QPL 2026 Tutorials

Causality in higher-order quantum theory

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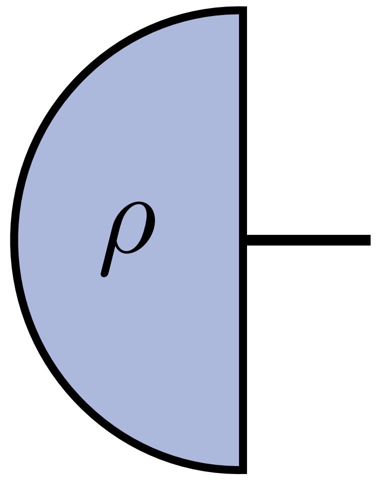
Outline

- Higher-order quantum theory
- Transforming quantum states: quantum channels
- Transforming a quantum channel: quantum superchannel
- Transforming multiple quantum channels: parallel, sequential and general superchannels
- The quantum switch
- Definite and indefinite causal order
- Transforming multiple quantum channels: QCCC, QCQC, and purifiable superchannels
- Advantages of indefinite causal order
- Simulation of indefinite causal order
- Implementation of indefinite causal order

Formalism: Higher-order operations

quantum data:

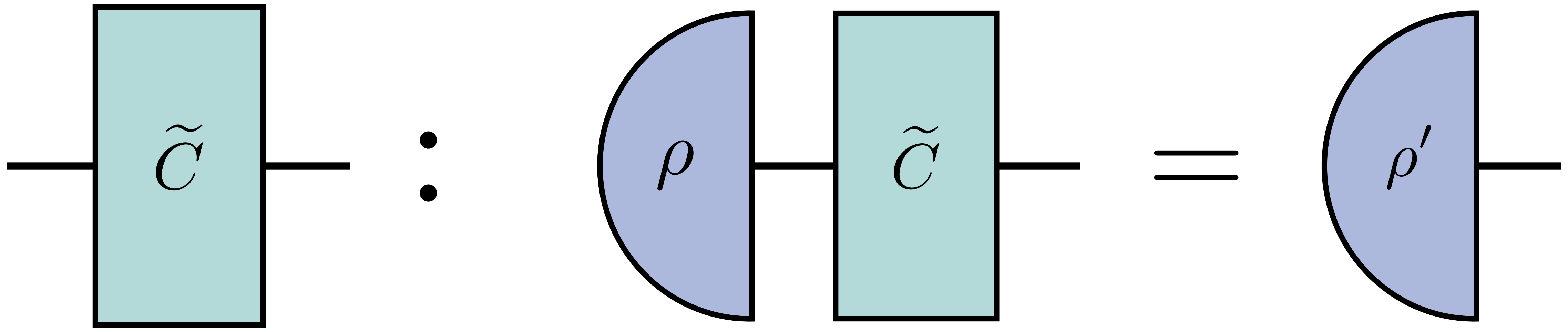
quantum states



$\rho \in \mathcal{L}(\mathcal{H}_{\text{in}})$

quantum functions:

quantum channels

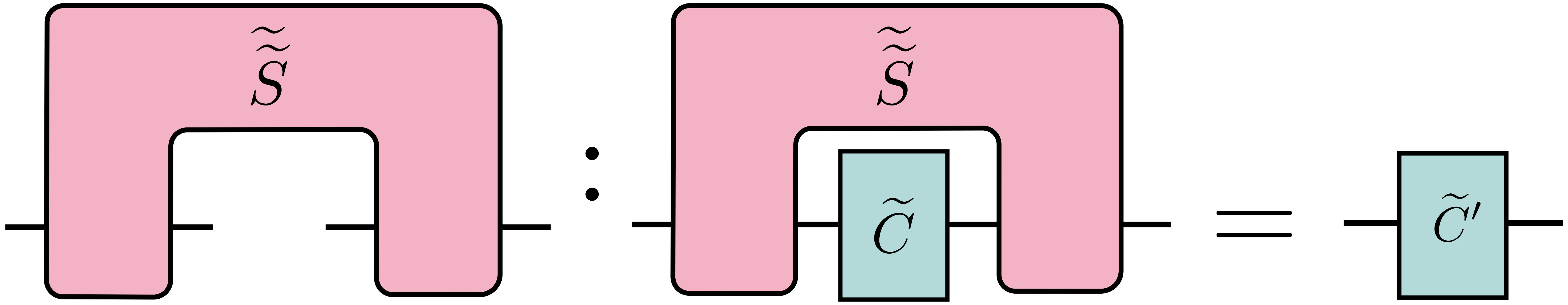


$\widetilde{C} : \mathcal{L}(\mathcal{H}_{\text{in}}) \to \mathcal{L}(\mathcal{H}_{\text{out}})$

$\widetilde{C}(\rho) = \rho'$

higher-order quantum functions:

quantum processes



"functions of functions"

$$\widetilde{\widetilde{S}} : [\mathcal{L}(\mathcal{H}_{\text{in}}) \to \mathcal{L}(\mathcal{H}_{\text{out}})] \to [\mathcal{L}(\mathcal{H}_{\text{in}'}) \to \mathcal{L}(\mathcal{H}_{\text{out}'})]$$

$$\widetilde{\widetilde{S}}(\widetilde{C}) = \widetilde{C}'$$

Formalism: Higher-order operations

REVIEW & TUTORIAL: Taranto et al., *Higher-order quantum operations*, PRX Quantum (2026)

- *Toward a general theory of quantum games*
  Gutoski and Watrous, STOC (2007)
- *Quantum circuit architecture*
  Chiribella et al., PRL (2008)
- *Transforming quantum operations: quantum supermaps*
  Chiribella et al., EPL (2008)
- *Theoretical framework for quantum networks*
  Chiribella et al., PRA (2009)
- *Quantum correlations with no causal order*
  Oreshkov et al., Nat. Commun. (2012)
- *Quantum computation without definite causal structure*
  Chiribella et al., PRA (2013)

transforming quantum states:

quantum channels

Quantum channels

Channel: The most general linear transformation that maps every quantum state into a quantum state, even when acting on part of a quantum state.

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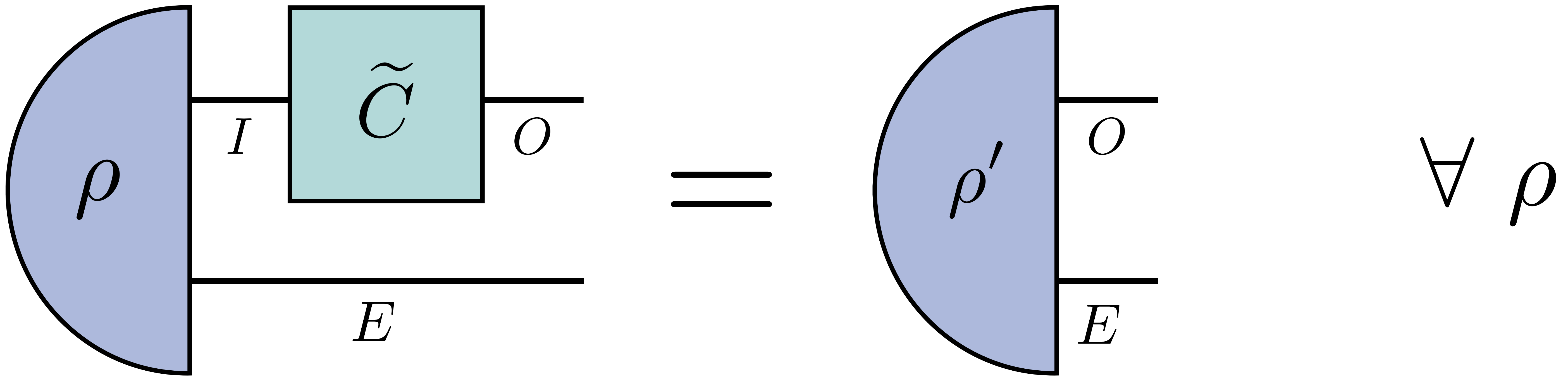

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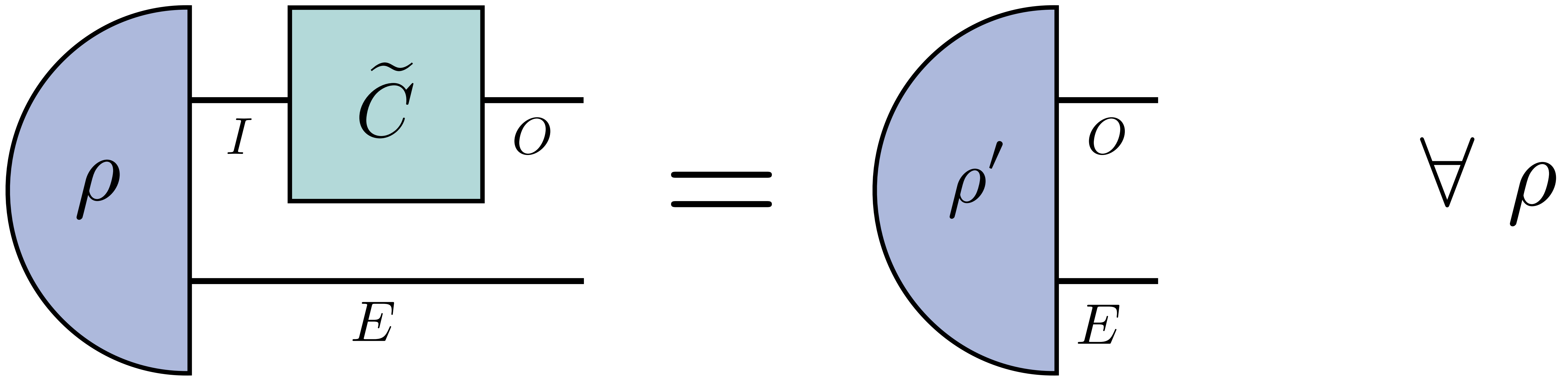

quantum state (operator)

$$\rho \in \mathcal{L}(\mathcal{H}_I \otimes \mathcal{H}_E)$$

quantum channel (map)

$$\widetilde{C} : \mathcal{L}(\mathcal{H}_I) \to \mathcal{L}(\mathcal{H}_O)$$

$$\widetilde{C} \otimes \widetilde{\mathbb{1}}(\rho) = \rho' \in \mathcal{L}(\mathcal{H}_O \otimes \mathcal{H}_E)$$

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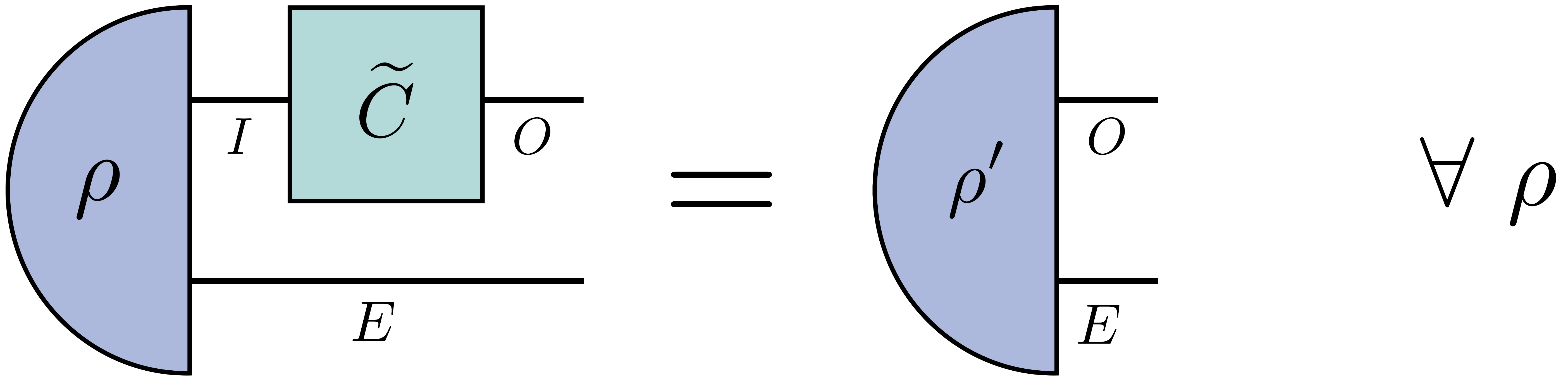

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$\rho \geq 0$ $\longrightarrow$ positive semidefinite (PSD)

$\widetilde{C} \otimes \widetilde{\mathbb{1}}(\rho) \geq 0$ $\longrightarrow$ completely positive (CP)

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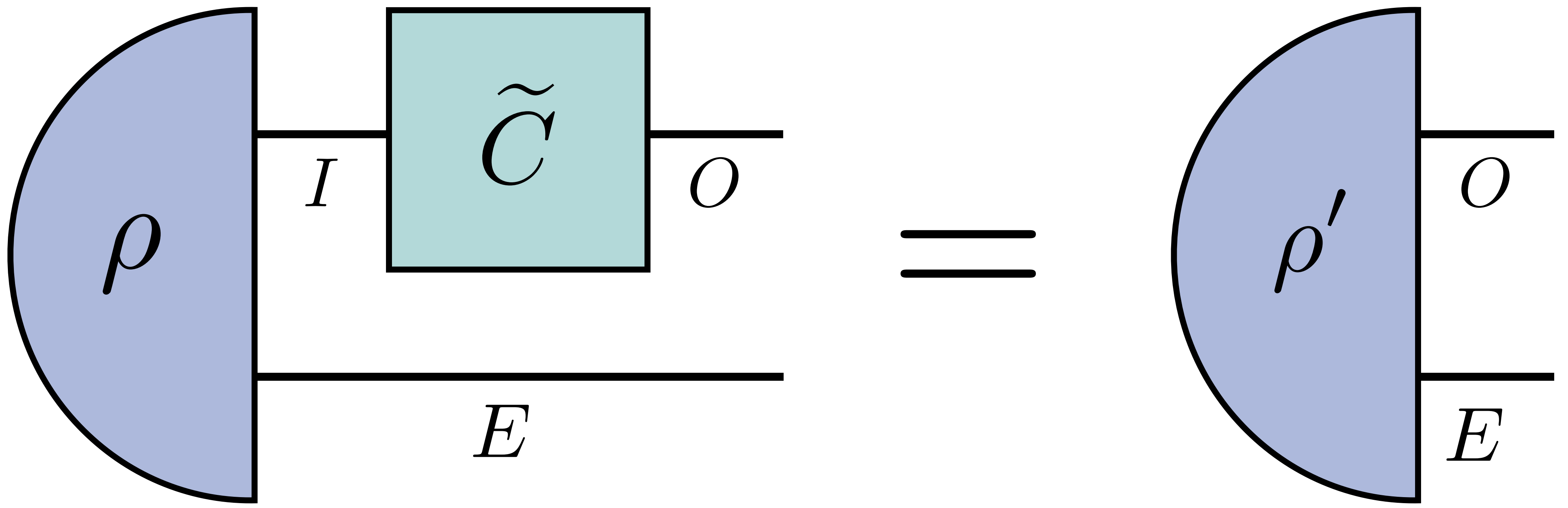

$\forall \rho$

quantum state (operator)

$$\rho \in \mathcal{L}(\mathcal{H}_I \otimes \mathcal{H}_E)$$

$\rho \geq 0 \longrightarrow$ positive semidefinite (PSD)

$\mathrm{tr}(\rho) = 1 \longrightarrow$ fixed trace

quantum channel (map)

$$\widetilde{C} : \mathcal{L}(\mathcal{H}_I) \to \mathcal{L}(\mathcal{H}_O)$$

$$\widetilde{C} \otimes \widetilde{\mathbb{1}}(\rho) = \rho' \in \mathcal{L}(\mathcal{H}_O \otimes \mathcal{H}_E)$$

$\widetilde{C} \otimes \widetilde{\mathbb{1}}(\rho) \geq 0 \longrightarrow$ completely positive (CP)

$\mathrm{tr}\left[\widetilde{C} \otimes \widetilde{\mathbb{1}}(\rho)\right] = 1 \longrightarrow$ completely trace preserving (CTP)

Quantum channels

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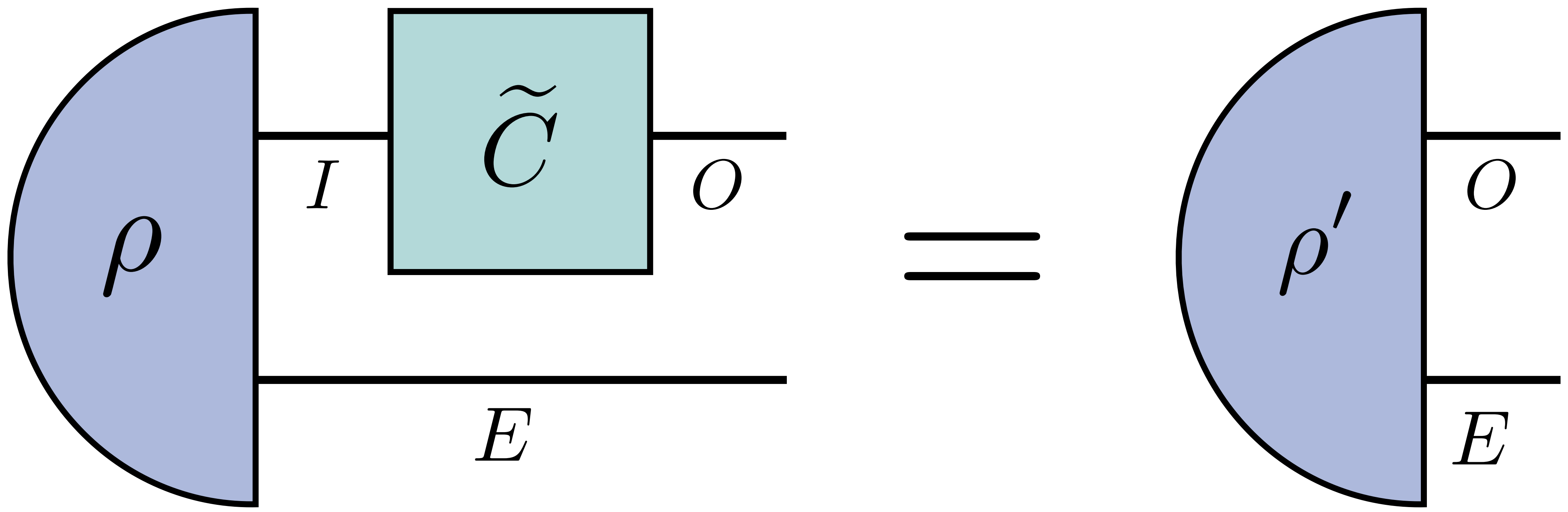

