

ZXLive

test passing | lint passing | Rolling Release passing | download rolling binaries

The ZX-calculus gives us a handy way to represent and work with quantum computations. ZXLive is an interactive tool for working with ZX. Draw graphs or load circuits and apply ZX rules. Intended for experimenting, building proofs, helping to write papers, showing off, or simply learning about ZX and quantum computing. It is powered by the pyzx open source library under the hood. The documentation is available at https://zxlive.readthedocs.io/



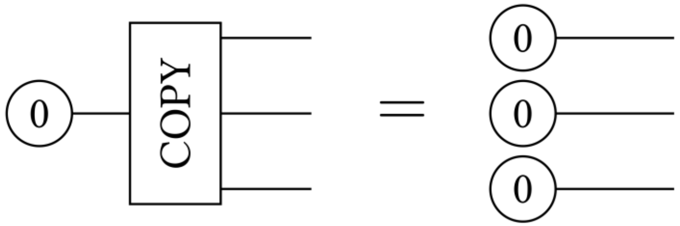
Quantum Error Correction

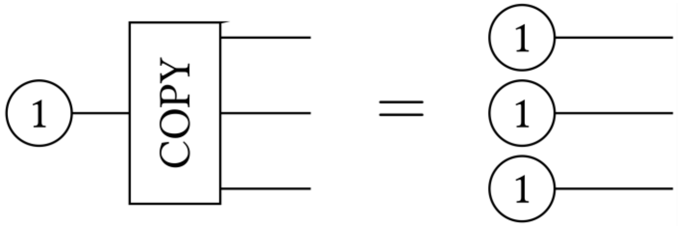
Quantum Error Correction: Talk roadmap

- The three-bit repetition code
- The three-qubit repetition code
- Stabilizer codes and their ZX diagrams

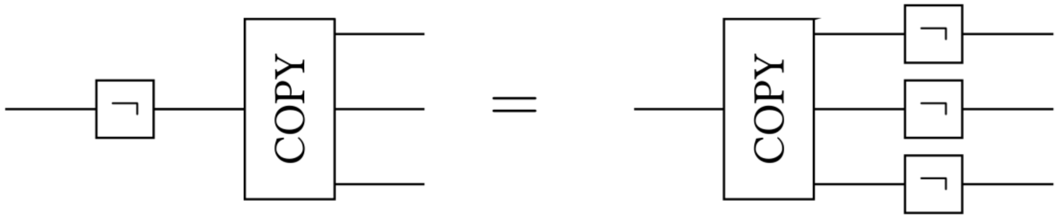
The three-bit repetition code

- Encode 0 as 000 and 1 as 111.





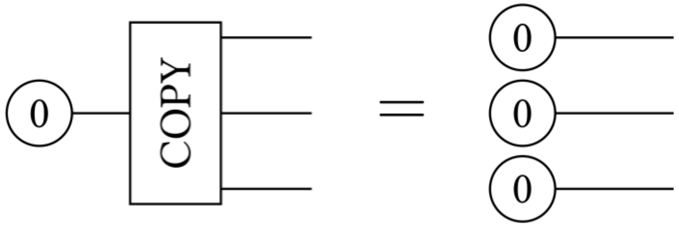
- To do the logical NOT gate, do the NOT gate on all 3 physical bits.

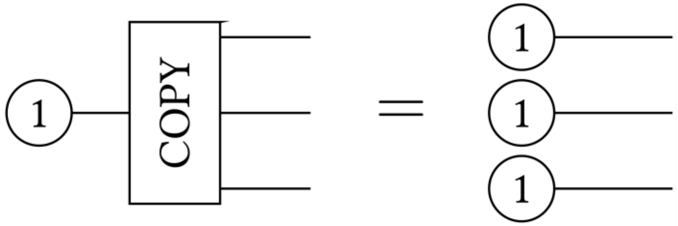


- To check if up to one bit flipped, i.e. an error occurred:
  - Check that the parity of any two bits is zero: $b_1 \oplus b_2 = 0$
  - Because: $0 \oplus 0 = 0$ and $1 \oplus 1 = 0$ BUT $0 \oplus 1 = 1$ and $1 \oplus 0 = 1$

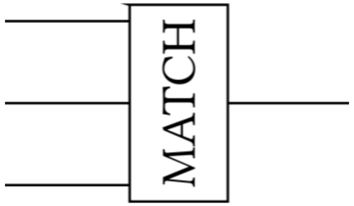
The three-bit repetition code

- Encode 0 as 000 and 1 as 111.

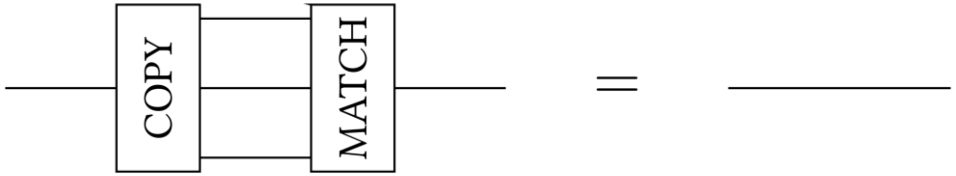




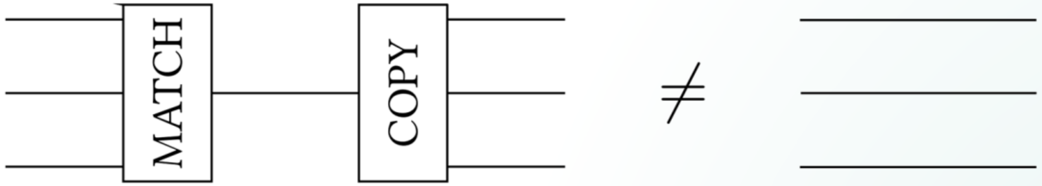
- To decode this information: match the three bits



Encode and then decode is equal to identity.

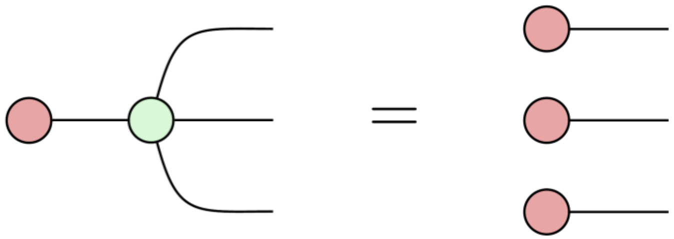


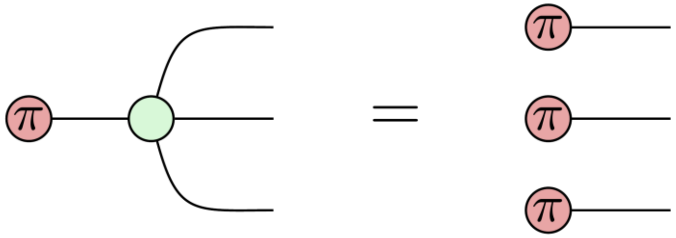
However, decode then encode is not identity.



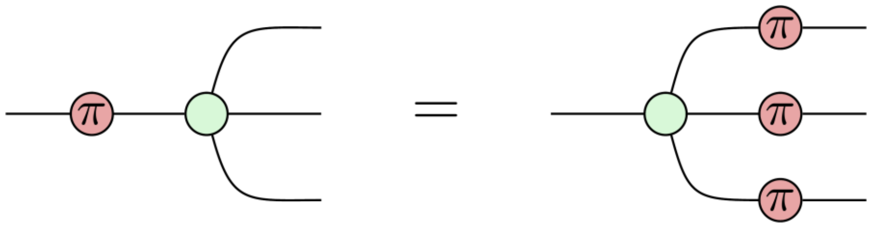
The three-qubit repetition code

- Encode $|0\rangle$ as $|000\rangle$ and $|1\rangle$ as $|111\rangle$.

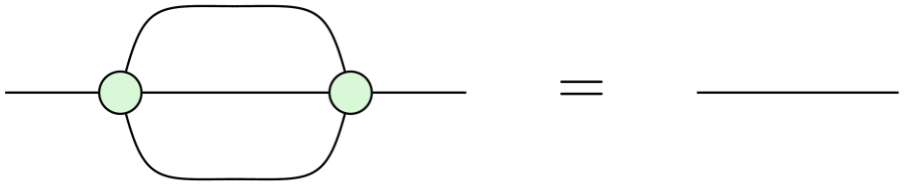




- To do the logical NOT gate, do the NOT gate on all 3 physical bits.



- Encode then decode is identity on the logical bit.