$\forall\ \rho$

quantum channels:

$$\widetilde{C} : \mathcal{L}(\mathcal{H}_I) \to \mathcal{L}(\mathcal{H}_O)$$

completely positive trace-preserving (CPTP)
linear maps

Choi representation

Choi operator: Every linear map can be represented by a linear operator, acting on the joint input and output space of the map, defined as

$$\widetilde{C} : \mathcal{L}(\mathcal{H}_I) \to \mathcal{L}(\mathcal{H}_O) \qquad \longrightarrow \text{ linear map}$$

$$C := \sum_{ij} |i\rangle\langle j| \otimes \widetilde{C}(|i\rangle\langle j|) \in \mathcal{L}(\mathcal{H}_I \otimes \mathcal{H}_O) \qquad \longrightarrow \text{ linear operator}$$

Choi representation

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Choi inverse: Consequently, the output of a linear map can be expressed as function of the Choi operator of the map and its input according to

$$\widetilde{C}(\rho) = \rho' = \mathrm{tr}_I[C\,(\rho^T \otimes \mathbb{1}_O)] =: \rho * C \quad \longrightarrow \text{ "link product"}$$

$$(\widetilde{A} \circ \widetilde{B})[\rho] = A * B * \rho$$

Quantum channels

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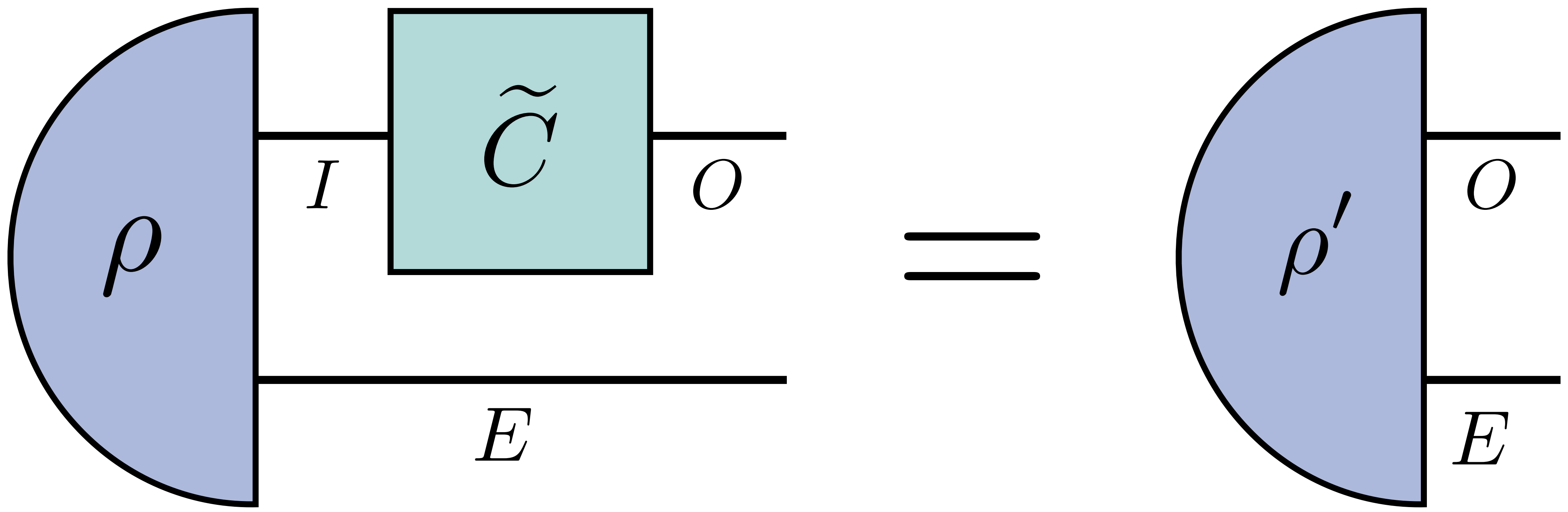

linear map

$$\widetilde{C} \otimes \widetilde{\mathbb{1}}(\rho) = \rho'$$

Choi operator

$$C * \rho = \rho'$$

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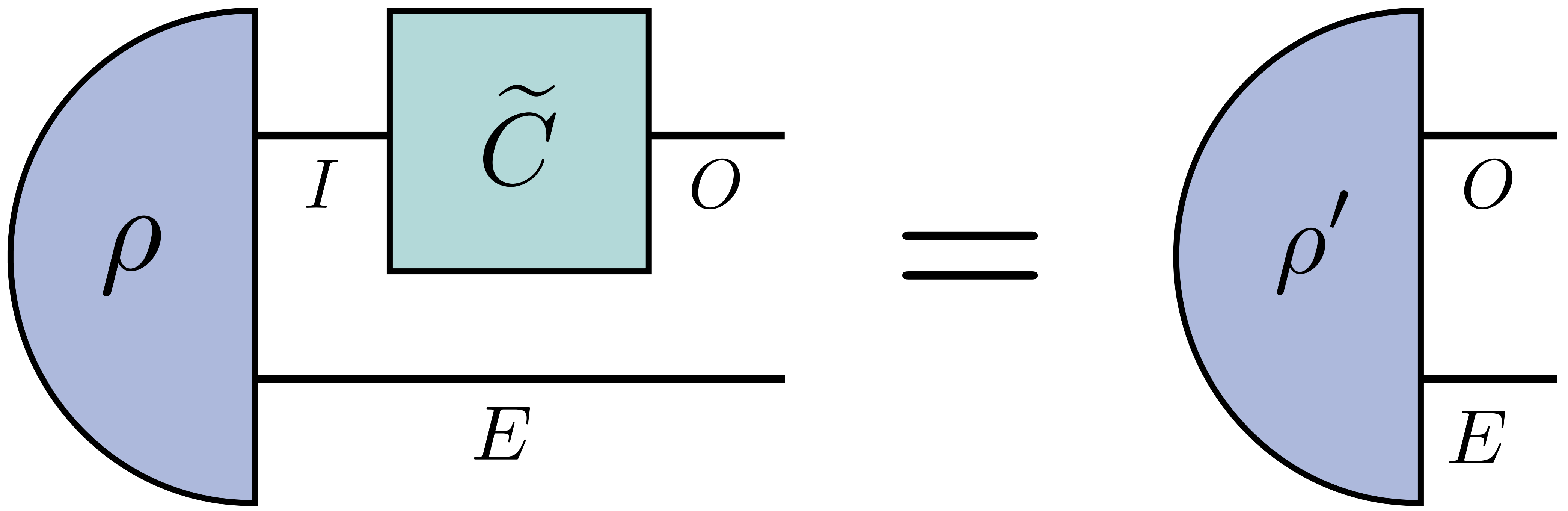

linear map

$$\widetilde{C} \otimes \widetilde{\mathbb{1}}(\rho) = \rho'$$

Choi operator

$$C * \rho = \rho'$$

$\widetilde{C} \otimes \widetilde{\mathbb{1}}(\rho) \geq 0$ $\longrightarrow$ completely positive (CP) $\longleftarrow$ $C \geq 0$

Quantum channels

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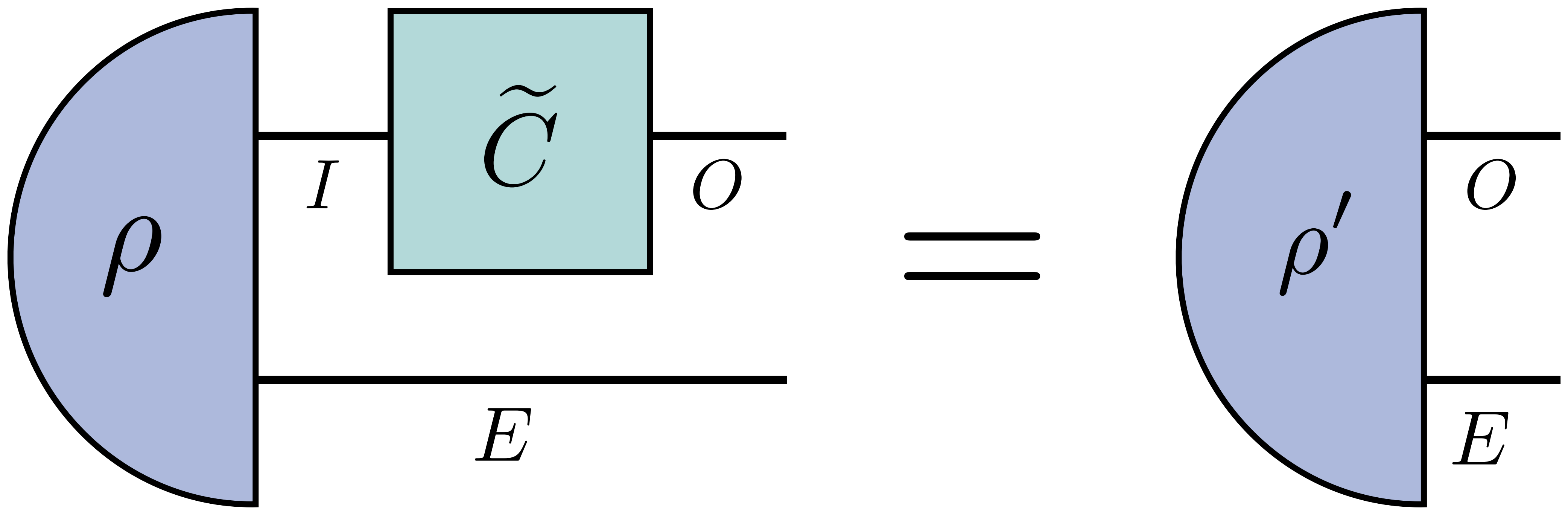

linear map				Choi operator
$\widetilde{C} \otimes \widetilde{\mathbb{1}}(\rho) = \rho'$				$C * \rho = \rho'$
$\widetilde{C} \otimes \widetilde{\mathbb{1}}(\rho) \geq 0$	$\longrightarrow$	completely positive (CP)	$\longleftarrow$	$C \geq 0$
$\mathrm{tr}\left[\widetilde{C} \otimes \widetilde{\mathbb{1}}(\rho)\right] = 1$	$\longrightarrow$	completely trace preserving (CTP)	$\longleftarrow$	$\mathrm{tr}_O\, C = \mathbb{1}_I$

transforming quantum channels:

quantum superchannels

Quantum superchannels

Superchannel: The most general linear higher-order transformation that maps every quantum channel into a quantum channel, even when acting on part of a quantum channel.

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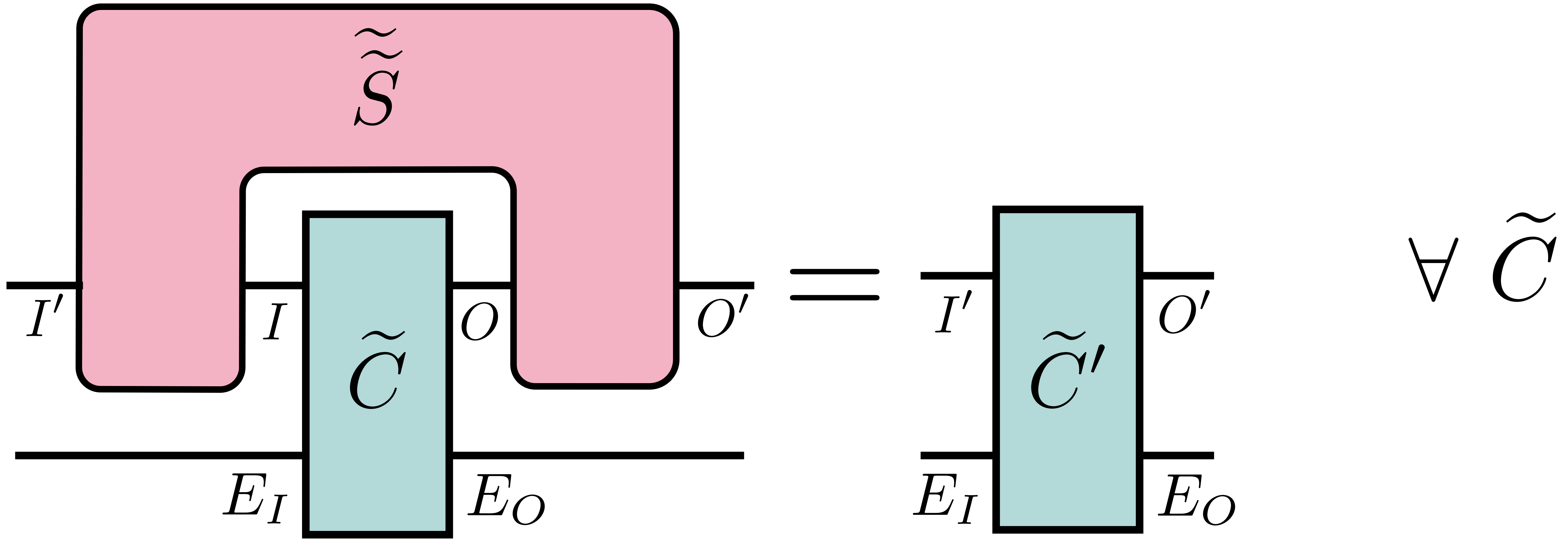

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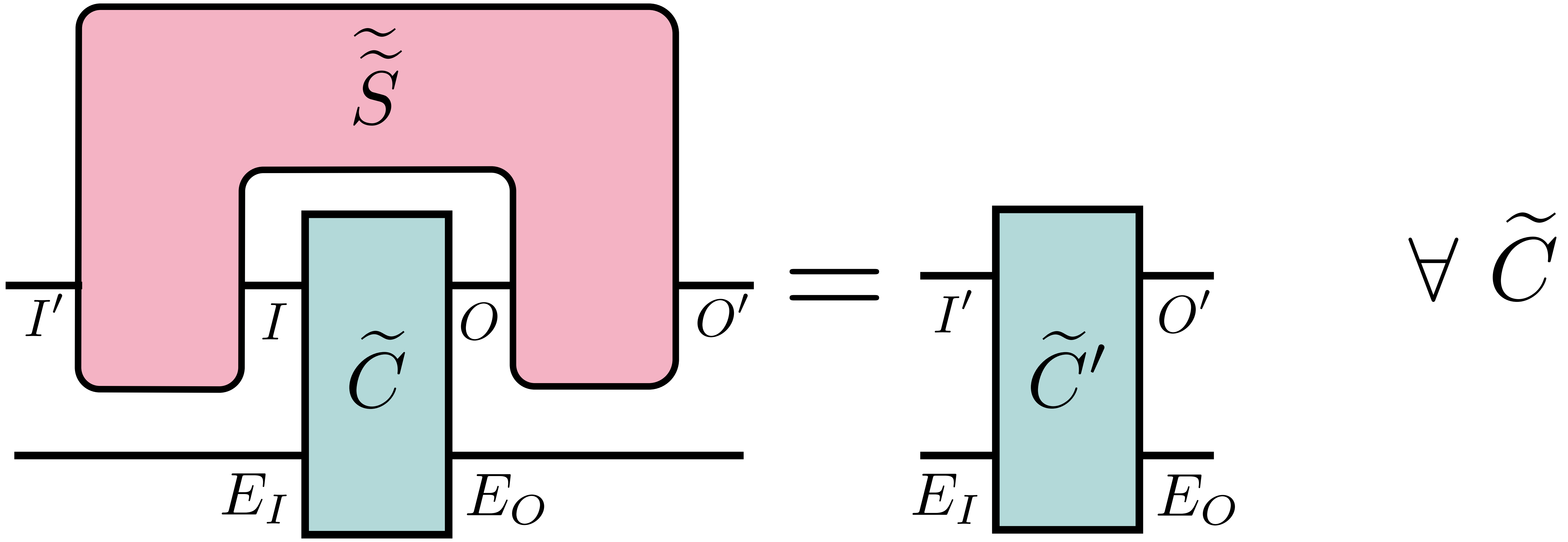

quantum channel (map)

$$\widetilde{C} : \mathcal{L}(\mathcal{H}_I \otimes \mathcal{H}_{E_I}) \to \mathcal{L}(\mathcal{H}_O \otimes \mathcal{H}_{E_O})$$

quantum superchannel (HO map)

$$\widetilde{\widetilde{S}} : [\mathcal{L}(\mathcal{H}_I) \to \mathcal{L}(\mathcal{H}_O)] \to [\mathcal{L}(\mathcal{H}_{I'}) \to \mathcal{L}(\mathcal{H}_{O'})]$$

$$\widetilde{\widetilde{S}} \otimes \widetilde{\widetilde{\mathbb{1}}}(\widetilde{C}) = \widetilde{C}' : \mathcal{L}(\mathcal{H}_{I'} \otimes \mathcal{H}_{E_I}) \to \mathcal{L}(\mathcal{H}_{O'} \otimes \mathcal{H}_{E_O})$$

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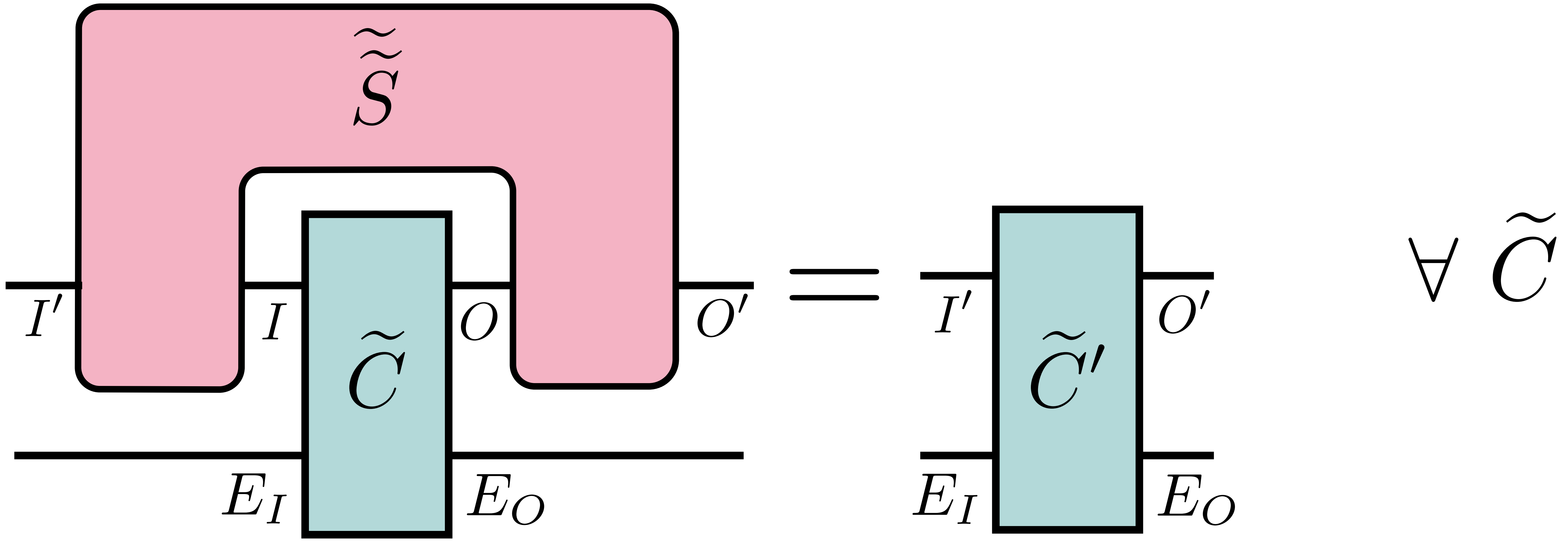

quantum channel (map)

$$\widetilde{C} : \mathcal{L}(\mathcal{H}_I \otimes \mathcal{H}_{E_I}) \to \mathcal{L}(\mathcal{H}_O \otimes \mathcal{H}_{E_O})$$

$\widetilde{C}$ is CPTP

quantum superchannel (HO map)

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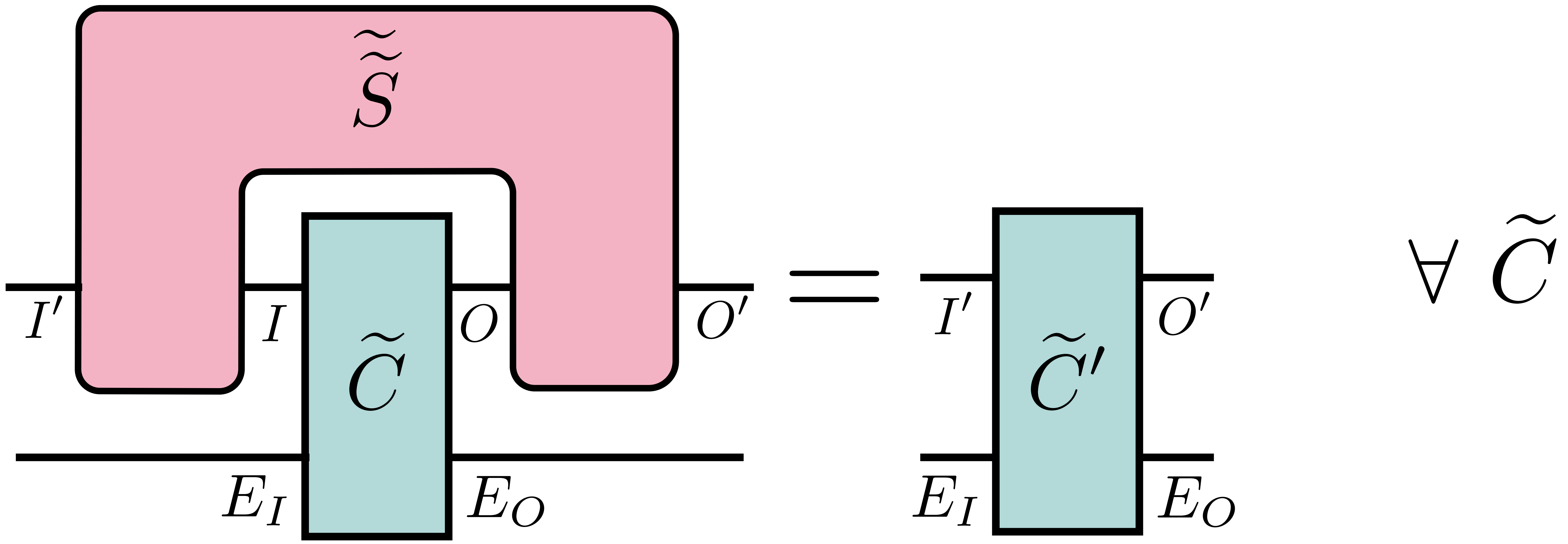
$\widetilde{\widetilde{S}}$ is completely CPTP preserving

Quantum superchannels

Superchannel: The most general linear higher-order transformation that maps every quantum channel into a quantum channel, even when acting on part of a quantum channel.



quantum superchannels:

$$\widetilde{\widetilde{S}} : [\mathcal{L}(\mathcal{H}_I) \to \mathcal{L}(\mathcal{H}_O)] \to [\mathcal{L}(\mathcal{H}_{I'}) \to \mathcal{L}(\mathcal{H}_{O'})]$$

completely CPTP preserving
linear higher-order maps

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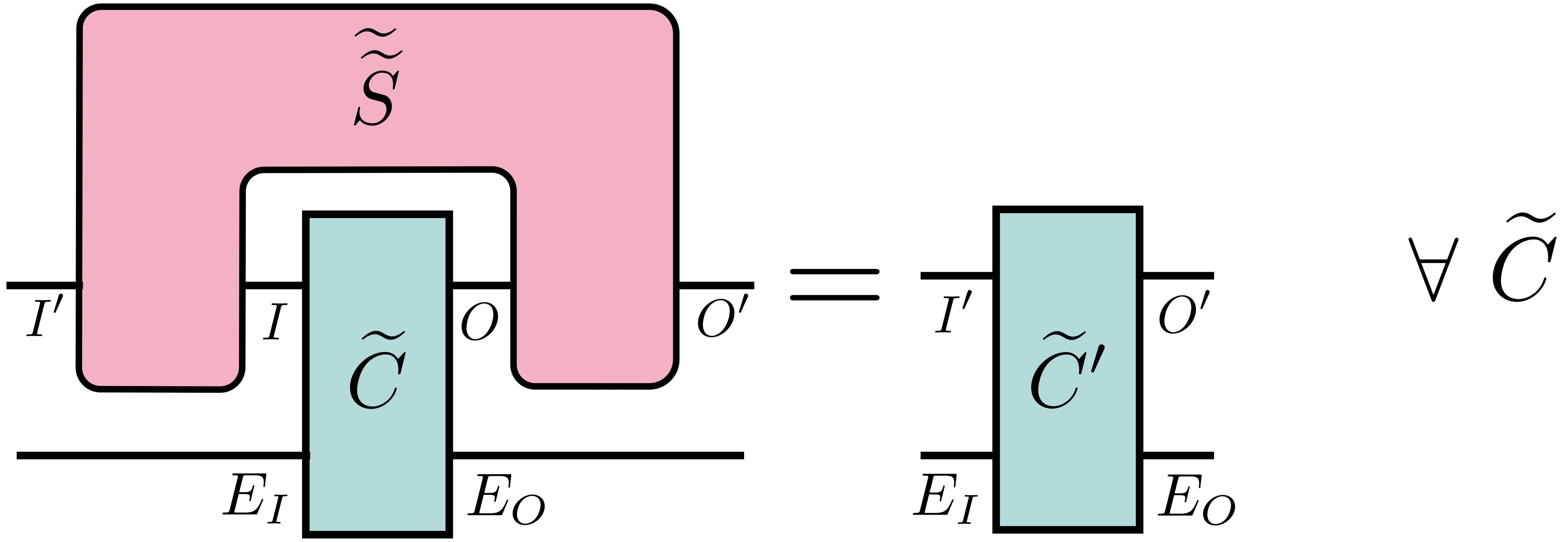

quantum channel (operator)

$$C \in \mathcal{L}(\mathcal{H}_I \otimes \mathcal{H}_{E_I} \otimes \mathcal{H}_O \otimes \mathcal{H}_{E_O})$$

quantum superchannel (map)

$$\widetilde{S} : \mathcal{L}(\mathcal{H}_I \otimes \mathcal{H}_O) \to \mathcal{L}(\mathcal{H}_{I'} \otimes \mathcal{H}_{O'})$$

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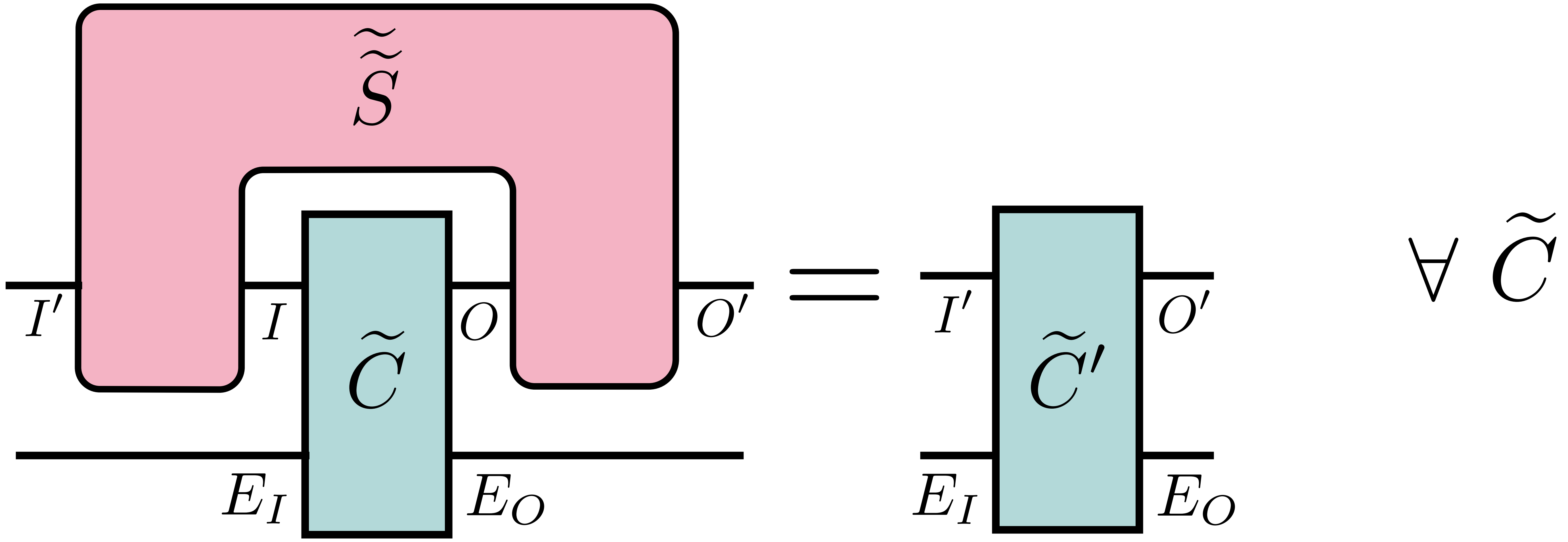

quantum channel (operator)

$$C \in \mathcal{L}(\mathcal{H}_I \otimes \mathcal{H}_{E_I} \otimes \mathcal{H}_O \otimes \mathcal{H}_{E_O})$$

$$C \geq 0$$

$$\mathrm{tr}_{OE_O}(C) = \mathbb{1}_{IE_I}$$

quantum superchannel (map)

$$\widetilde{S} : \mathcal{L}(\mathcal{H}_I \otimes \mathcal{H}_O) \to \mathcal{L}(\mathcal{H}_{I'} \otimes \mathcal{H}_{O'})$$

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$$\widetilde{S} \otimes \widetilde{\mathbb{1}}(C) \geq 0 \qquad \forall\ C$$

$$\mathrm{tr}_{OE_O}[\widetilde{S} \otimes \widetilde{\mathbb{1}}(C)] = \mathbb{1}_{IE_I}$$

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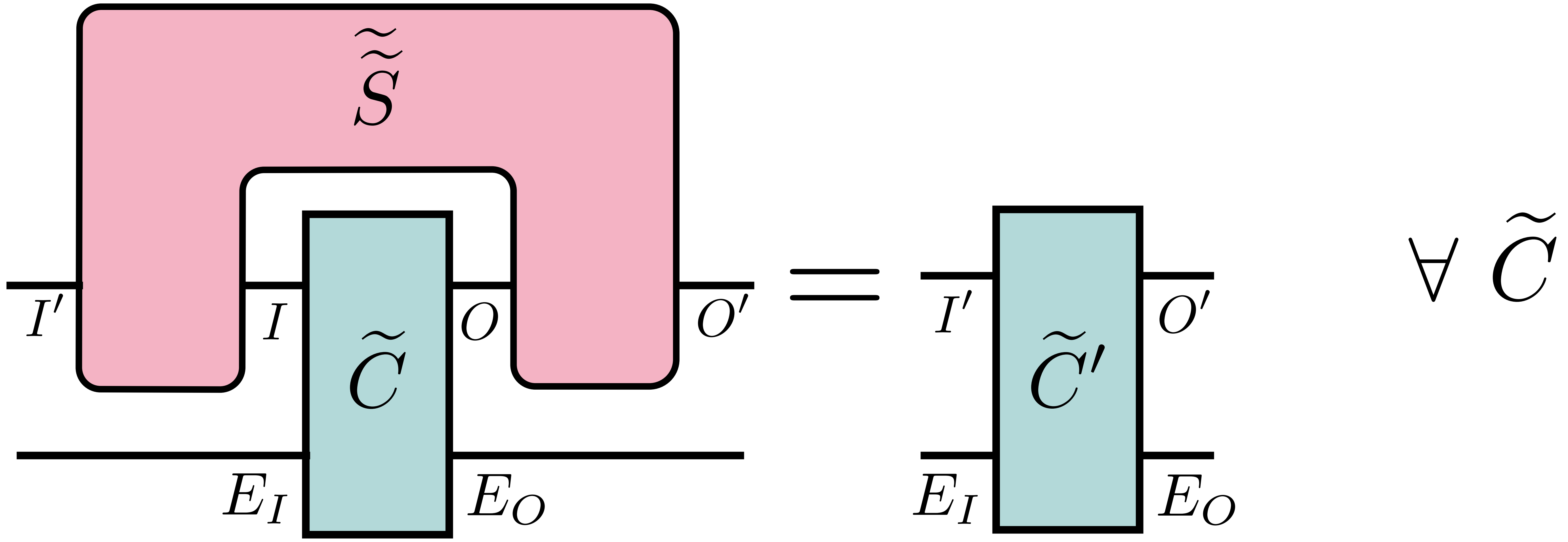

quantum superchannel (operator)

$$S \in \mathcal{L}(\mathcal{H}_{I'} \otimes \mathcal{H}_I \otimes \mathcal{H}_O \otimes \mathcal{H}_{O'})$$

$$S * C = C'$$

$$S * C \geq 0 \qquad \forall\ C$$

$$\mathrm{tr}_{OE_O}(S * C) = \mathbb{1}_{IE_I}$$

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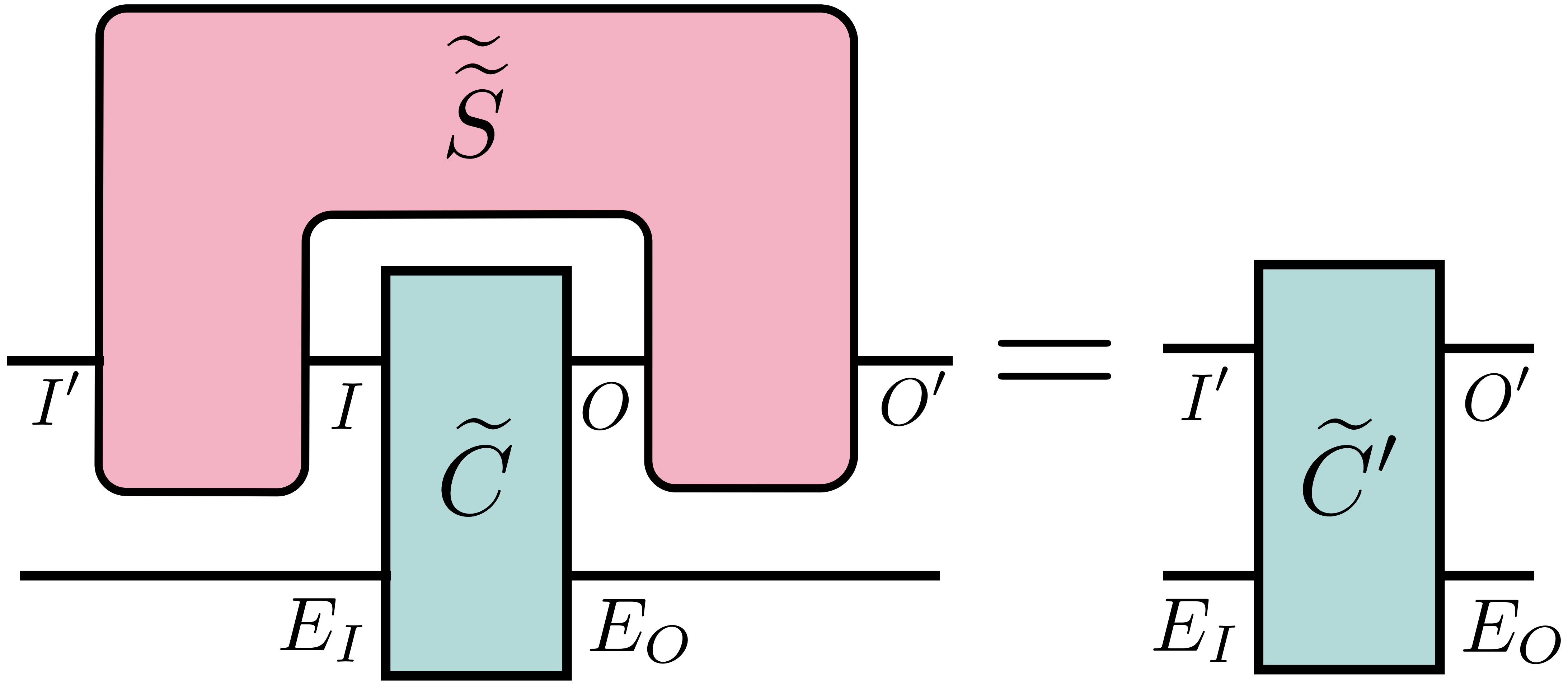

$\forall\ \tilde{C}$

quantum superchannel (operator)