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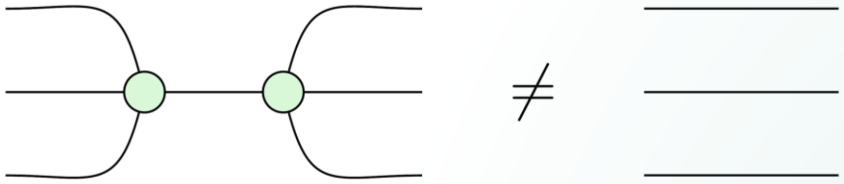
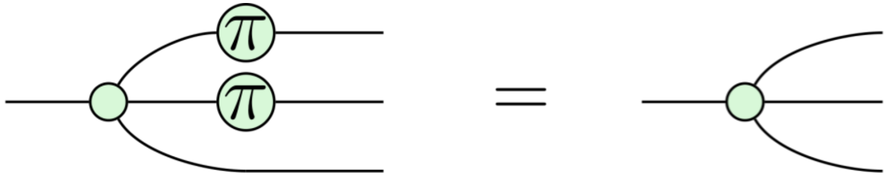
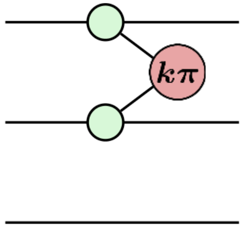


The three-qubit repetition code

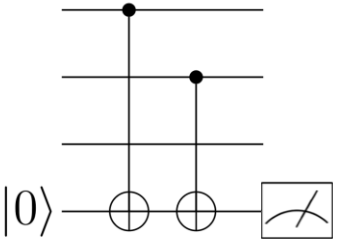
- All states in the code space are stabilized by ZZI.



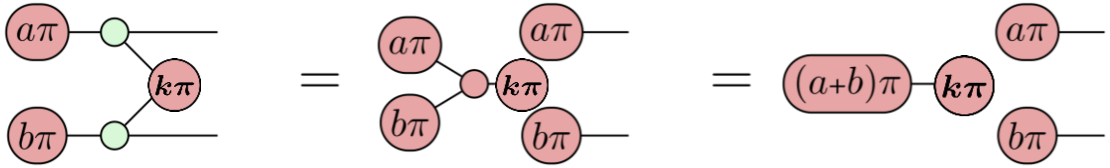
- We can measure this stabilizer without collapsing logical information



As a circuit:



- This measures the parity of the first two qubits



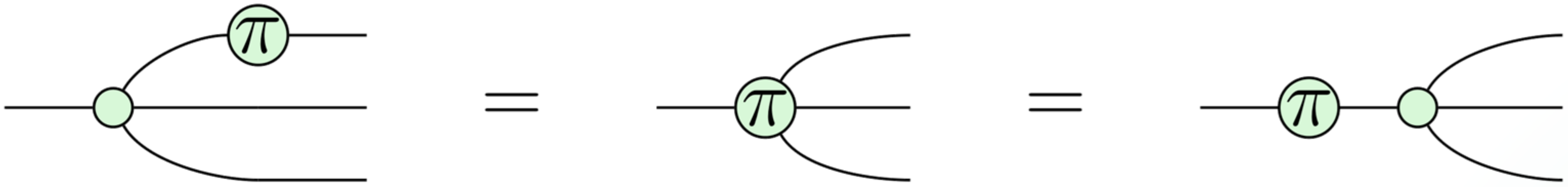
$$\implies \quad k = a \oplus b$$

This detects a bit flip similar to the classical case

The three-qubit repetition code

- While this code protects against X errors (i.e. bit flips), it fails to protect against Z errors (i.e. phase flips).

- Proof: Physical Z error is equal to a logical Z operation



- To fix: We need more stabilizers:
  - In this case, involving Pauli X or Y (as they anti-commute with Z)

- In general, we get stabilizer codes with many stabilizers

$[\![n, k, d]\!]$ Stabiliser Codes

n: number of physical qubits
k: number of logical qubits
d: distance of the code

are fixed by $n - k$ independent Pauli operators, called the **stabilizer generators.**

These fix the code, but do not fix the mapping of logical states to physical states.

That is fixed by specifying $2k$ more Pauli operators, called the **logical operators**.

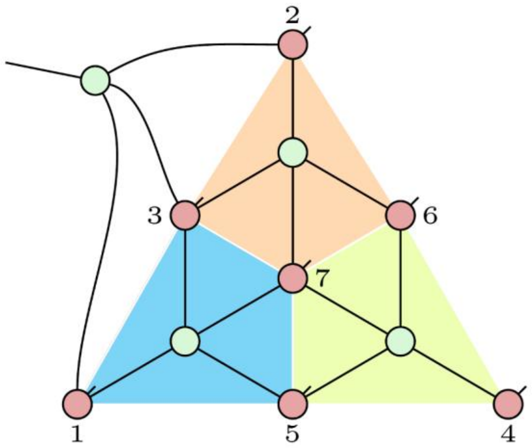
In total, these $n + k$ Pauli operators fix the **encoder map** E.

ZX Diagrams for Stabilizer Codes

- As a running example, we will consider the code. It is typically defined by

$$H_X = \begin{bmatrix} 1&0&1&0&1&0&1 \\ 0&1&1&0&0&1&1 \\ 0&0&0&1&1&1&1 \end{bmatrix} \qquad H_Z = \begin{bmatrix} 1&0&1&0&1&0&1 \\ 0&1&1&0&0&1&1 \\ 0&0&0&1&1&1&1 \end{bmatrix}$$

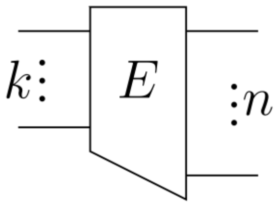
- And typically depicted as the following (without the overlaid ZX diagram)

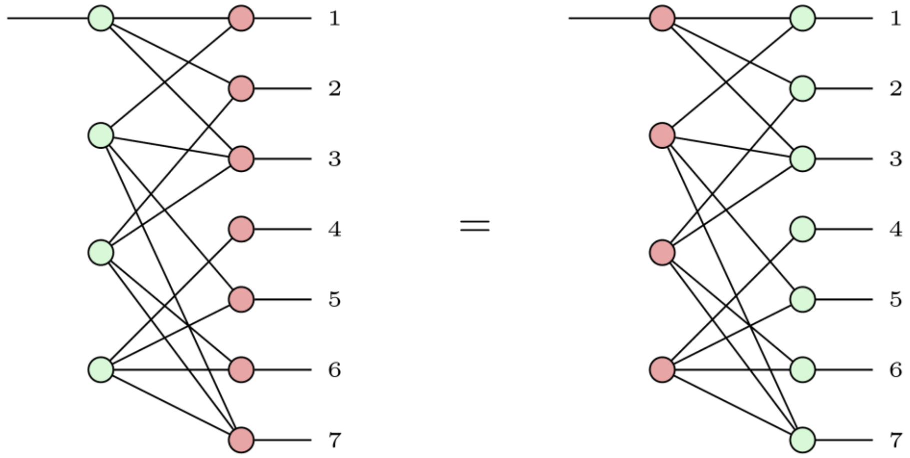


ZX Diagrams for Stabilizer Codes

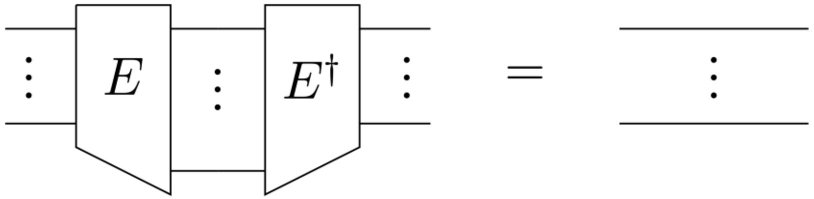
Example: the $[\![7,1,3]\!]$ code

- The encoder E encodes k logical qubits into n physical qubits.





- $E^\dagger$, its one-sided inverse, is the ideal decoder.