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$$S * C = C'$$

$$S * C \geq 0 \qquad \forall\ C$$

$$\mathrm{tr}_{OE_O}(S * C) = \mathbb{1}_{IE_I}$$

$\rightarrow$

$\tilde{\tilde{S}}$

completely CPTP preserving

$\leftarrow$

quantum superchannel (operator)

$$S \in \mathcal{L}(\mathcal{H}_{I'} \otimes \mathcal{H}_I \otimes \mathcal{H}_O \otimes \mathcal{H}_{O'})$$

$$S \geq 0$$

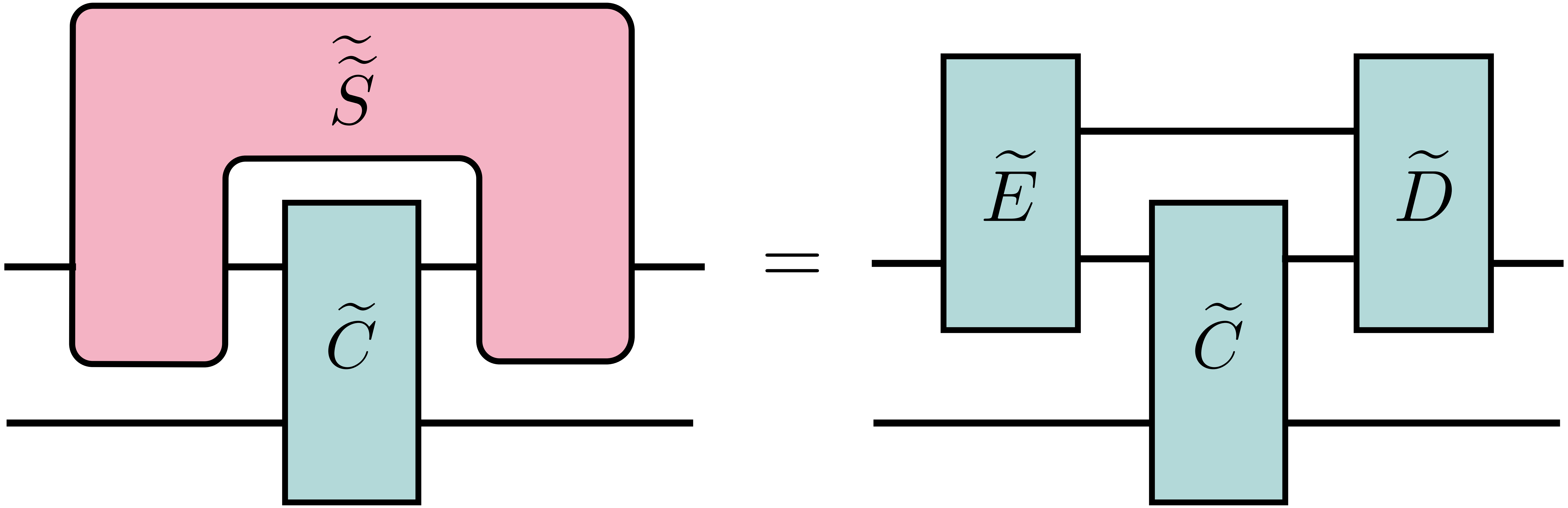
$$\mathrm{tr}_{O'}(S) = \mathrm{tr}_{O'O}(S) \otimes \frac{\mathbb{1}_O}{d_O}$$

$$\mathrm{tr}_{IOO'}(S) = d_I \mathbb{1}_{I'}$$

quantum realization theorem for superchannels

Quantum superchannels

Quantum realization: Every superchannel (completely CPTP preserving linear higher-order map) can be decomposed into a quantum circuit. That is, for every superchannel there exists an "encoder" channel and a "decoder" channel such that



$$\forall \text{ superchannel } \widetilde{\widetilde{S}},\ \exists \text{ channels } \widetilde{E}, \widetilde{D} \text{ such that}$$

$$\widetilde{\widetilde{S}}(\widetilde{C}) = (\widetilde{D} \otimes \widetilde{\mathbb{1}}) \circ (\widetilde{\mathbb{1}} \otimes \widetilde{C}) \circ (\widetilde{E} \otimes \widetilde{\mathbb{1}}) \quad \forall\, \widetilde{C}$$

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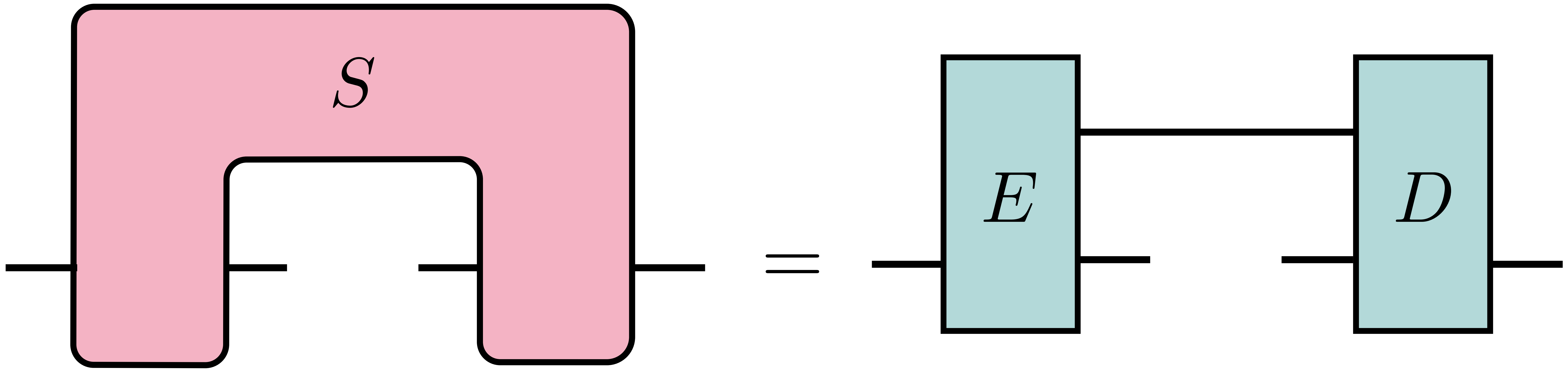
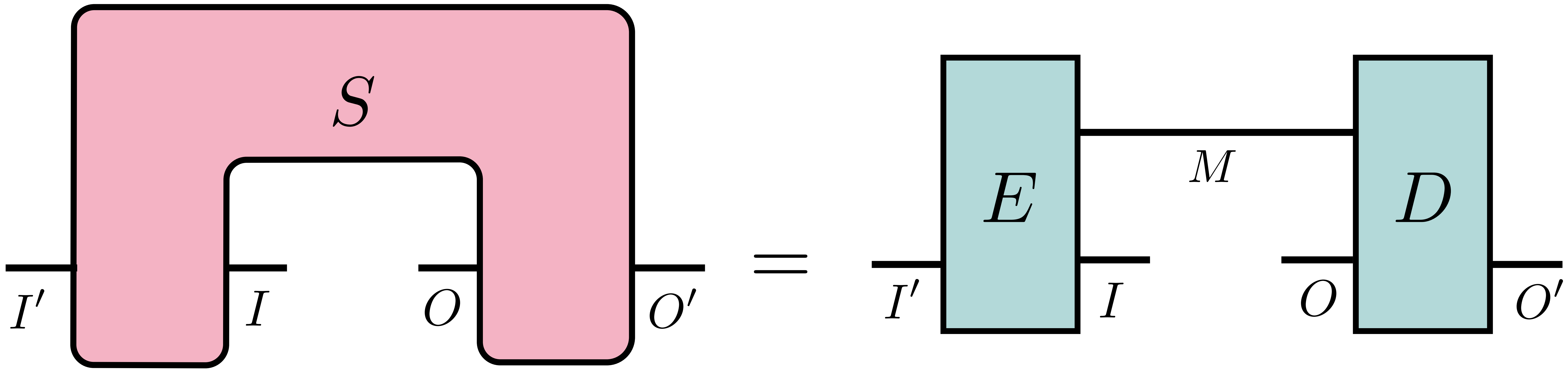


$$\forall \text{ superchannel } S, \ \exists \text{ channels } E, D \text{ such that}$$

$$S = E * D$$

Quantum superchannels



S **as a function of** E, D

E, D **as a function of** S

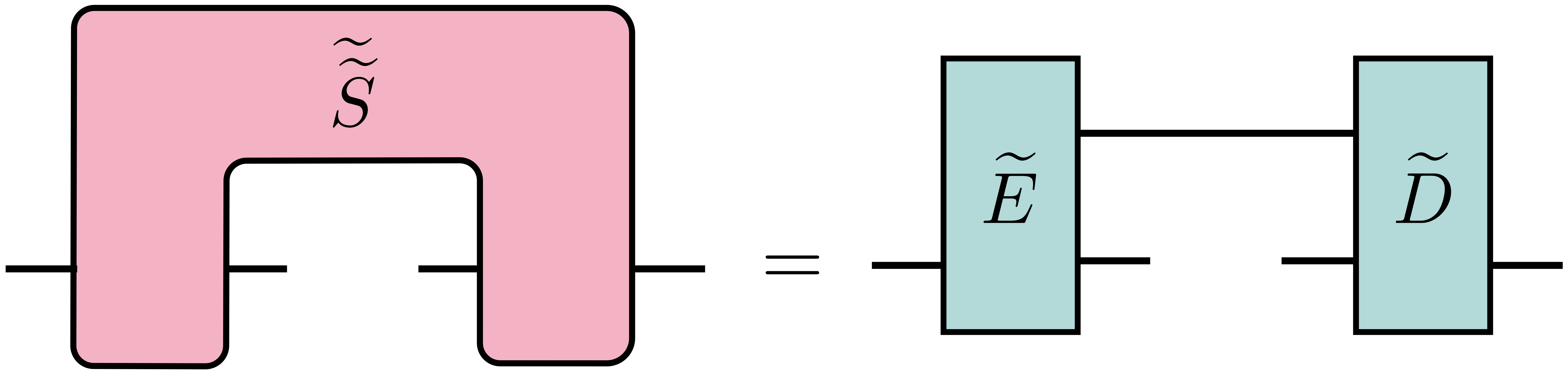
$$S = E * D$$
$$= \mathrm{tr}_M[(E^{T_M} \otimes \mathbb{1}_{OO'})(\mathbb{1}_{I'I} \otimes D)]$$

$$E = \frac{1}{d_O}\left(\mathbb{1}_{I'I} \otimes \sqrt{\mathrm{tr}_{OO'}(S)^T}\right) \sum_{ij} |ii\rangle\langle jj| \left(\mathbb{1}_{I'I} \otimes \sqrt{\mathrm{tr}_{OO'}(S)^T}\right)$$

$$D = d_O \left(\sqrt{\mathrm{tr}_{OO'}(S)^{-1}} \otimes \mathbb{1}_{OF}\right) S \left(\sqrt{\mathrm{tr}_{OO'}(S)^{-1}} \otimes \mathbb{1}_{OF}\right)$$

Quantum superchannels

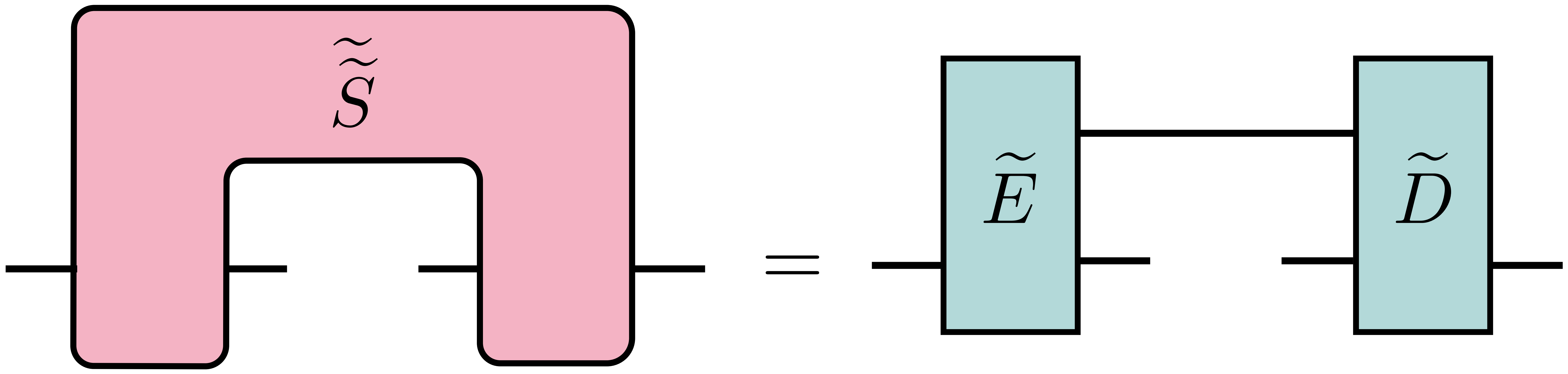
Quantum circuit realization: Every superchannel (completely CPTP preserving linear higher-order map) can be decomposed into a quantum circuit. That is, for every superchannel there exists an "encoder" channel and a "decoder" channel such that



Then why represent it as a higher-order map?

Quantum superchannels

Quantum circuit realization: Every superchannel (completely CPTP preserving linear higher-order map) can be decomposed into a quantum circuit. That is, for every superchannel there exists an "encoder" channel and a "decoder" channel such that



Proof that this is the most general such transformation

linear characterization

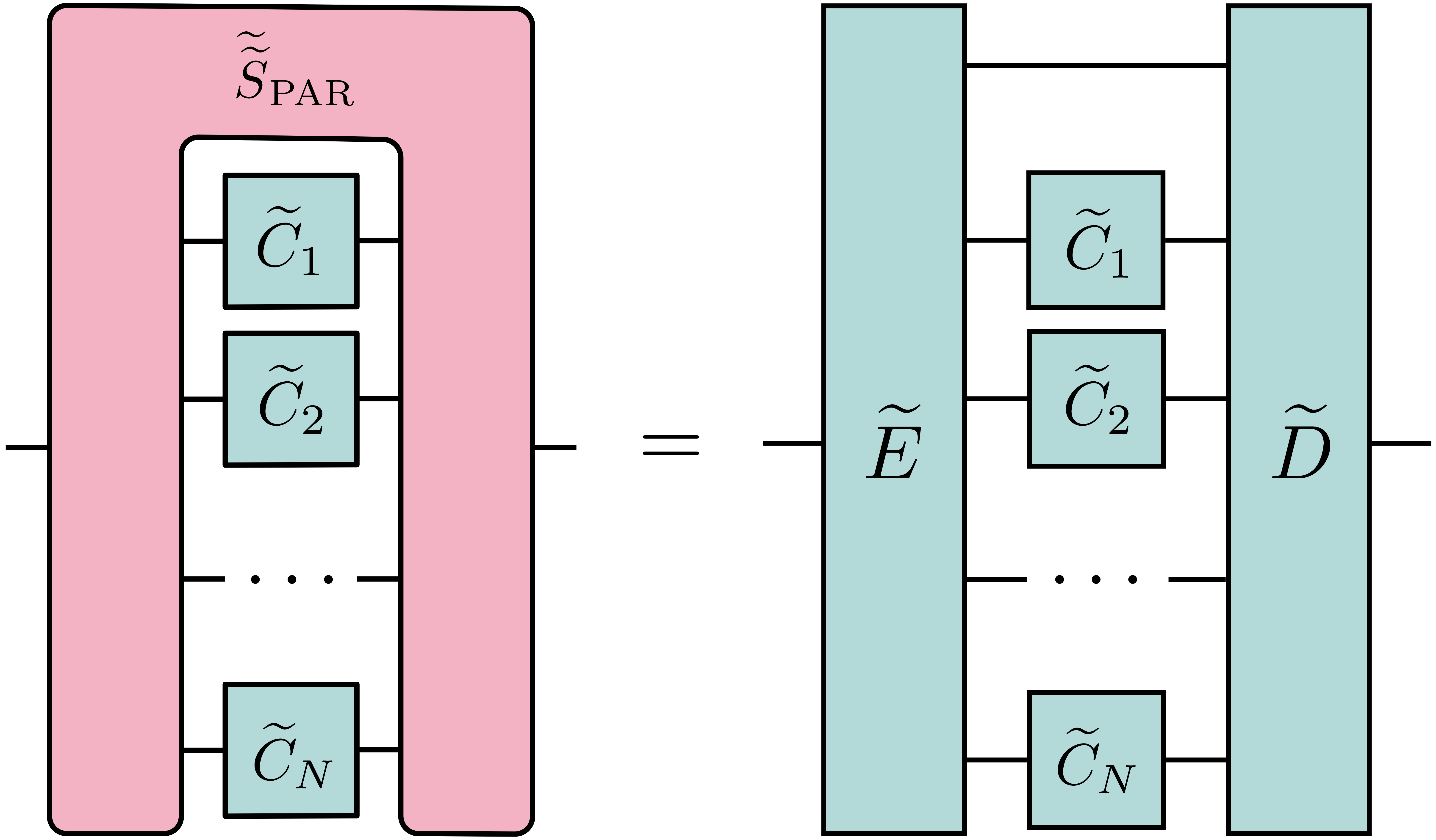
nonlinear characterization

transforming multiple quantum channels:

many-slot superchannels

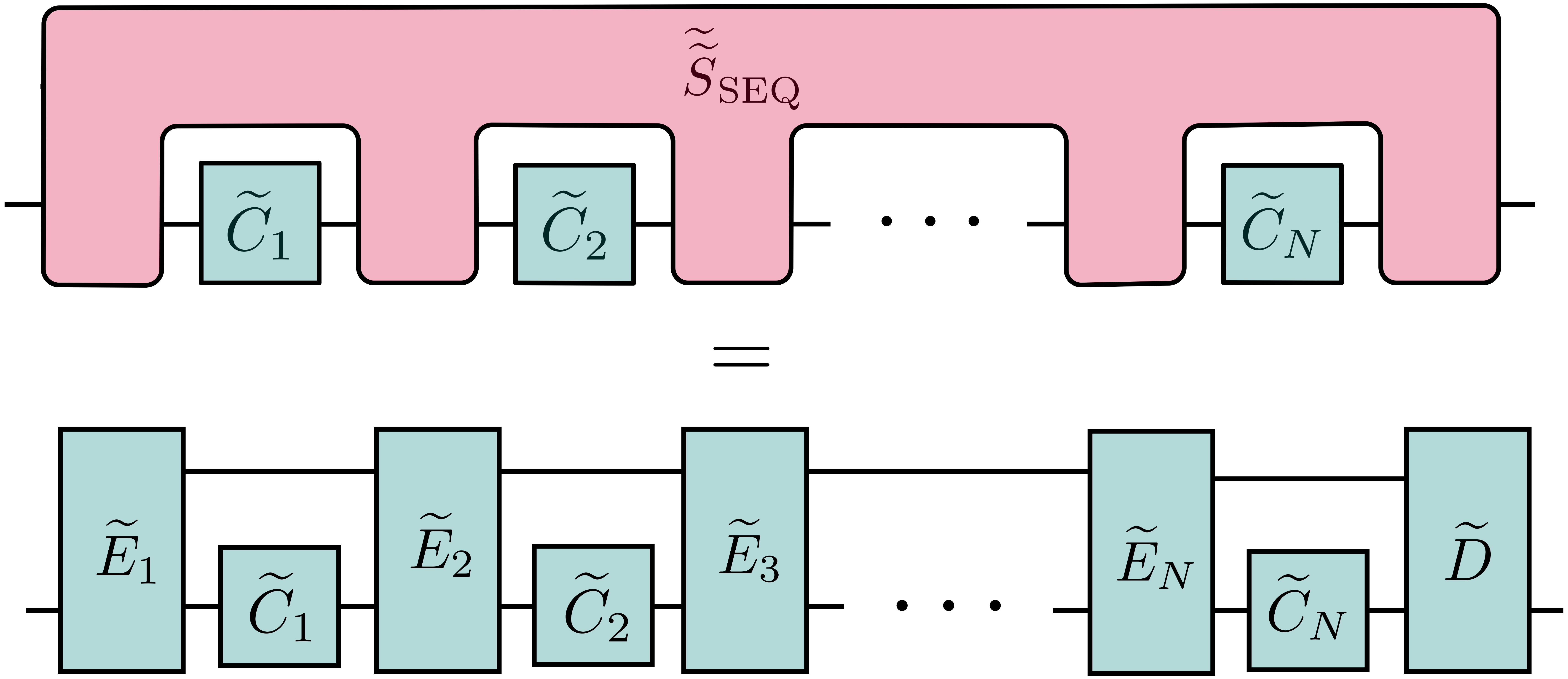
Parallel superchannels

Parallel superchannels: Parallel superchannels are superchannels that act on their input channels in a parallel, non-adaptive form. This implies they can be implemented with a single encoder and single decoder channel.