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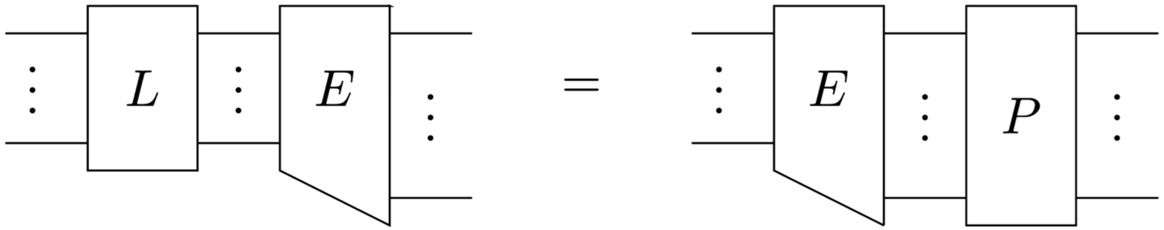
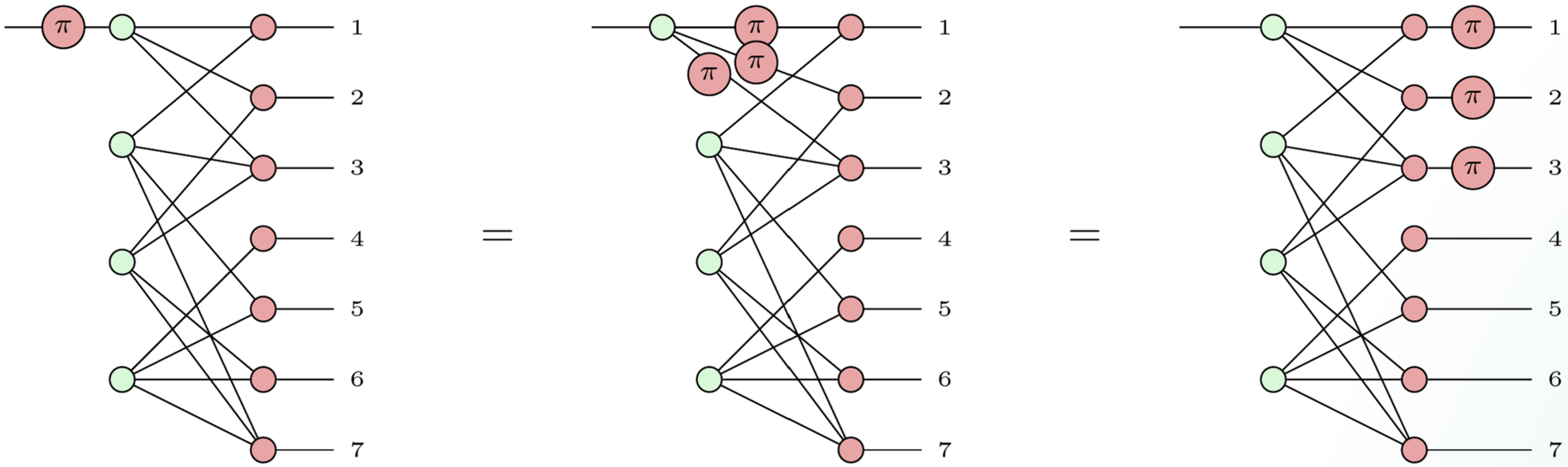
ZX Diagrams for Stabilizer Codes

Example: the $[\![7,1,3]\!]$ code

An operation P on the physical qubits implements a logical operation if it preserves the code space

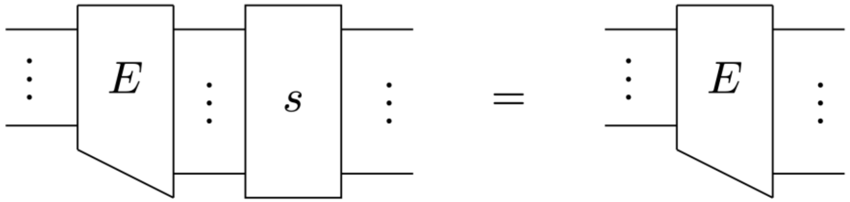




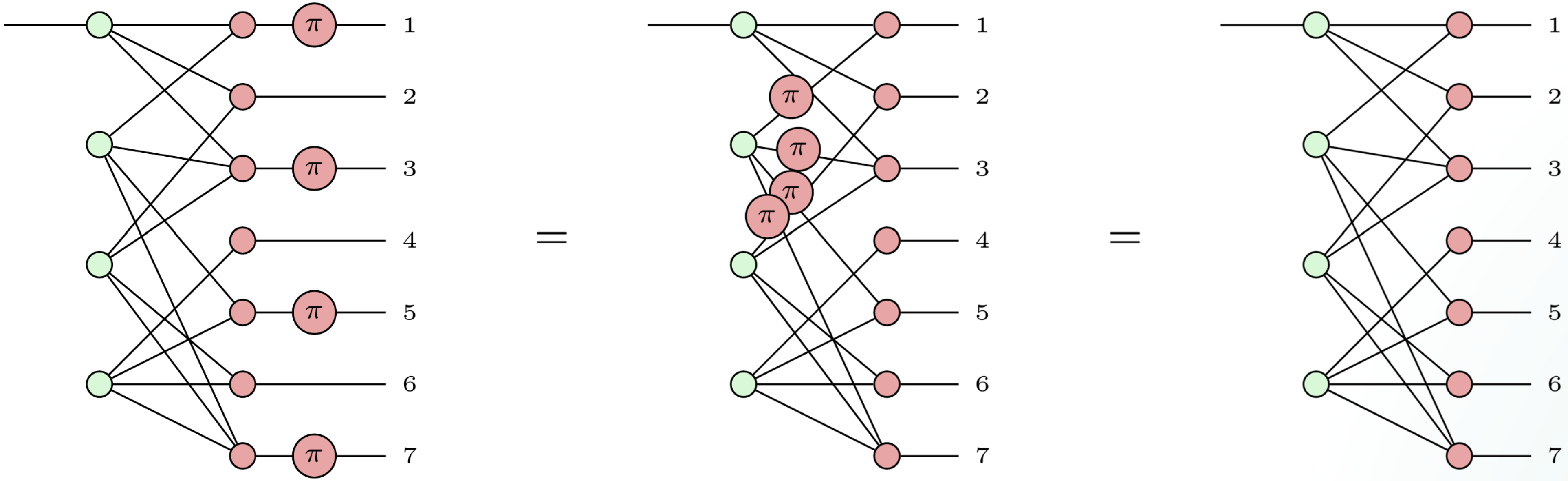
ZX Diagrams for Stabilizer Codes

Example: the $[\![7,1,3]\!]$ code

Any Pauli operator s in stabilizer group stabilizes E.



$$H_X = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$



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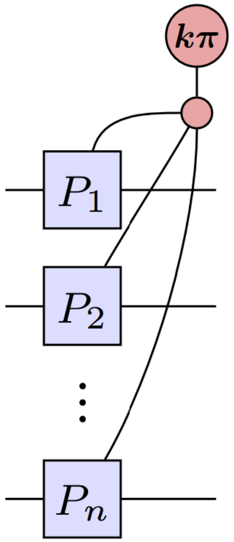
ZX Diagrams for Stabilizer Codes

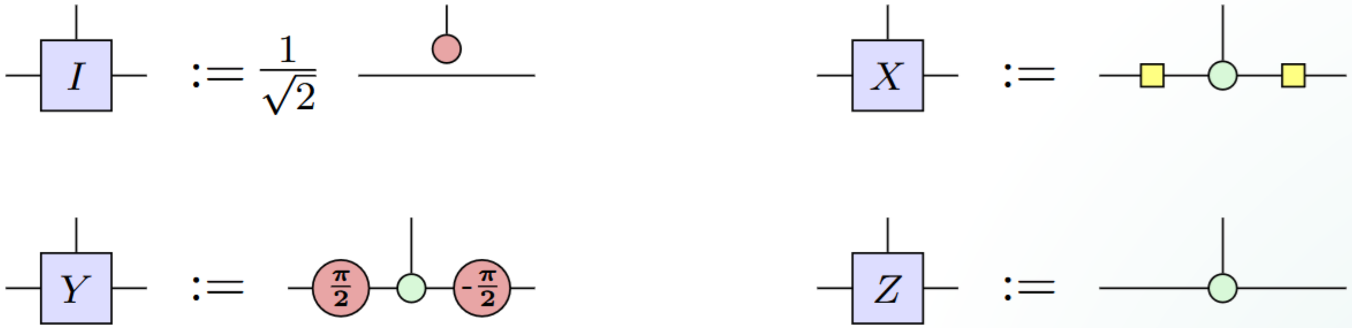
Example: the $[\![7,1,3]\!]$ code

We can repeatedly measure stabilizers to detect errors

Measuring the Pauli stabilizer $P_1 \dots P_n$:



Where the Pauli box P_i is given by:



ZX Diagrams for Stabilizer Codes

Example: the $[\![7,1,3]\!]$ code

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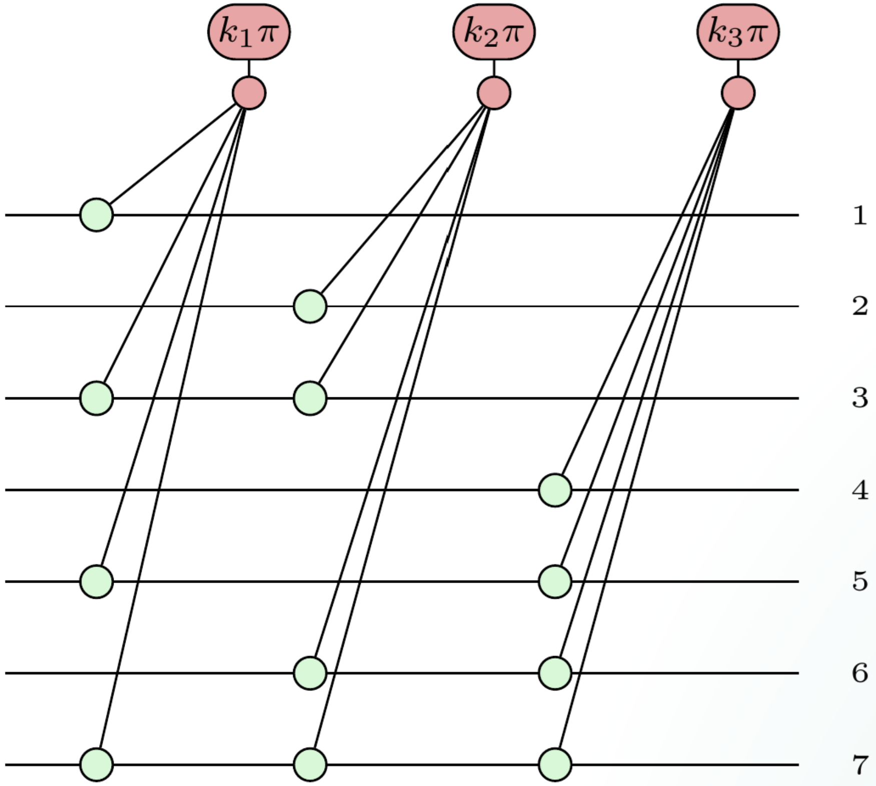
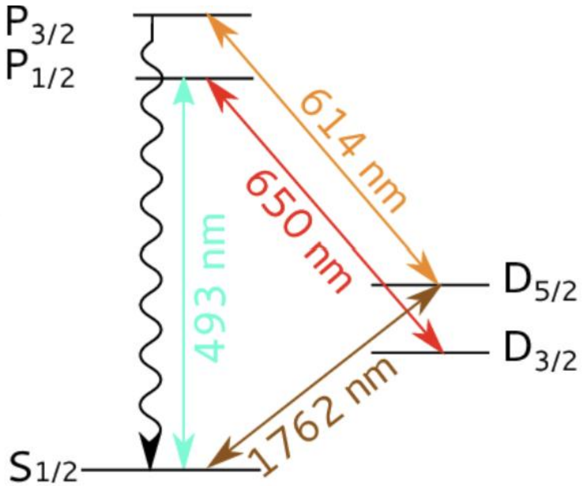


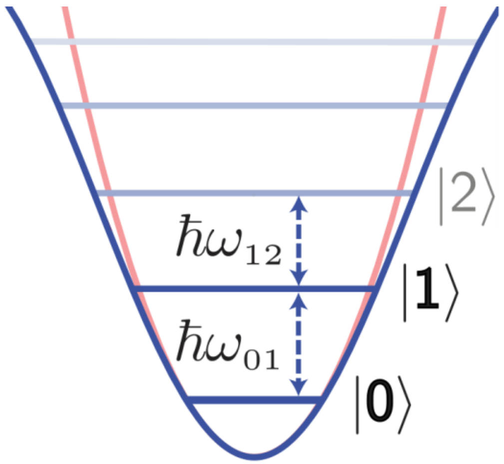
Higher dimensional ZX calculi

Higher dimensional ZX calculi - Motivation

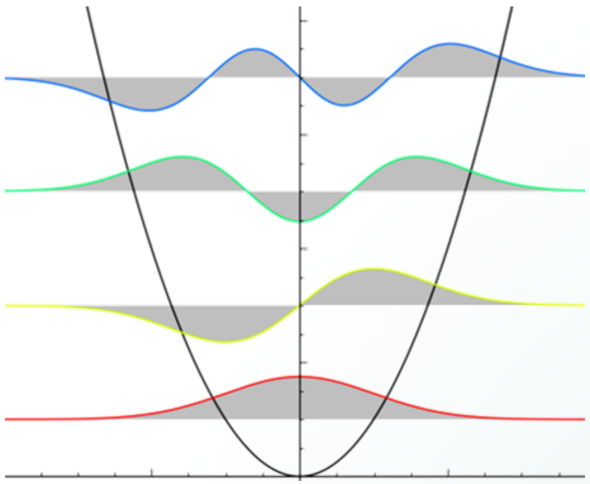
- Nature is not binary; qubits are an abstraction
- Most hardware is natively higher-dimensional



Trapped ions & neutral atoms
Multi-level hyperfine states



Superconducting
Higher transmon energy levels



Photonic
Quantum harmonic oscillator

Higher dimensional ZX calculi - Motivation

- Nature is not binary; qubits are an abstraction
- Most hardware is natively higher-dimensional
- Hardware-efficient quantum error correction
  - GKP qubits, and cat qubits
- Algorithmic and circuit efficiency
  - n-controlled toffoli gate without ancilla: $O(n^2)$ for qubits vs $O(n)$ for qutrits
  - Richer transversal gate sets & improved magic state distillation routines
- We can generalize the ZX-based techniques to higher dimensions

Higher dimensional ZX calculi: Talk roadmap

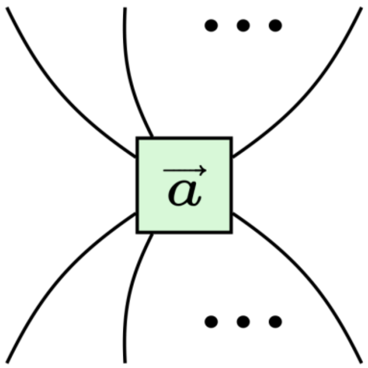
- Qudit ZX
- Mixed-dimensional ZX
- Infinite-dimensional ZX
  - Discrete variables
  - Continuous variables

Qudit ZX calculus

A qubit is a 2 dimensional vector: $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$

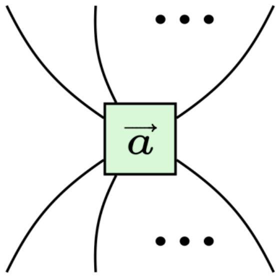
A qudit is a d-dimensional vector (for d ≥ 1):

$$|\psi\rangle = a_0\,|0\rangle + a_1\,|1\rangle + a_2\,|2\rangle + \cdots + a_{d-1}\,|d-1\rangle$$

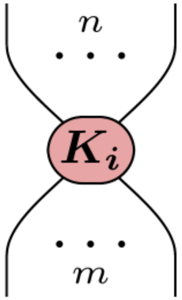


$$\xmapsto{[\![\cdot]\!]} \quad \sum_{j=0}^{d-1} a_j\,|j\rangle^{\otimes m}\,\langle j|^{\otimes n}$$

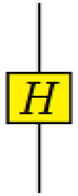
Qudit ZX calculus: Generators



$$\xmapsto{[\![\cdot]\!]} \quad \sum_{j=0}^{d-1} a_j \, |j\rangle^{\otimes m} \, \langle j|^{\otimes n}$$