Sequential superchannels

Sequential superchannels: Sequential superchannels are superchannels that act on their input channels in a sequential, adaptive form. This implies they can be implemented with a sequence of quantum channels in a quantum circuit.



Parallel and Sequential Superchannels

Parallel superchannels:

$$S_{\mathrm{PAR}} := E * D$$

Sequential superchannels:

$$S_{\mathrm{SEQ}} := E_1 * E_1 * \ldots * E_N * D$$

Parallel and Sequential Superchannels

Notation (trace-and-replace map): $X \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B), \;\; {}_BX := \mathrm{tr}_B(X) \otimes \frac{\mathbb{1}_B}{d_B}$

Parallel superchannels:

$$S_{\mathrm{PAR}} := E * D$$

Sequential superchannels:

$$S_{\mathrm{SEQ}} := E_1 * E_1 * \ldots * E_N * D$$

Parallel and Sequential Superchannels

Notation (trace-and-replace map): $X \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B), \ {}_B X := \mathrm{tr}_B(X) \otimes \frac{\mathbb{1}_B}{d_B}$

Parallel superchannels:

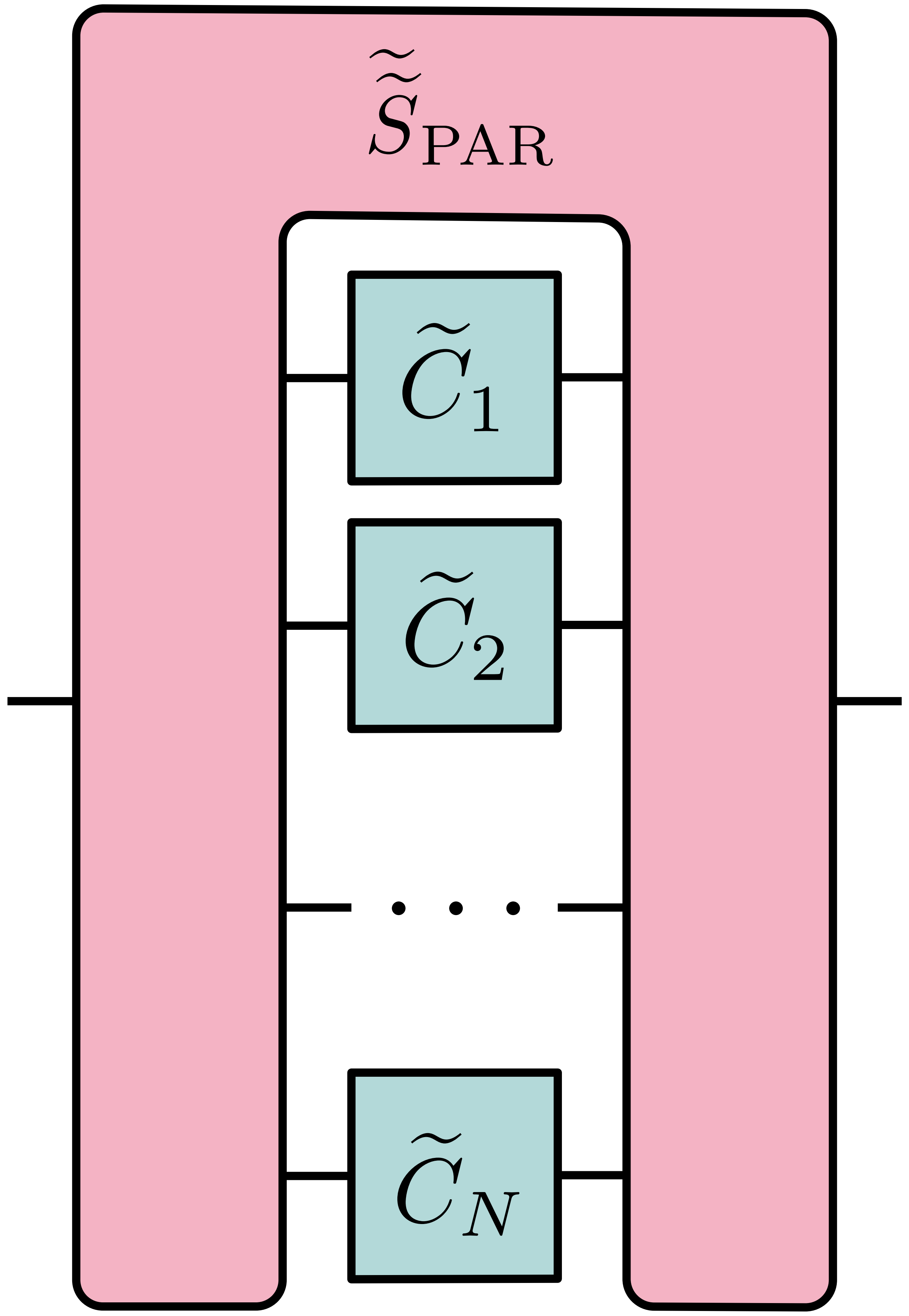
$$S_{\mathrm{PAR}} = E * D \quad \iff \quad \begin{aligned} S_{\mathrm{PAR}} &\geq 0 \\ \mathrm{tr}(S_{\mathrm{PAR}}) &= d_{I'} d_{O_1} \dots d_{O_N} \\ {}_{O'}S_{\mathrm{PAR}} &= {}_{O_1 \dots O_N O'}S_{\mathrm{PAR}} \\ {}_{I_1 \dots I_N O_1 \dots O_N O'}S_{\mathrm{PAR}} &= {}_{I' I_1 \dots I_N O_1 \dots O_N O'}S_{\mathrm{PAR}} \end{aligned}$$

Sequential superchannels:

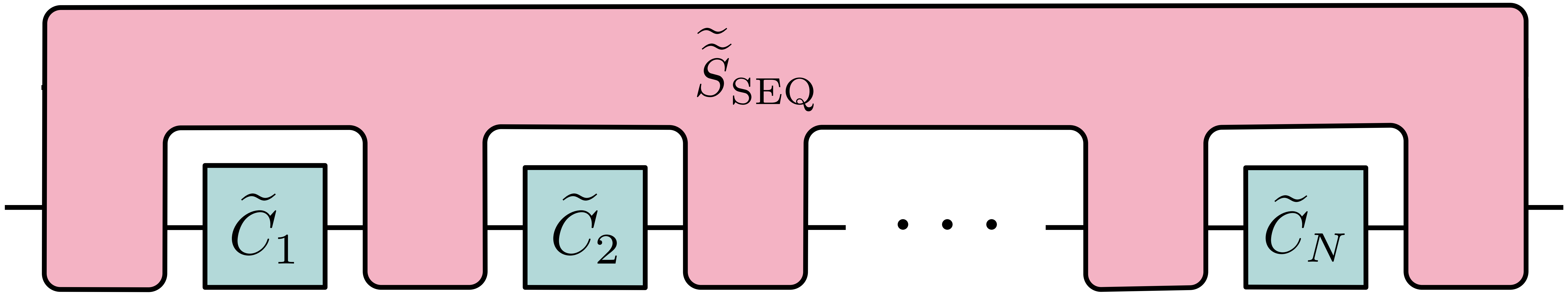
$$S_{\mathrm{SEQ}} = E_1 * E_2 * \dots * E_N * D \quad \iff \quad \begin{aligned} S_{\mathrm{SEQ}} &\geq 0 \\ \mathrm{tr}(S_{\mathrm{SEQ}}) &= d_{I'} d_{O_1} \dots d_{O_N} \\ {}_{O'}S_{\mathrm{SEQ}} &= {}_{O_N O'}S_{\mathrm{SEQ}} \\ {}_{I_N O_N O'}S_{\mathrm{SEQ}} &= {}_{O_{N-1} I_N O_N O'}S_{\mathrm{SEQ}} \\ &\dots \\ {}_{I_1 O_1 \dots I_N O_N O'}S_{\mathrm{SEQ}} &= {}_{I' I_1 O_1 \dots I_N O_N O'}S_{\mathrm{SEQ}} \end{aligned}$$

Parallel and Sequential Superchannels

Parallel and sequential superchannels = Implementable via quantum circuits



$\mathrm{PAR} \subset \mathrm{SEQ}$



is there a more general superchannel?

General Superchannels

Superchannel: The most general linear higher-order transformation that maps all tensor products of quantum channels into a quantum channel, even when acting on part of the quantum channels.

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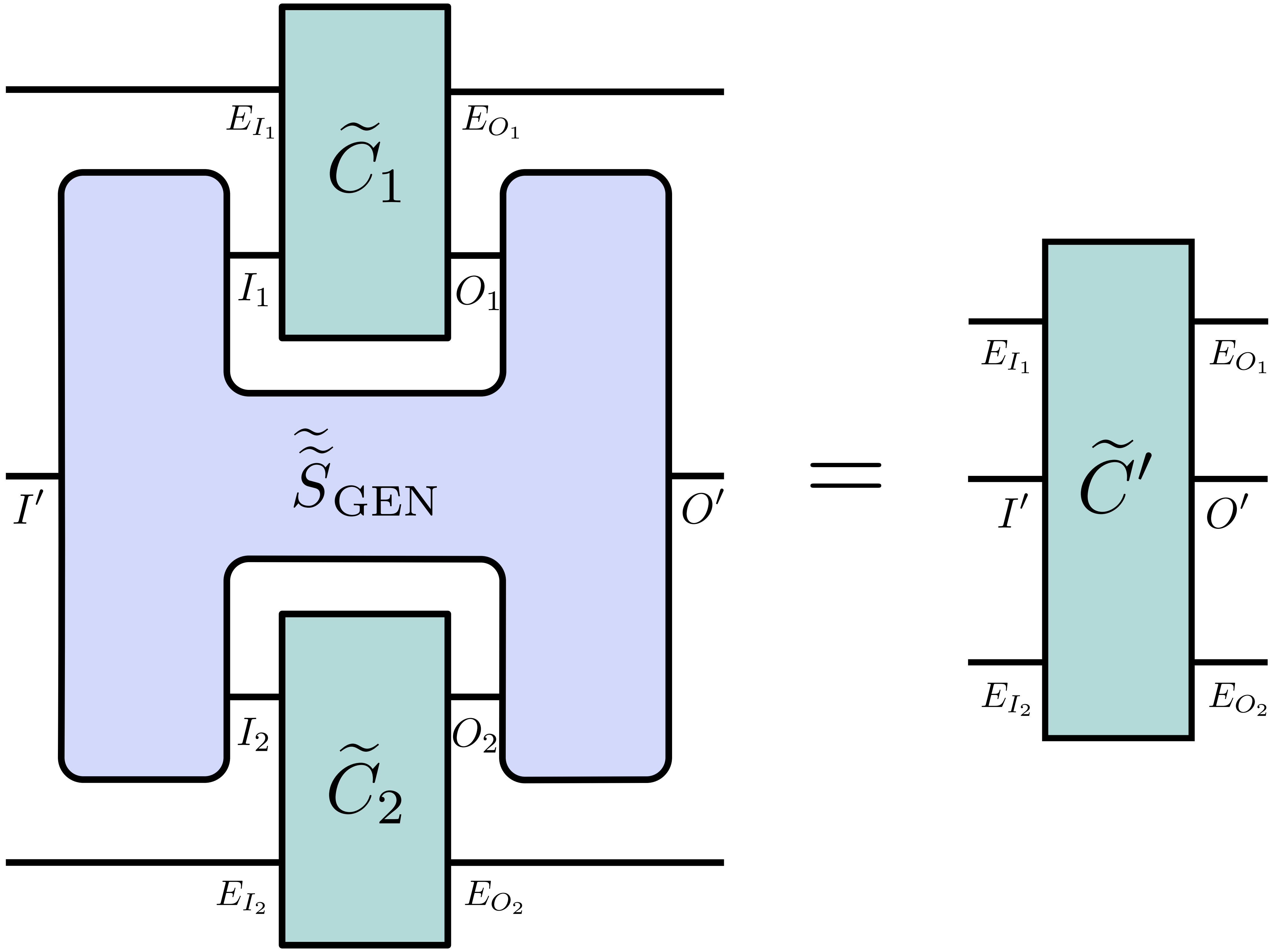


$$\widetilde{\widetilde{S}}_{\text{GEN}} : \bigotimes_i [\mathcal{L}(\mathcal{H}_{I_i}) \to \mathcal{L}(\mathcal{H}_{O_i})] \to [\mathcal{L}(\mathcal{H}_{I'}) \to \mathcal{L}(\mathcal{H}_{O'})]$$

$$\widetilde{\widetilde{S}}_{\text{GEN}} \otimes \widetilde{\widetilde{\mathbb{1}}}(\widetilde{C}_1 \otimes \ldots \otimes \widetilde{C}_N) = \widetilde{C}'$$

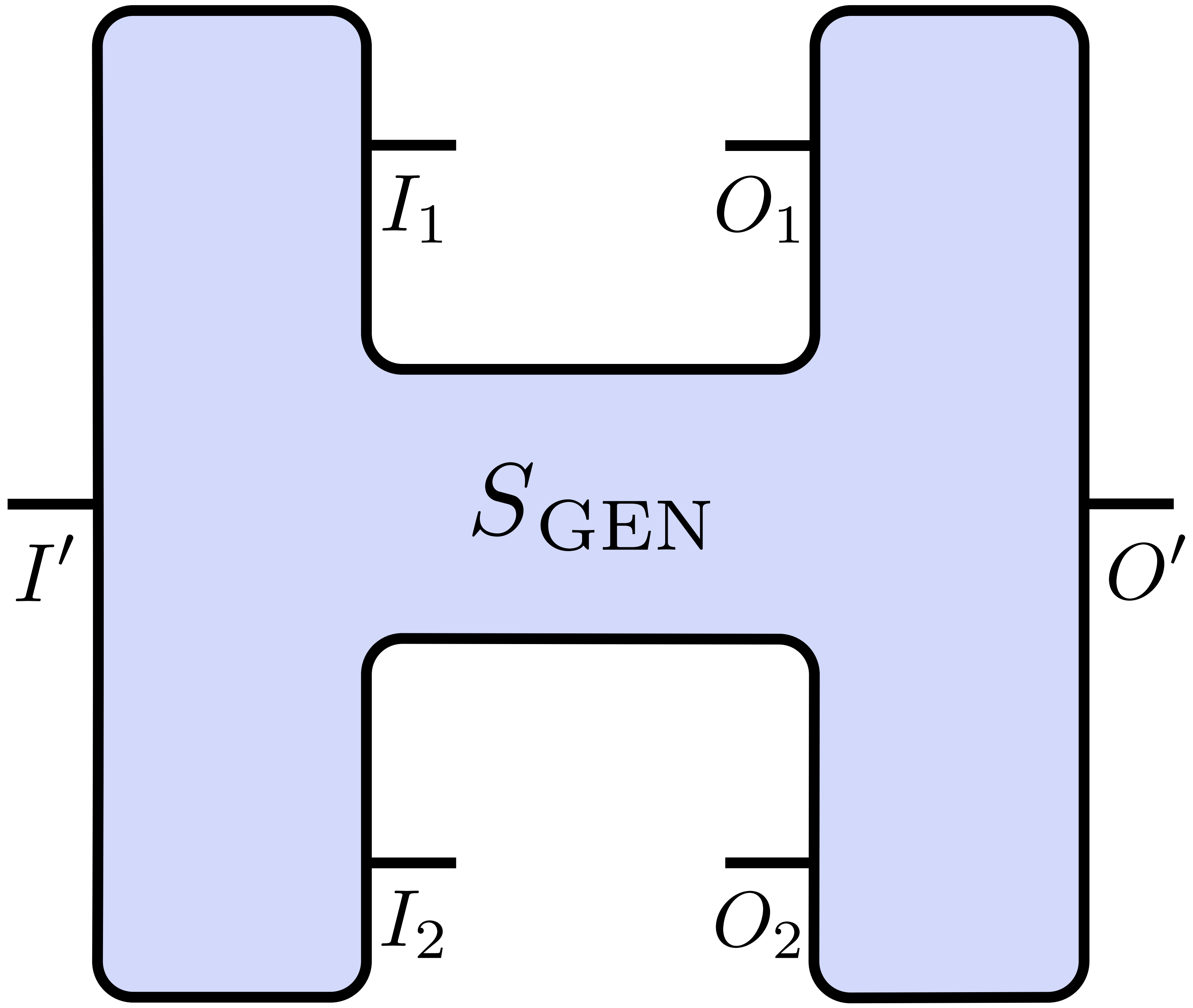
$\forall\ \widetilde{C}_1, \widetilde{C}_2$

completely CPTP preserving
linear
higher-order maps

(generalizes to any finite number of slots)

General Superchannels

Superchannel: The most general linear higher-order transformation that maps all tensor products of quantum channels into a quantum channel, even when acting on part of the quantum channels.



$$S_{\text{GEN}} * (C_1 \otimes C_2) = C' \;\; \forall\, C_1, C_2$$

(generalizes to any finite number of slots)