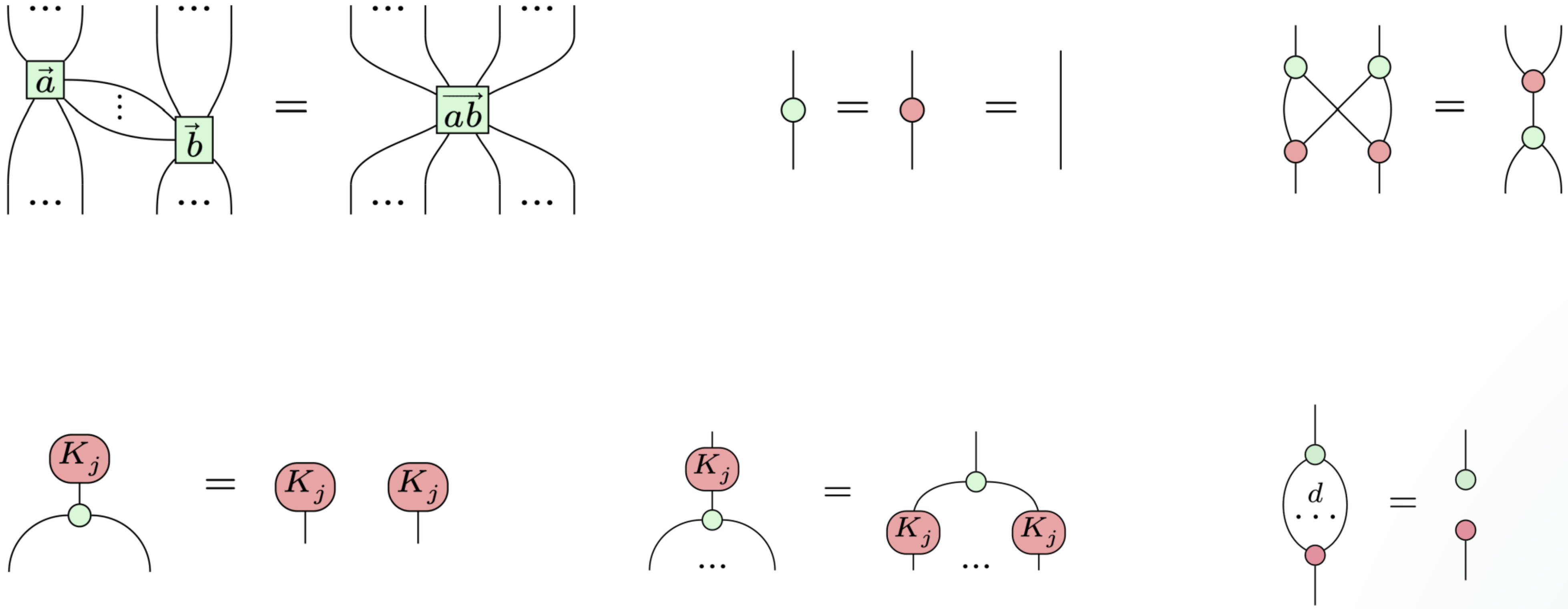


$$\xmapsto{[\![\cdot]\!]} \quad \sum_{\substack{i+j_1+\cdots+j_m \\ \equiv k_1+\cdots+k_n \pmod{d}}} |j_1, \cdots, j_m\rangle \, \langle k_1, \cdots, k_n|$$



$$\xmapsto{[\![\cdot]\!]} \quad \frac{1}{\sqrt{d}} \sum_{k,j=0}^{d-1} \omega^{jk} \, |j\rangle \, \langle k|$$

Qudit ZX calculus: Rules



Overview of ZX Completeness Results

	Qubits (d=2)	Qupits (d odd prime)	Qudits (d >= 2)	FHilb Mixed-dimensions	Infinite dimensional
Stabilizer	· Backens, 2014				
Approx. Universal	· Jeandel, Perdrix & Vilmart, 2018				
Universal	· Ng, Wang & Hadzihasanovic, 2018 · Jeandel, Perdrix & Vilmart, 2018				

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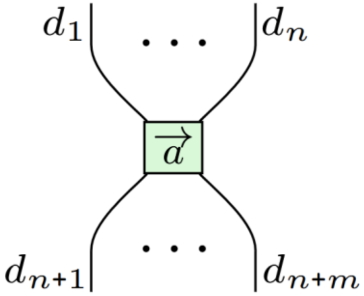
Universal	· Ng, Wang & Hadzihasanovic, 2018 · Jeandel, Perdrix & Vilmart, 2018	Follows from the qudits results on the right	· ZXW – Poór et. al., 2023 · open question for ZX		

Mixed-dimensional ZX calculus

Each wire is labelled by its qudit dimension.

Mixed-dimensional Z spider is defined as:

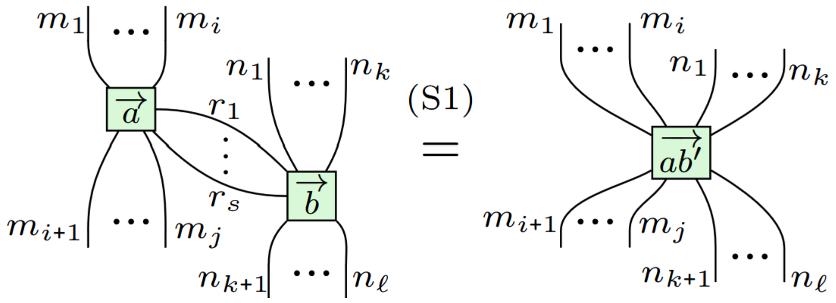


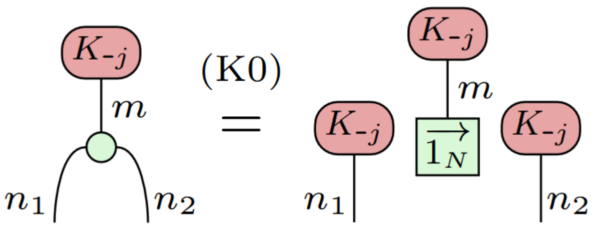
$$\xmapsto{[\![\cdot]\!]} \quad \sum_{j=0}^{\min\{d_i\}_i-1} a_j \left|j,\cdots,j\right\rangle \left\langle j,\cdots,j\right|$$

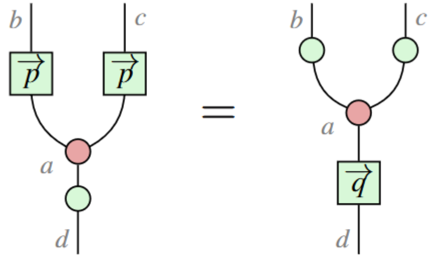
X spider is same as the qudit version.

Mixed-dimensional ZX calculus: Rules

All the rules of qudit ZX calculus hold. Non-qudit rules include:





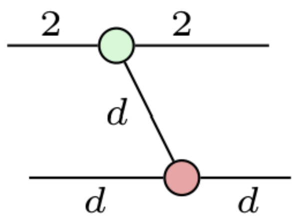


(PC)

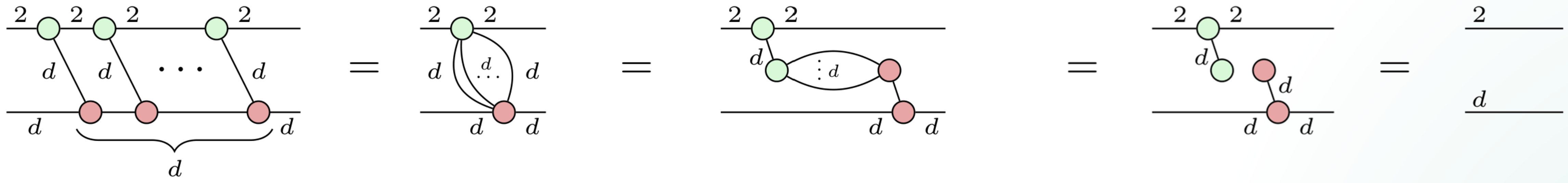
where $\vec{q}$ is a vector such that for all $0 \leq i < b$ and $0 \leq j < c$ we have $p_i p_j = q_{i+j \pmod{a}}$.

Mixed-dimensional ZX calculus: Example

Qubit-controlled CNOT on a qudit:



When applied d-times, is equal to the identity:



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Universal	· Ng, Wang & Hadzihasanovic, 2018 · Jeandel, Perdrix & Vilmart, 2018	Follows from the qudits results on the right	· ZXW – Poór et. al., 2023 · open question for ZX	· ZX – Poór, Shaikh & Wang, 2024	

Infinite-dimensional ZX: Discrete-variables

- "Like a qudit with $d = \infty$"

- A single-mode state is a superposition of infinite basis: $\sum_{n=0}^{\infty} a_n |n\rangle$

- Example: Coherent state

  State emitted by an ideal laser

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

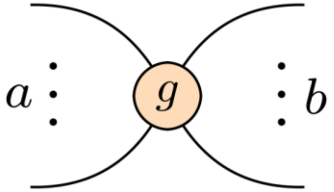
Probability of measuring n photons:

$$P(n) = |\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!}$$

Infinite-dimensional ZX: Discrete-variables

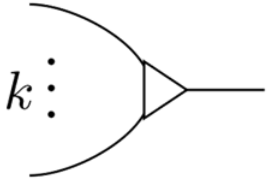
Generators:

Z spider (also called the Fock spider)



$$\xrightarrow{[\![\cdot]\!]} \quad \sum_{n=0}^{\infty} g(n)\, |n\rangle^{\otimes b} \langle n|^{\otimes a}$$

W node

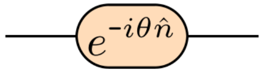


$$\xrightarrow{[\![\cdot]\!]} \quad \sum_{n_1,\ldots,n_k \geqslant 0} \sqrt{\frac{(\sum_i n_i)!}{\prod_i n_i!}} \left|\sum_i n_i\right\rangle\left\langle n_1, \ldots, n_k\right|$$

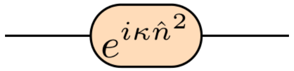
Infinite-dimensional ZX: Discrete-variables

Common Gates:

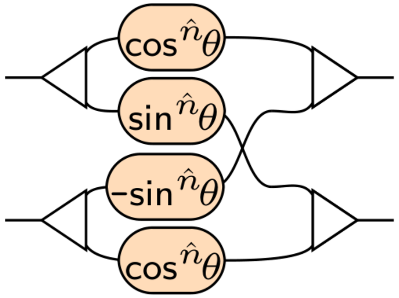
Rotation
$e^{i\theta\hat{n}}$



Kerr
$e^{i\kappa\hat{n}^2}$

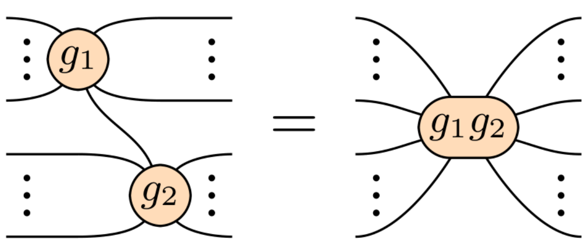


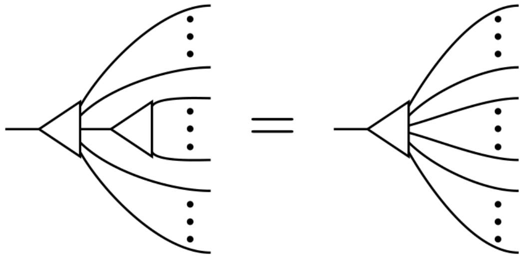
Beam splitter
$e^{i\theta(\hat{a}_1^\dagger\hat{a}_2+\hat{a}_1\hat{a}_2^\dagger)}$

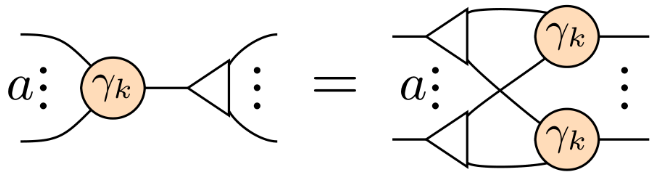


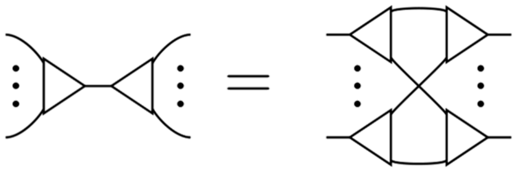
Infinite-dimensional ZX: Discrete-variables - Rules

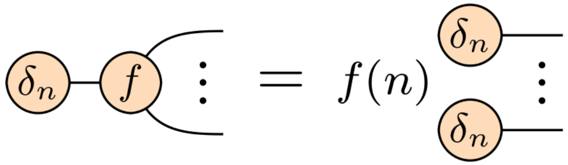
Generalizes qubit ZW calculus

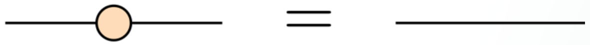








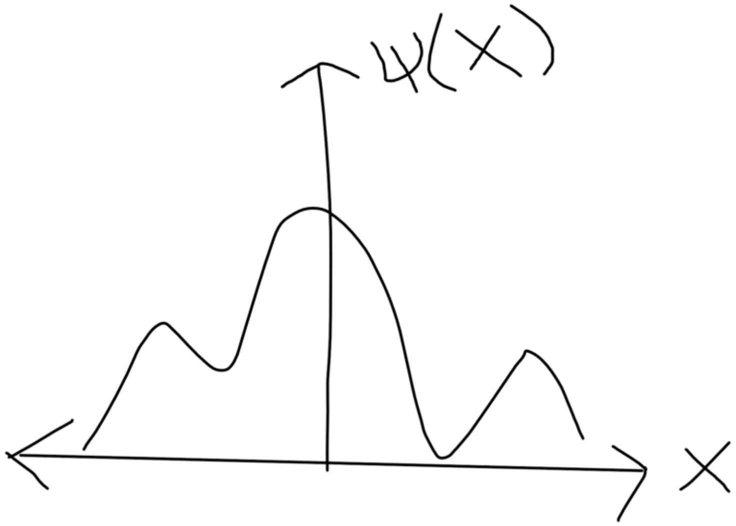




Infinite-dimensional ZX: Continuous-variables

- Consider a particle on a line, i.e. $\mathbb{R}$

- The position wavefunction of the particle is a superposition over $\mathbb{R}$

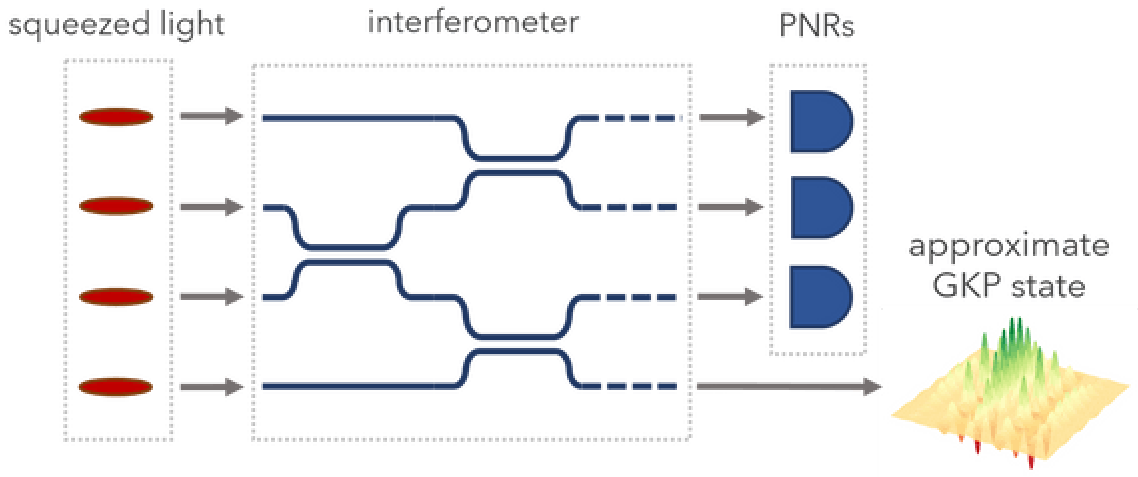


$$\int_{\mathbb{R}} \psi(x) \, |x\rangle \; dx$$

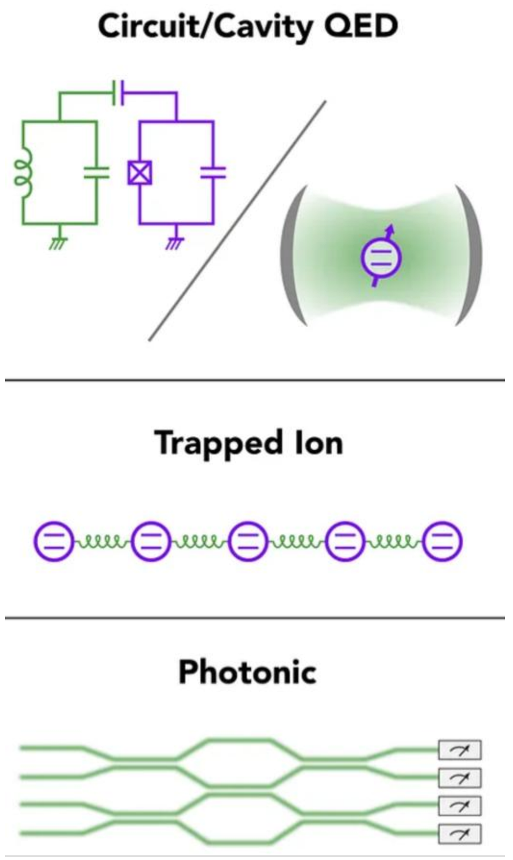
Why continuous-variable quantum computation?