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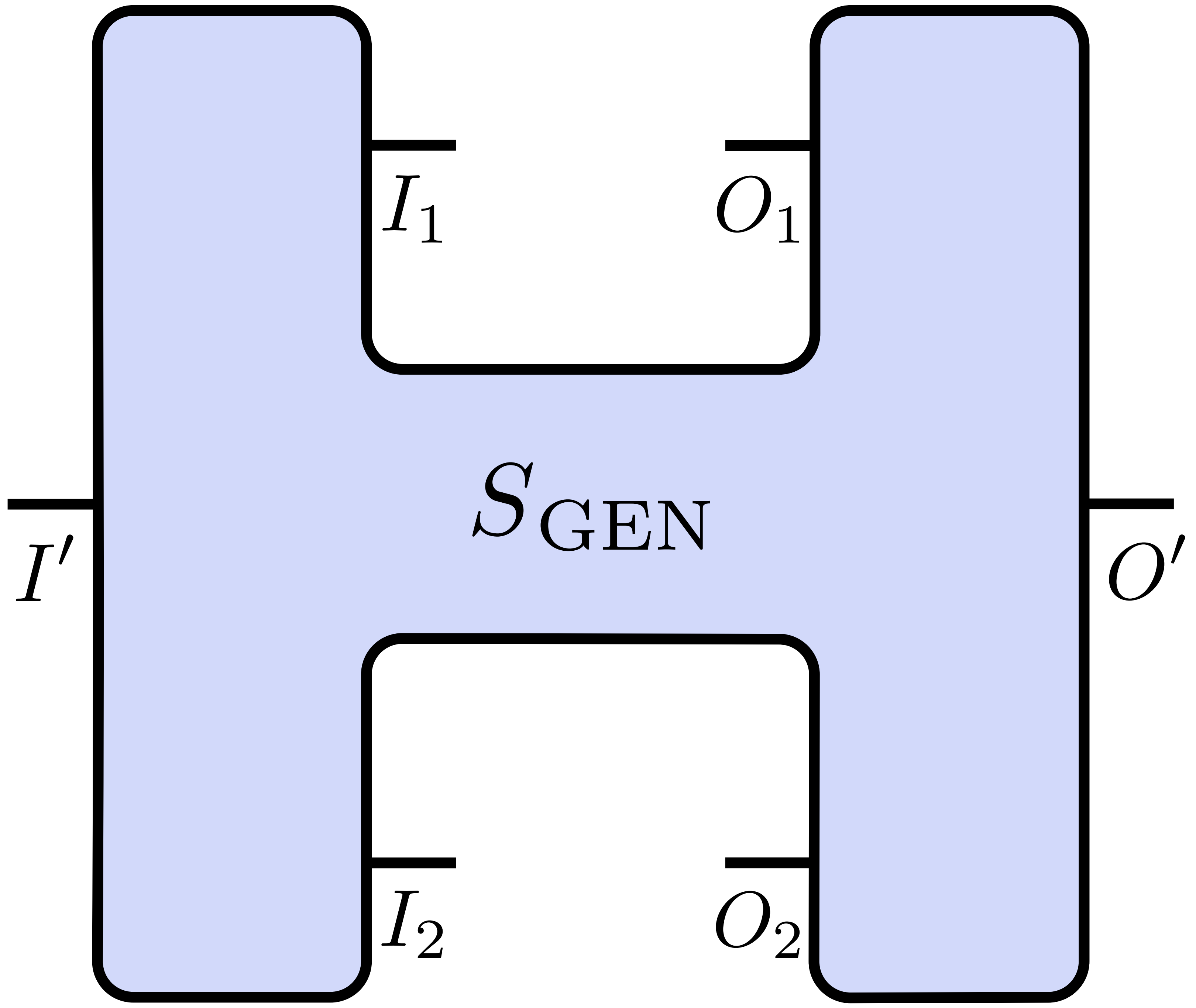


$$S_{\text{GEN}} * (C_1 \otimes C_2) = C' \;\; \forall\, C_1, C_2$$

$$\iff$$

$$S_{\text{GEN}} \geq 0$$

$$\text{tr}(S_{\text{GEN}}) = d_{I'} d_{O_1} d_{O_2}$$

$${}_{I_1O_1O'}S_{\text{GEN}} = {}_{I_1O_1O_2O'}S_{\text{GEN}}$$

$${}_{I_2O_2O'}S_{\text{GEN}} = {}_{O_1I_2O_2O'}S_{\text{GEN}}$$

$${}_{O'}S_{\text{GEN}} = {}_{O_1O'}S_{\text{GEN}} + {}_{O_2O'}S_{\text{GEN}} - {}_{O_1O_2O'}S_{\text{GEN}}$$

$${}_{I_1O_1I_2O_2O'}S_{\text{GEN}} = {}_{I'I_1O_1I_2O_2O'}S_{\text{GEN}}$$

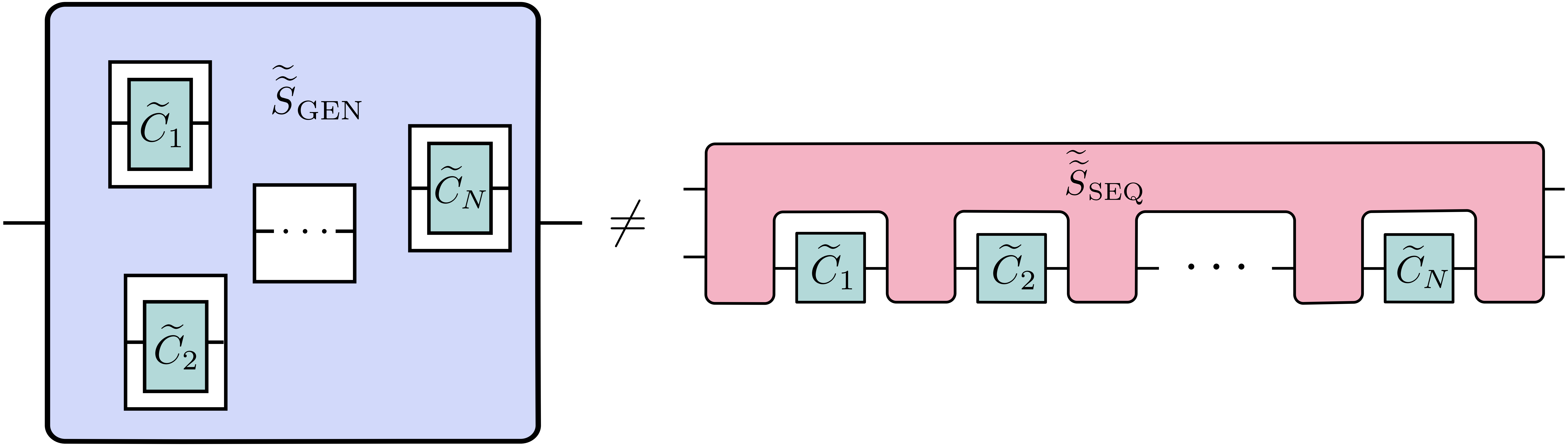
(generalizes to any finite number of slots)

is there a quantum circuit realization theorem
for general many-slot superchannels?

is there a quantum circuit realization theorem for general many-slot superchannels?

NO.

General Superchannels

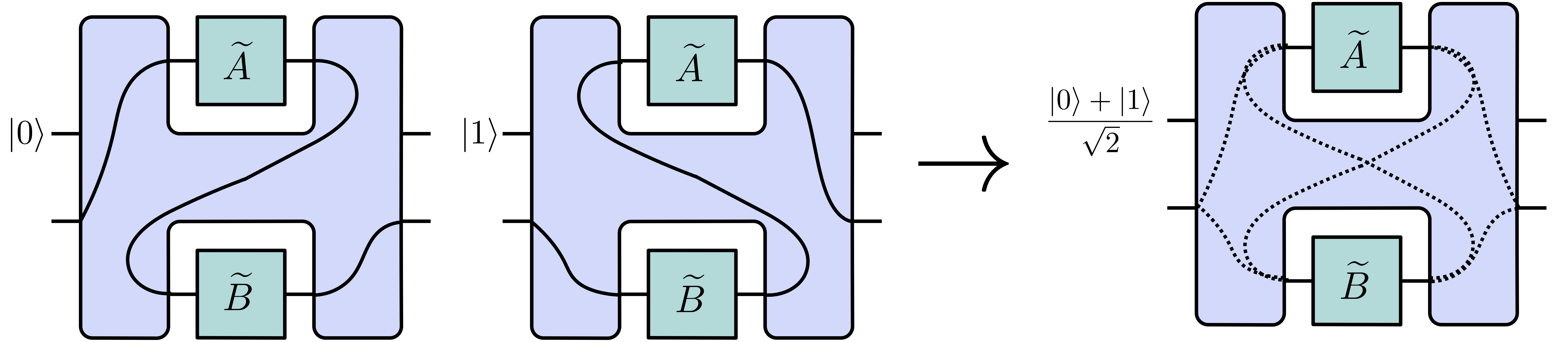


There exist many-slot general superchannels that do not satisfy the conditions of a sequential superchannel and, hence, cannot be decomposed as a **quantum circuit**.

indefinite causal order

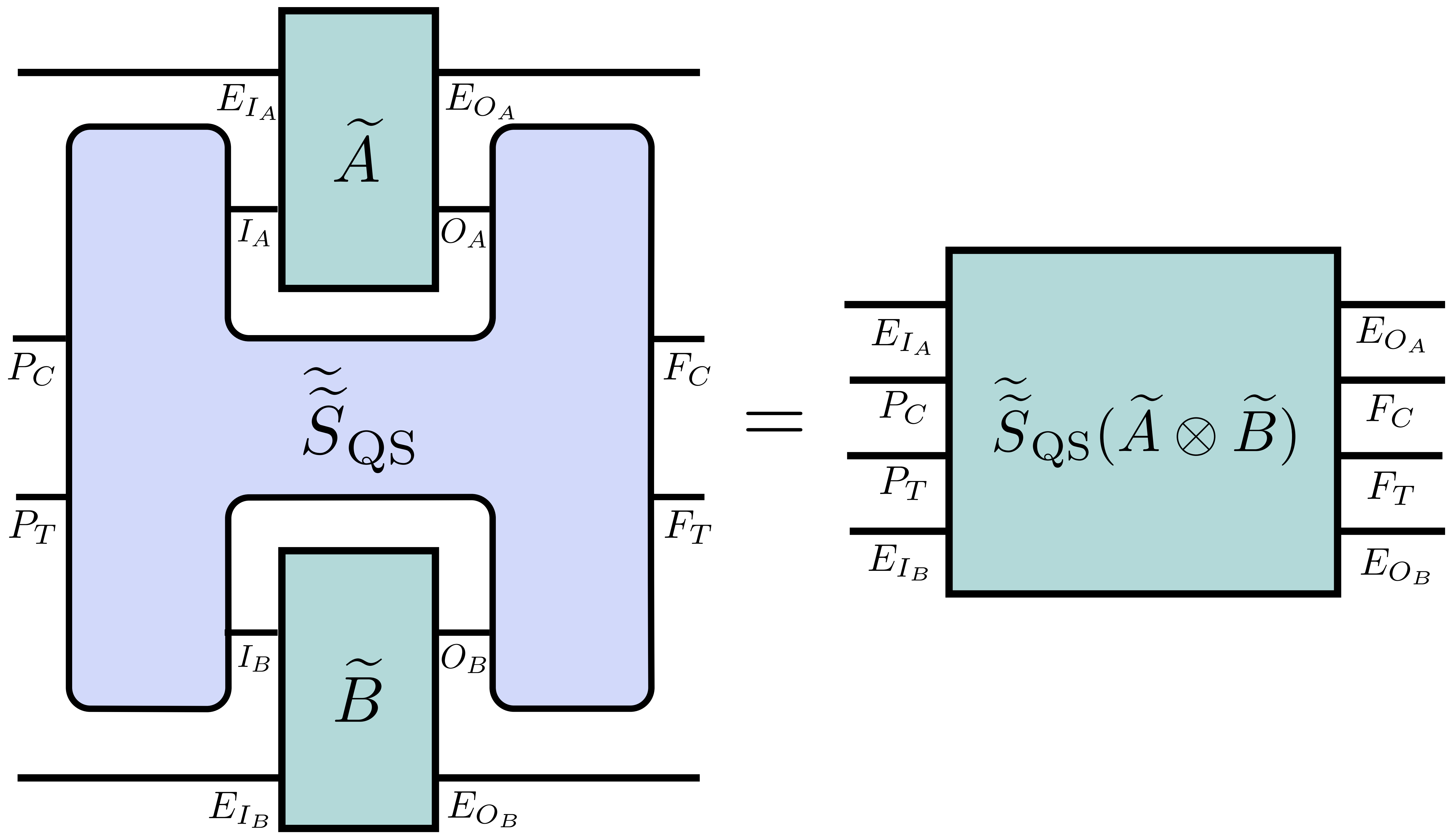
example of a general many-slot superchannel with indefinite causal order

The Quantum Switch



Chiribella et al., PRA (2013)

The Quantum Switch



Choi operator of the quantum switch superchannel:

$$S_{\mathrm{QS}} := |s\rangle\langle s|$$

$$|s\rangle := |0\rangle_{P_C} |\mathbb{1}\rangle_{P_T I_A} |\mathbb{1}\rangle_{O_A I_B} |\mathbb{1}\rangle_{O_B F_T} |0\rangle_{F_C} + |1\rangle_{P_C} |\mathbb{1}\rangle_{P_T I_B} |\mathbb{1}\rangle_{O_B I_A} |\mathbb{1}\rangle_{O_A F_T} |1\rangle_{F_C}$$

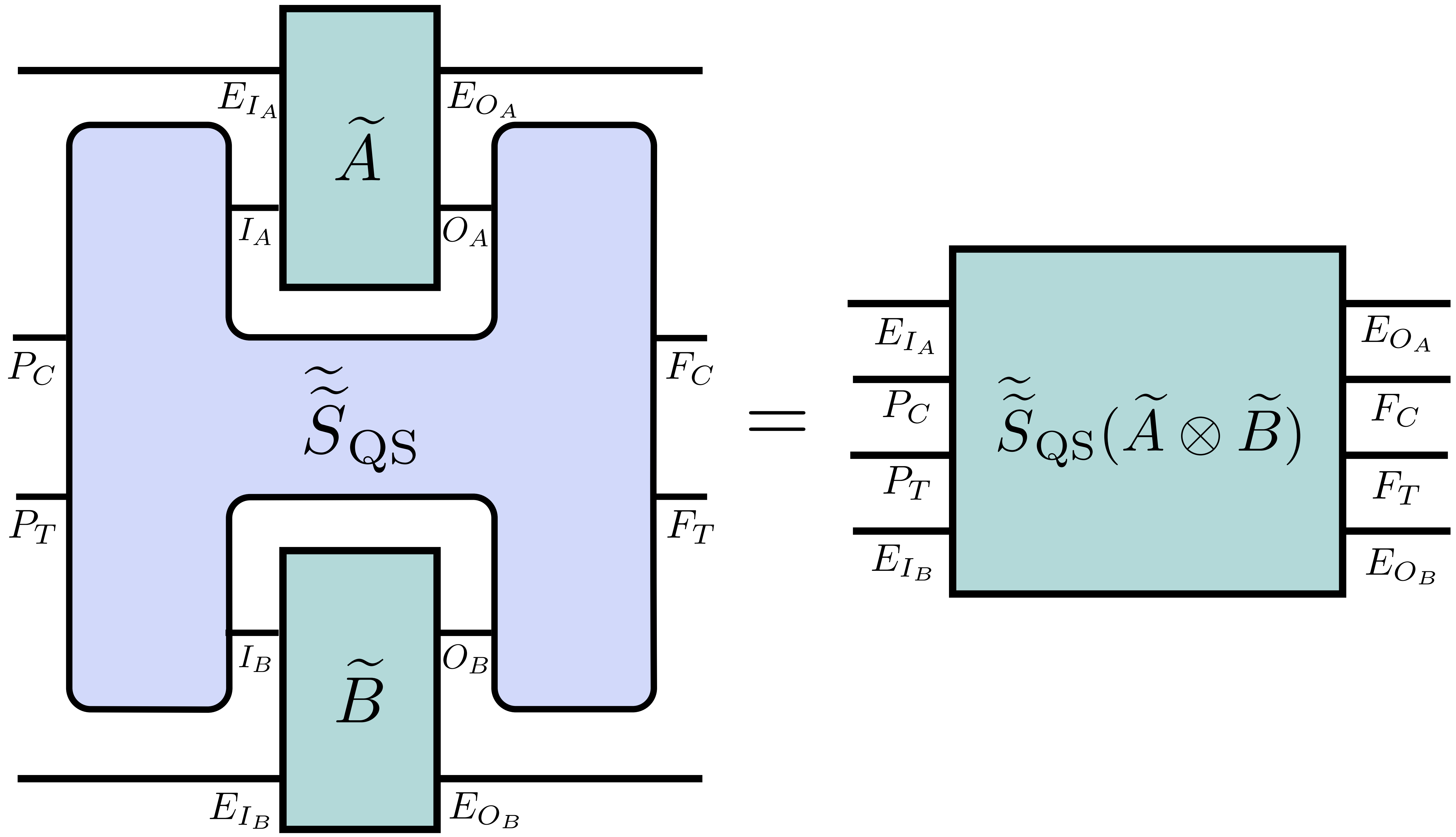
$$|\mathbb{1}\rangle := \sum_i |ii\rangle$$

$$|\mathbb{1}\rangle\langle\mathbb{1}|$$

(Choi operator of the identity channel)

Chiribella et al., PRA (2013)

The Quantum Switch



the quantum switch is not a sequential superchannel

Choi operator of the quantum switch superchannel:

$$S_{\mathrm{QS}} := |s\rangle\langle s|$$

$$\begin{aligned} |s\rangle := &|0\rangle_{P_C} |\mathbb{1}\rangle\rangle_{P_T I_A} |\mathbb{1}\rangle\rangle_{O_A I_B} |\mathbb{1}\rangle\rangle_{O_B F_T} |0\rangle_{F_C} \\ &+ |1\rangle_{P_C} |\mathbb{1}\rangle\rangle_{P_T I_B} |\mathbb{1}\rangle\rangle_{O_B I_A} |\mathbb{1}\rangle\rangle_{O_A F_T} |1\rangle_{F_C} \end{aligned}$$

Chiribella et al., PRA (2013)

advantage of the quantum switch

Quantum switch advantage

Discriminating commuting from anti-commuting unitaries

$$\widetilde{U} = \mathsf{U}\rho\mathsf{U}^\dagger \qquad \widetilde{V} = \mathsf{V}\rho\mathsf{V}^\dagger$$

TASK:

Hypothesis 1: $[\mathsf{U},\mathsf{V}] = \mathsf{UV} - \mathsf{VU} = 0$ **commute** $(\text{with } p = 1/2)$

Hypothesis 2: $\{\mathsf{U},\mathsf{V}\} = \mathsf{UV} + \mathsf{VU} = 0$ **anticommute** $(\text{with } p = 1/2)$

Rules: access to one query of each channel per round

Chiribella, PRA (2012) & Araújo et al., NJP (2015)