- Analog simulation of bosonic systems / quantum field theories

- Error correction with Bosonic codes



- Native to many platforms

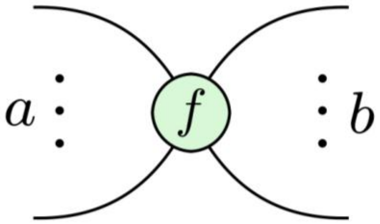


Left Figure: https://www.xanadu.ai/blog/riding-bosonic-qubits-towards-fault-tolerant-quantum-computation
Right Figure: https://medium.com/qiskit/introducing-bosonic-qiskit-a-package-for-simulating-bosonic-and-hybrid-qubit-bosonic-circuits-1e1e528287bb

Infinite-dimensional ZX: Continuous-variables

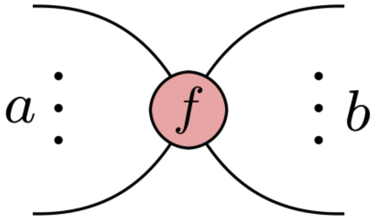
Generators

Z spider (Position)



$$\xmapsto{[\![\cdot]\!]} \quad \int f(x)\, |x\rangle^{\otimes b} \langle x|^{\otimes a}\, dx$$

X spider (Momentum)



$$\xmapsto{[\![\cdot]\!]} \quad \int f(p)\, |p\rangle^{\otimes b} \langle p|^{\otimes a}\, dp$$

Infinite-dimensional ZX: Continuous-variables

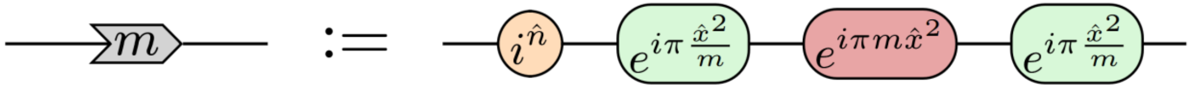
Common Gates

Pauli operators

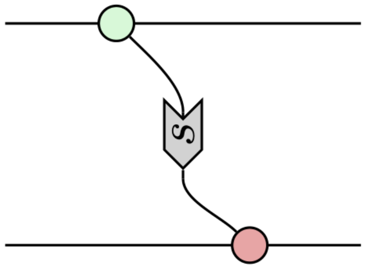
$\xmapsto{[\![\cdot]\!]} X(x)$ and

$\xmapsto{[\![\cdot]\!]} Z(p)$ where $\chi_a(x) = e^{iax}$

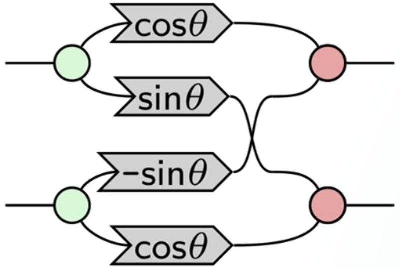
Squeezing



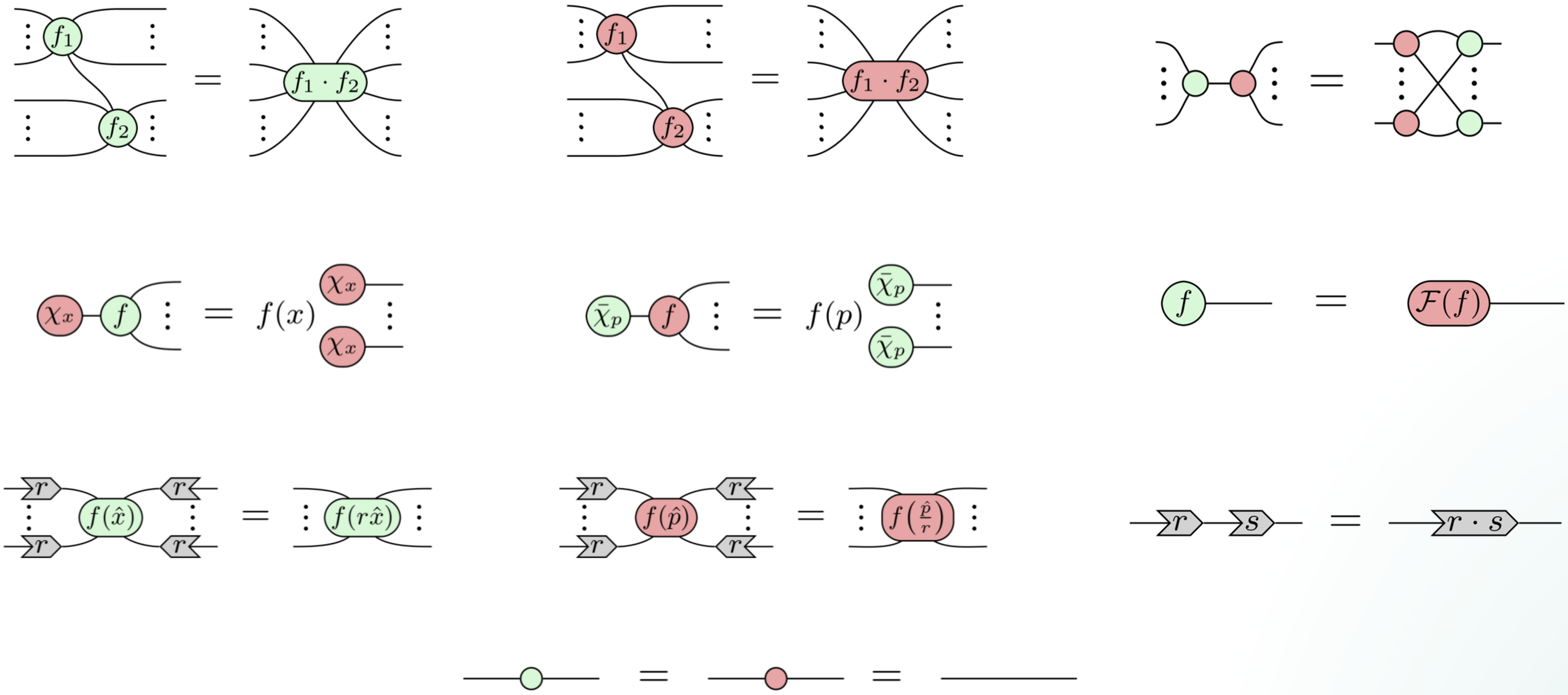
CSUM



Beam splitter

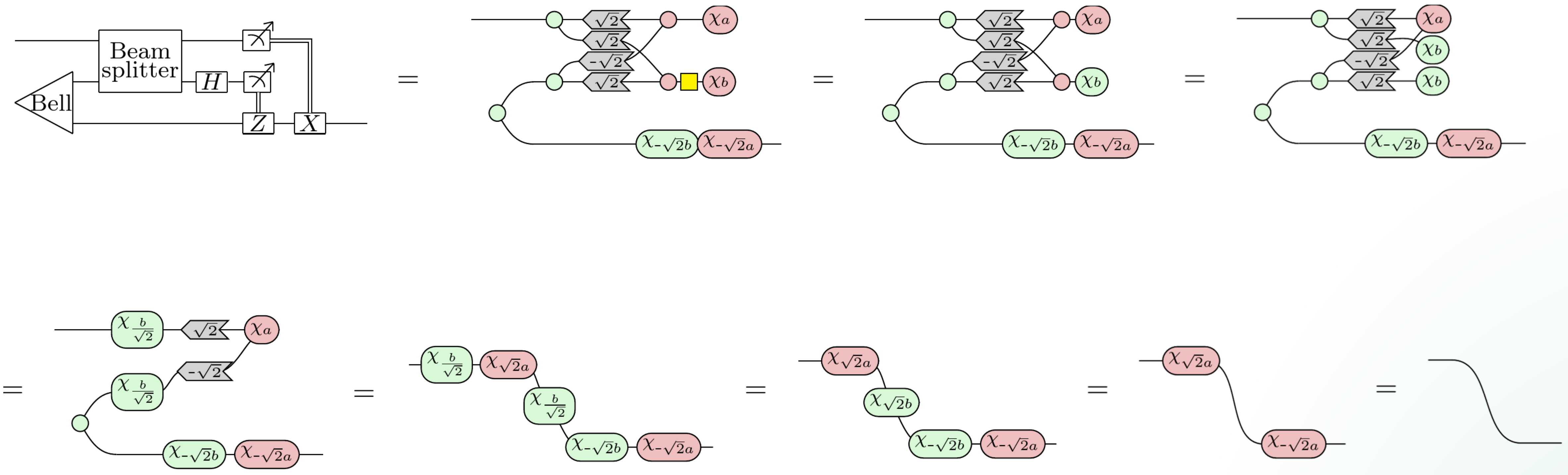


Infinite-dimensional ZX: Continuous-variables - Rules



Infinite-dimensional ZX: Continuous-variables

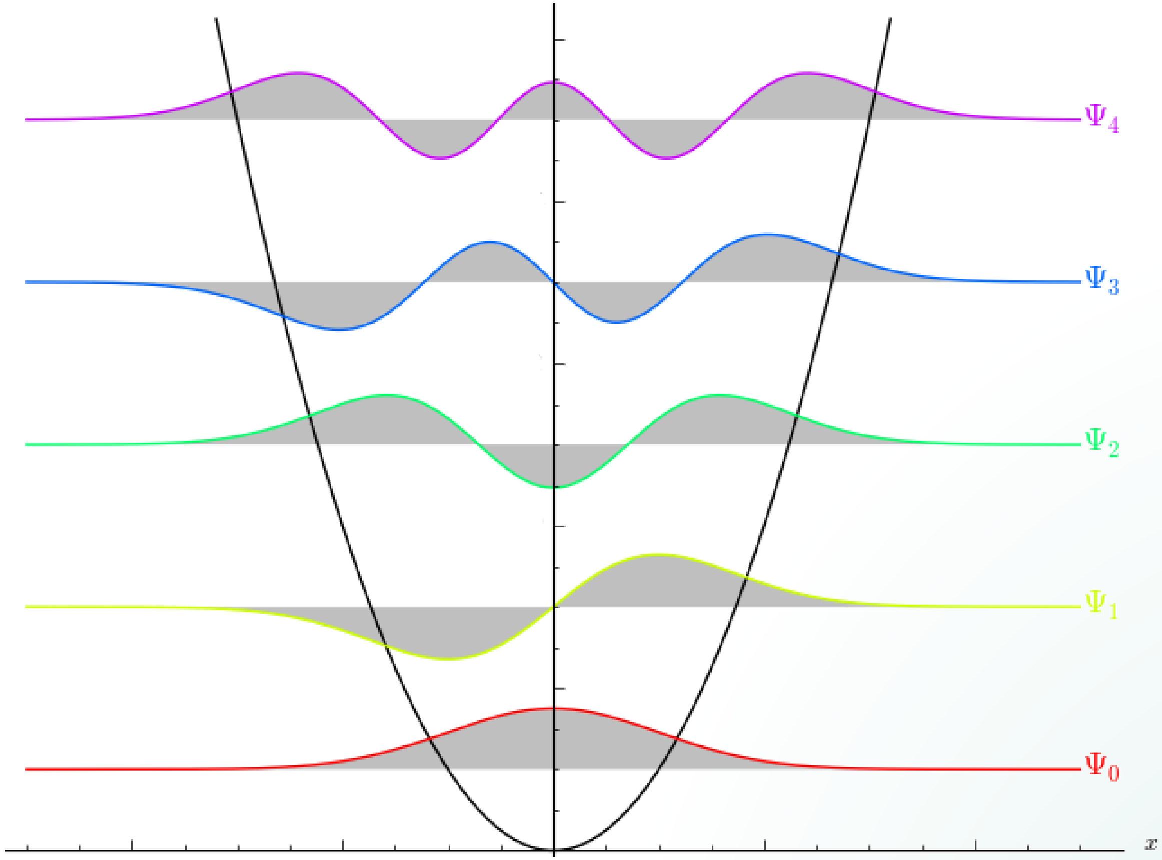
Example: Quantum teleportation



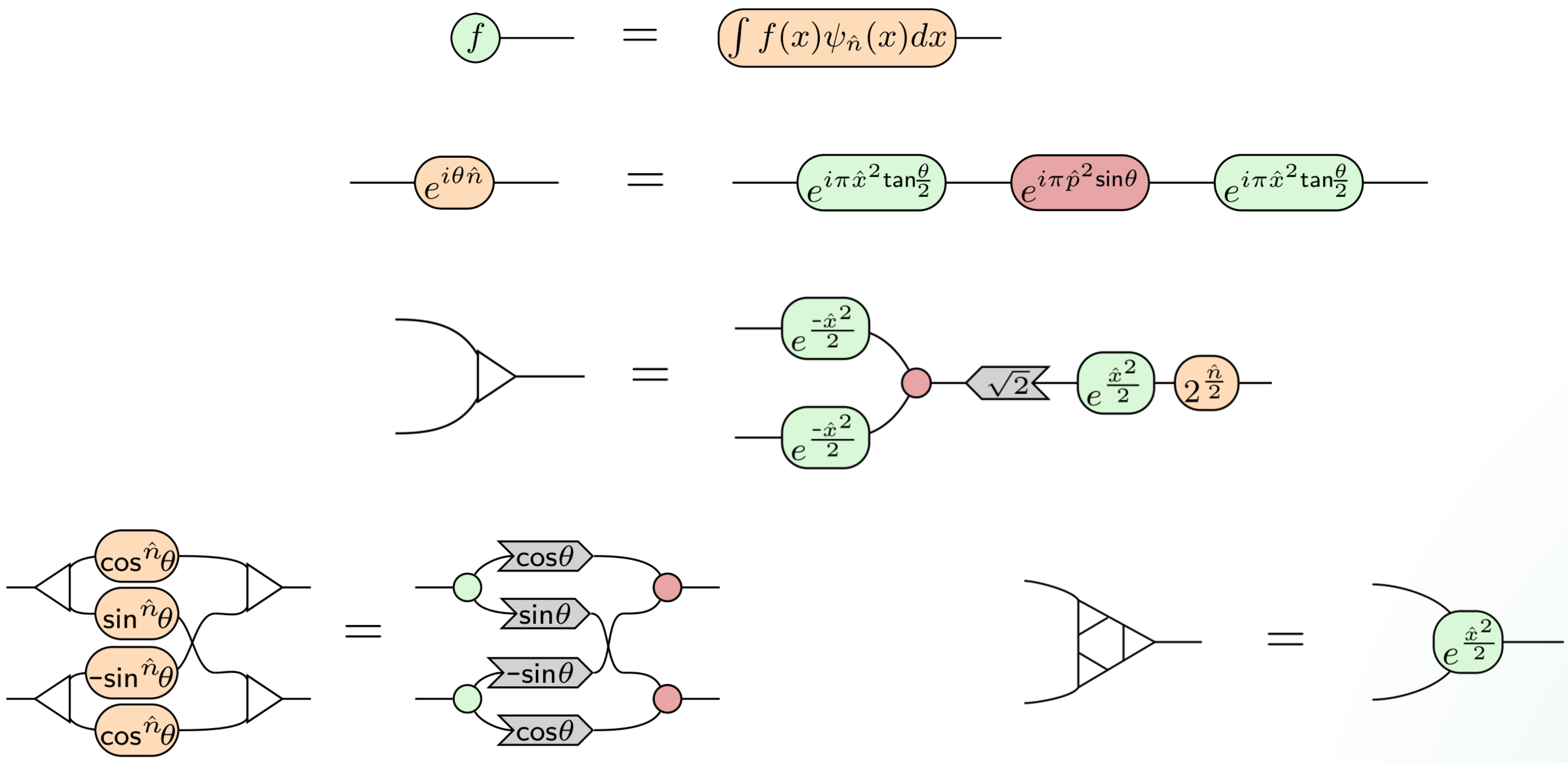
Relating continuous and discrete variables

$$|n\rangle = \int \psi_n(x)\,|x\rangle\;dx$$

where $\psi_n(x)$ is the n-th Hermite function.



ZX rules with continuous-discrete interaction



Overview of ZX Completeness Results

	Qubits (d=2)	Qupits (d odd prime)	Qudits (d >= 2)	FHilb Mixed-dimensions	Infinite dimensional
Stabilizer	· Backens, 2014	· d=3: Wang, 2018 · Booth & Carette, 2022			· Booth, Carette & Comfort, 2024 · Shaikh, Yeh & Gogioso, 2024
Approx. Universal	· Jeandel, Perdrix & Vilmart, 2018				
Universal	· Ng, Wang & Hadzihasanovic, 2018 · Jeandel, Perdrix & Vilmart, 2018	Follows from the qudits results on the right	· ZXW – Poór et. al., 2023 · open question for ZX	· ZX – Poór, Shaikh & Wang, 2024	

Thank you