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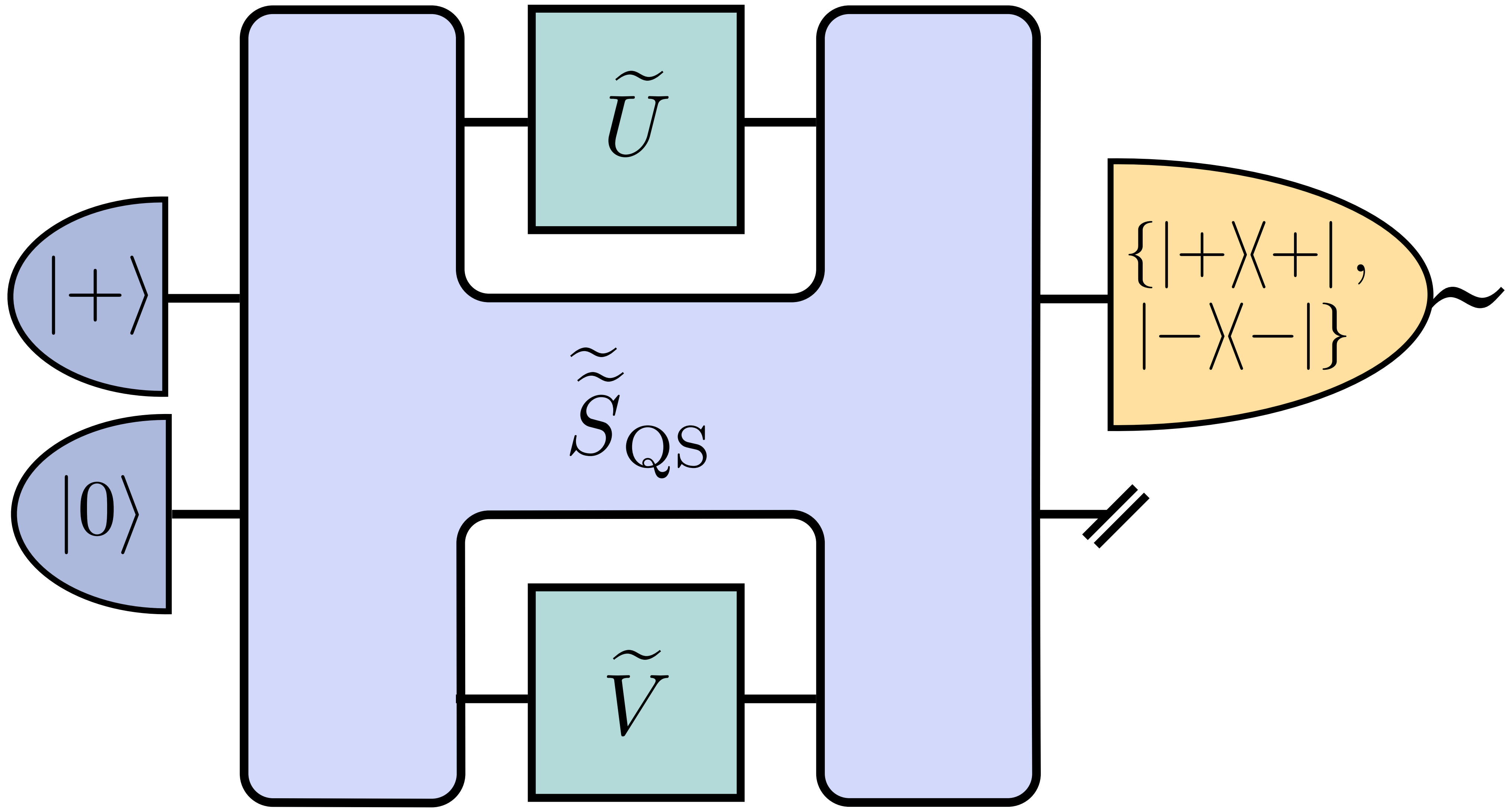

Chiribella, PRA (2012) & Araújo et al., NJP (2015)

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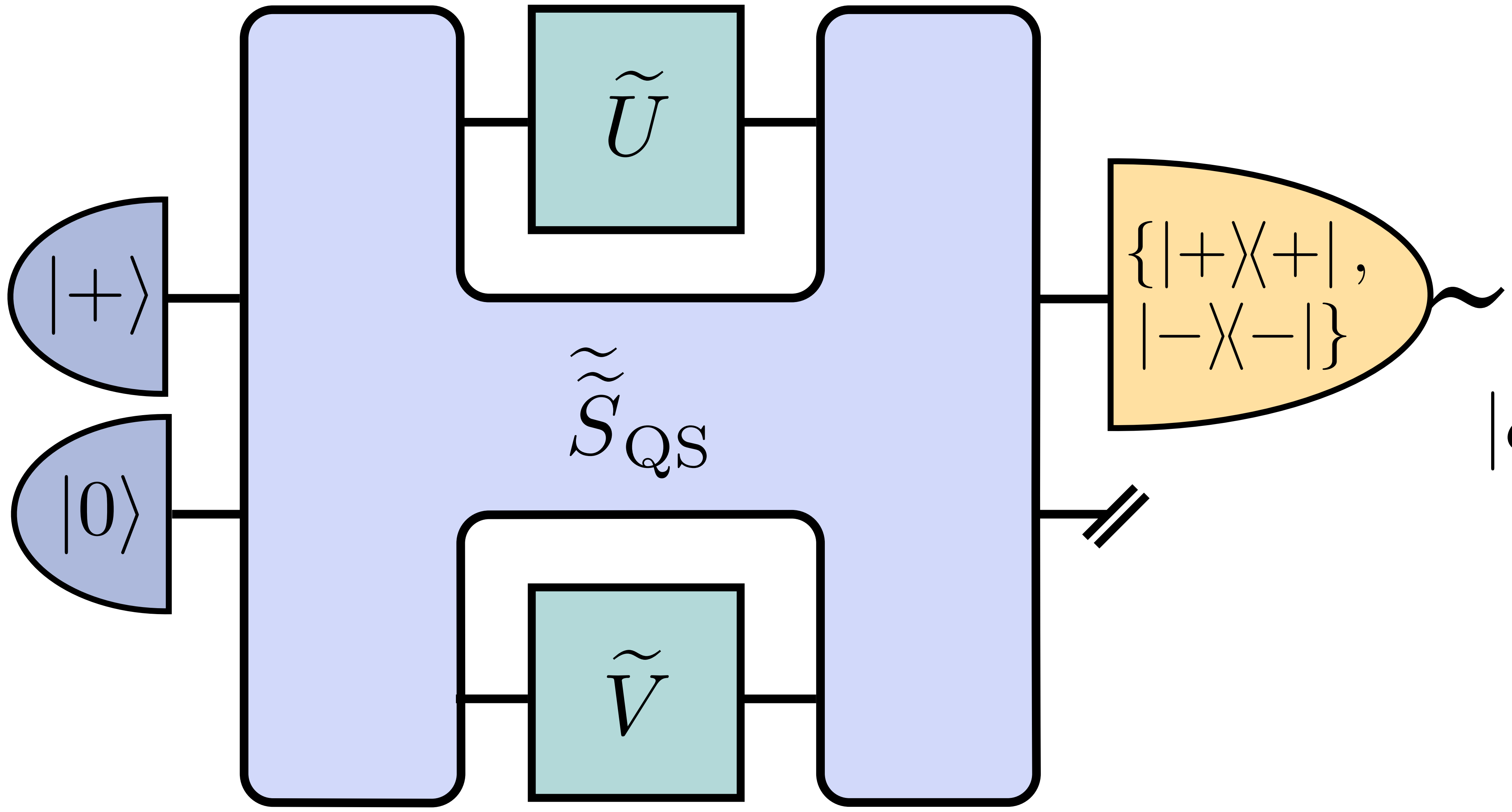

$$\widetilde{\widetilde{S}}_{\mathrm{QS}}(\widetilde{U}\otimes\widetilde{V})\left[|+\rangle\langle+|_{P_C}\otimes|0\rangle\langle 0|_{P_T}\right] = |\phi\rangle\langle\phi|_{F_C F_T}$$

$$|\phi\rangle_{F_C F_T} = \frac{1}{\sqrt{2}}\left(|+\rangle_{F_C}\otimes\{\mathsf{U},\mathsf{V}\}\,|0\rangle_{F_T} + |-\rangle_{F_C}\otimes[\mathsf{U},\mathsf{V}]\,|0\rangle_{F_T}\right)$$

Chiribella, PRA (2012) & Araújo et al., NJP (2015)

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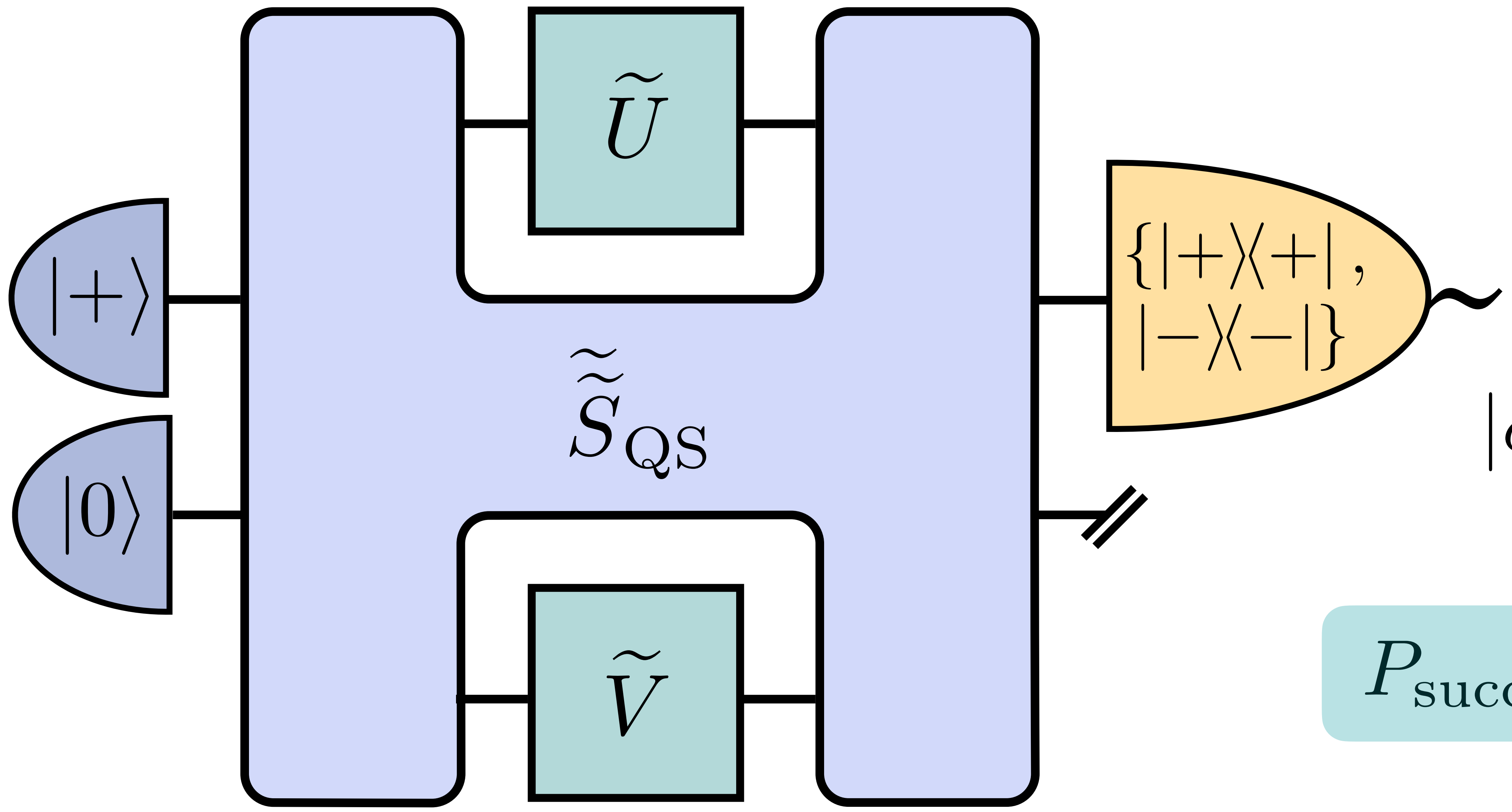

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$P_{\mathrm{succ}} = 1$

commute **anticommute**

Chiribella, PRA (2012) & Araújo et al., NJP (2015)

Quantum switch advantage

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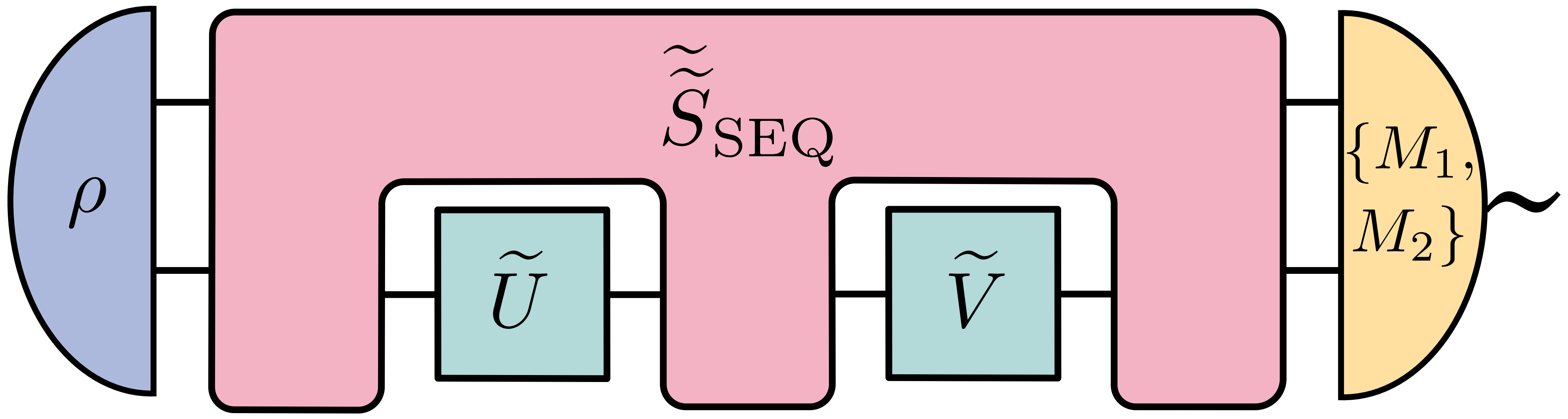

$\forall\ \widetilde{\widetilde{S}}_{\mathrm{SEQ}}$

$\forall\ \rho$

$\forall\ \{M_1, M_2\}$

$P_{\mathrm{succ}} < 1$

Chiribella, PRA (2012) & Araújo et al., NJP (2015)

Quantum switch advantage

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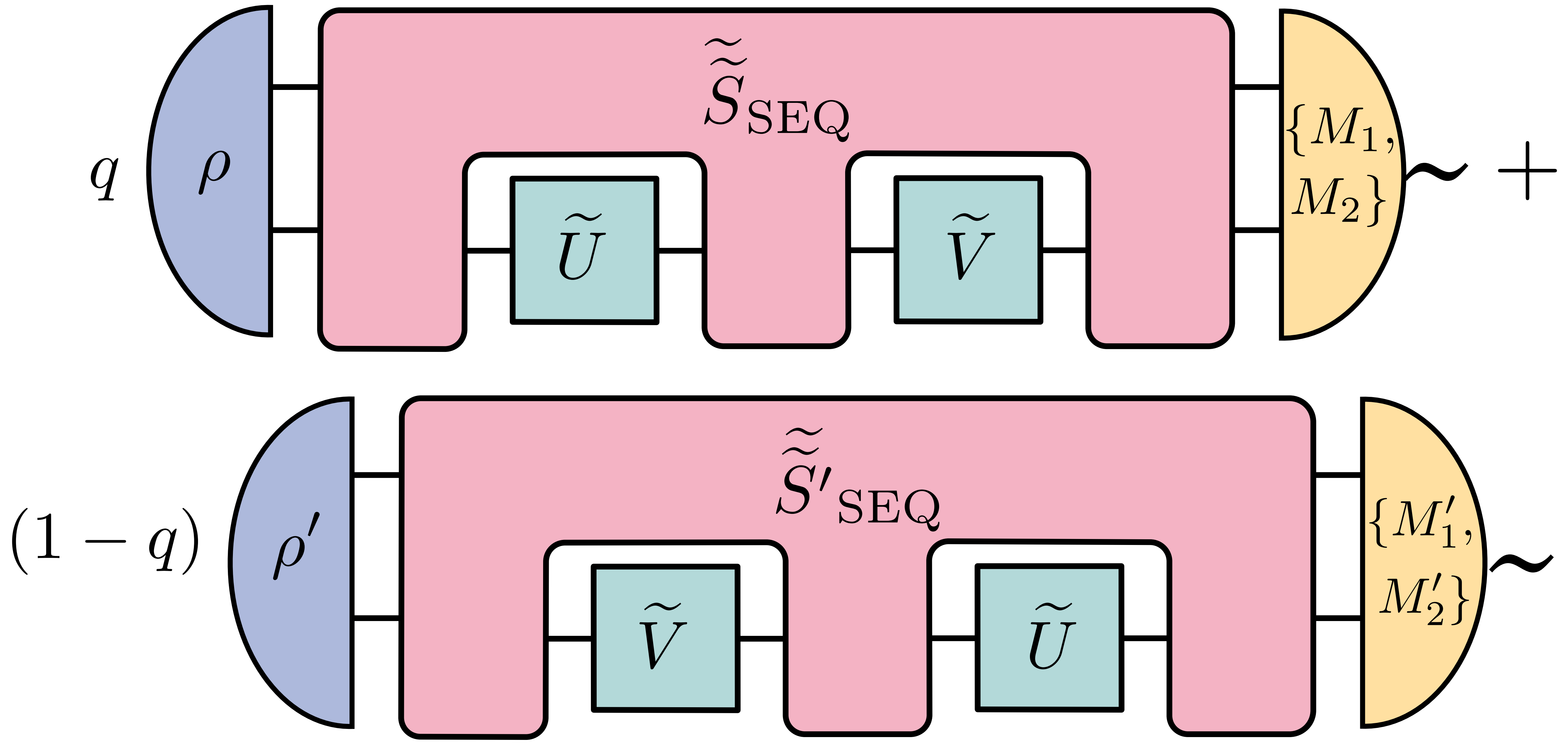


$\forall\ \widetilde{\widetilde{S}}_{\mathrm{SEQ}},\ \widetilde{\widetilde{S}}'_{\mathrm{SEQ}}$

$\forall\ \rho,\ \rho'$

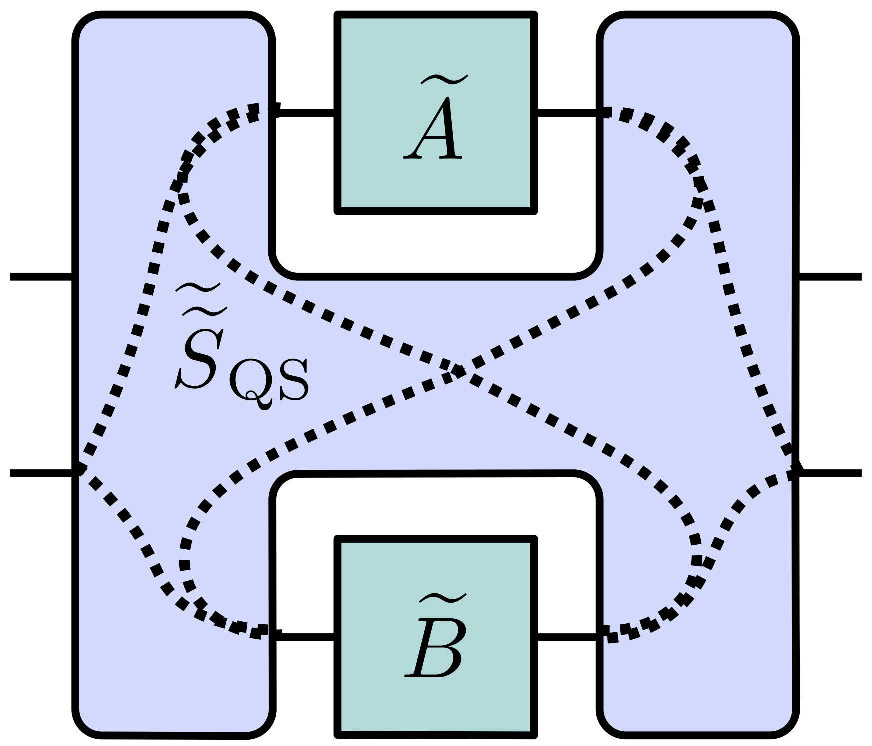
$\forall\ \{M_1, M_2\},\ \{M_1', M_2'\}$

$\forall\ q \in [0,1]$

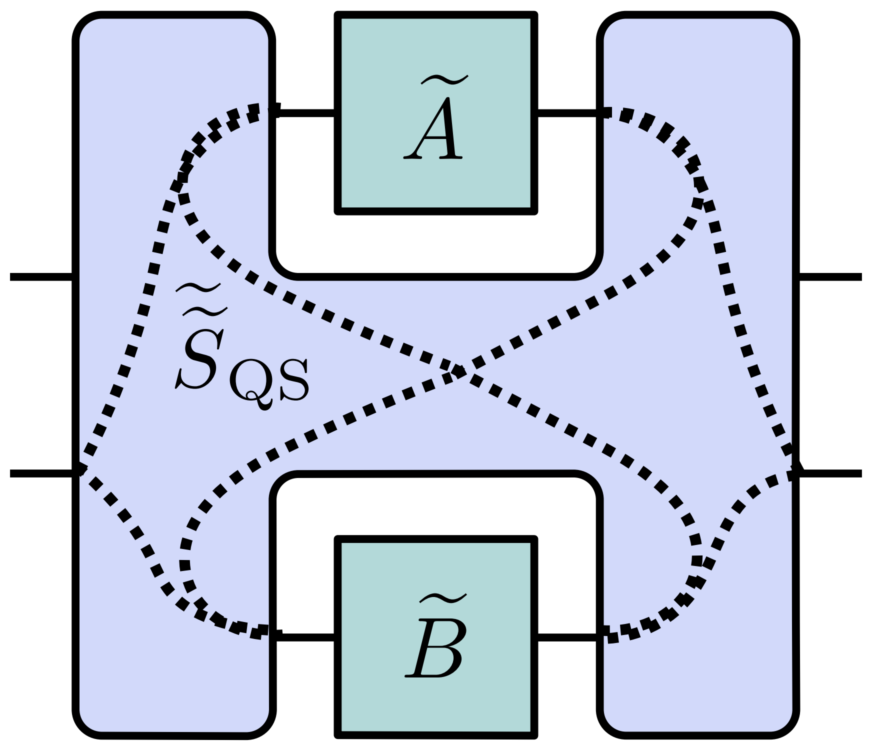
$P_{\mathrm{succ}} < 1$

≈ 0.9288

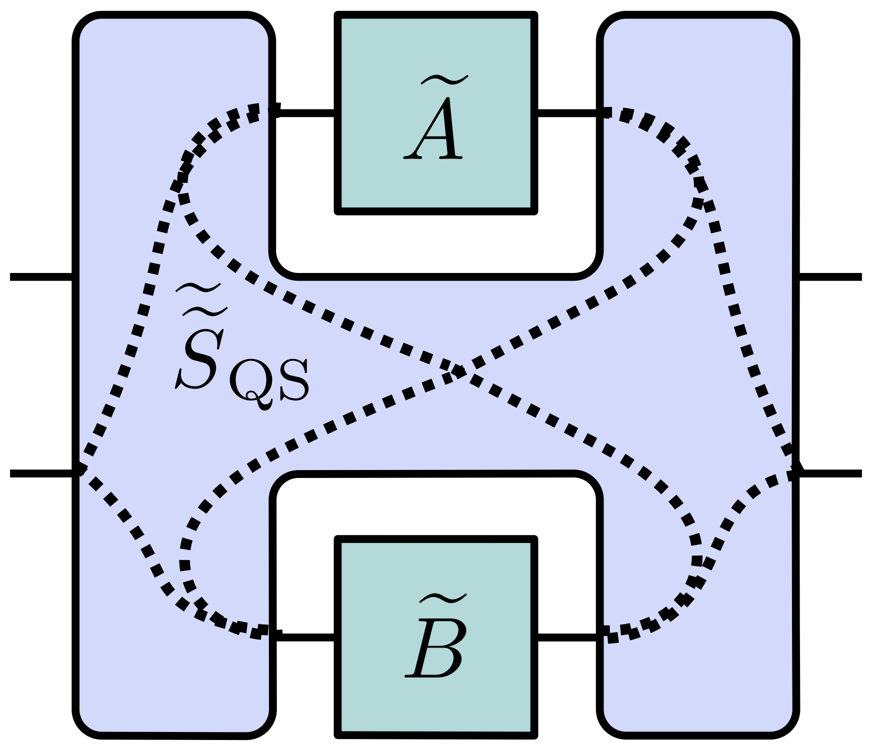
Chiribella, PRA (2012) & Araújo et al., NJP (2015)



The quantum switch is considered to be a superchannel
with **indefinite causal order**



are there other forms of indefinite causal order?



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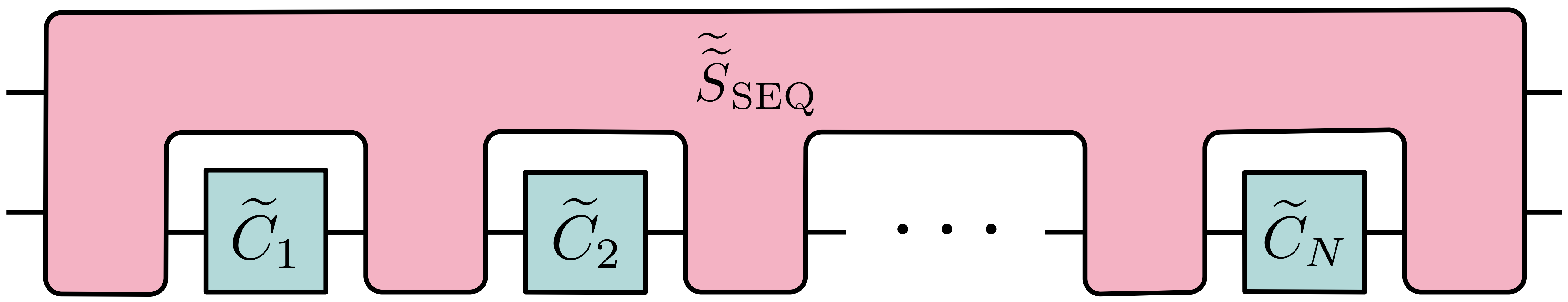
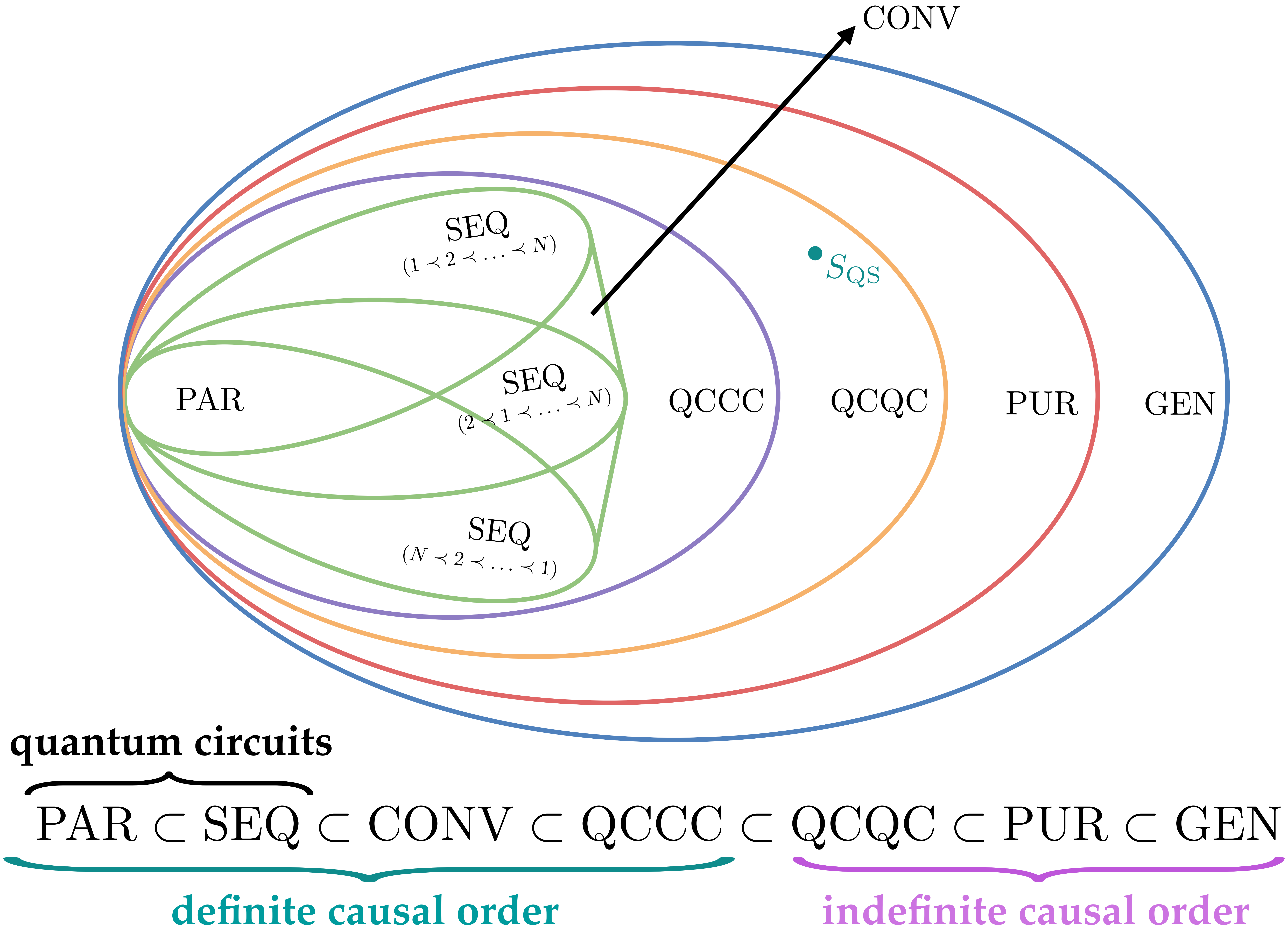
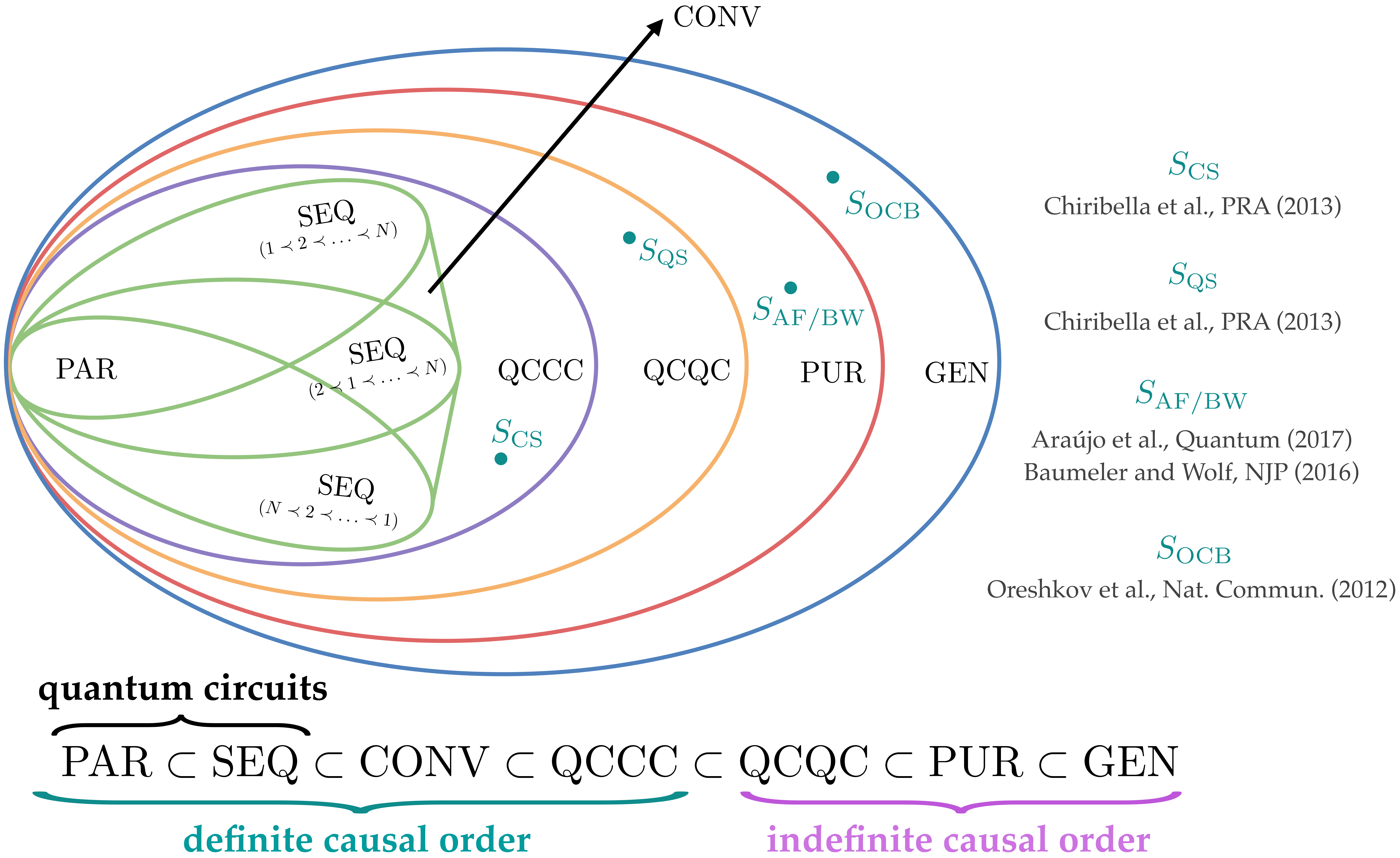



CONV
SEQ $(1 \prec 2 \prec \ldots \prec N)$
S_{QS}
PAR
SEQ $(2 \prec 1 \prec \ldots \prec N)$
QCCC
QCQC
PUR
GEN
SEQ $(N \prec 2 \prec \ldots \prec 1)$
quantum circuits
PAR ⊂ SEQ ⊂ CONV ⊂ QCCC ⊂ QCQC ⊂ PUR ⊂ GEN
definite causal order
indefinite causal order


CONV
SEQ
$(1 \prec 2 \prec \ldots \prec N)$
SEQ
$(2 \prec 1 \prec \ldots \prec N)$
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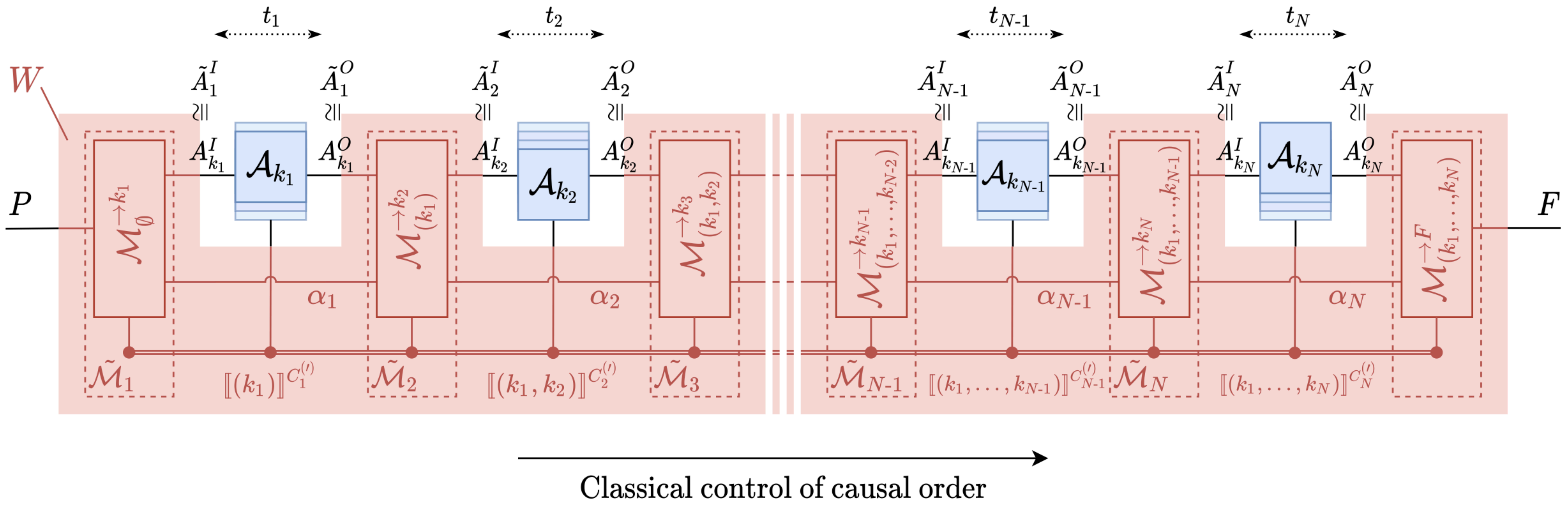
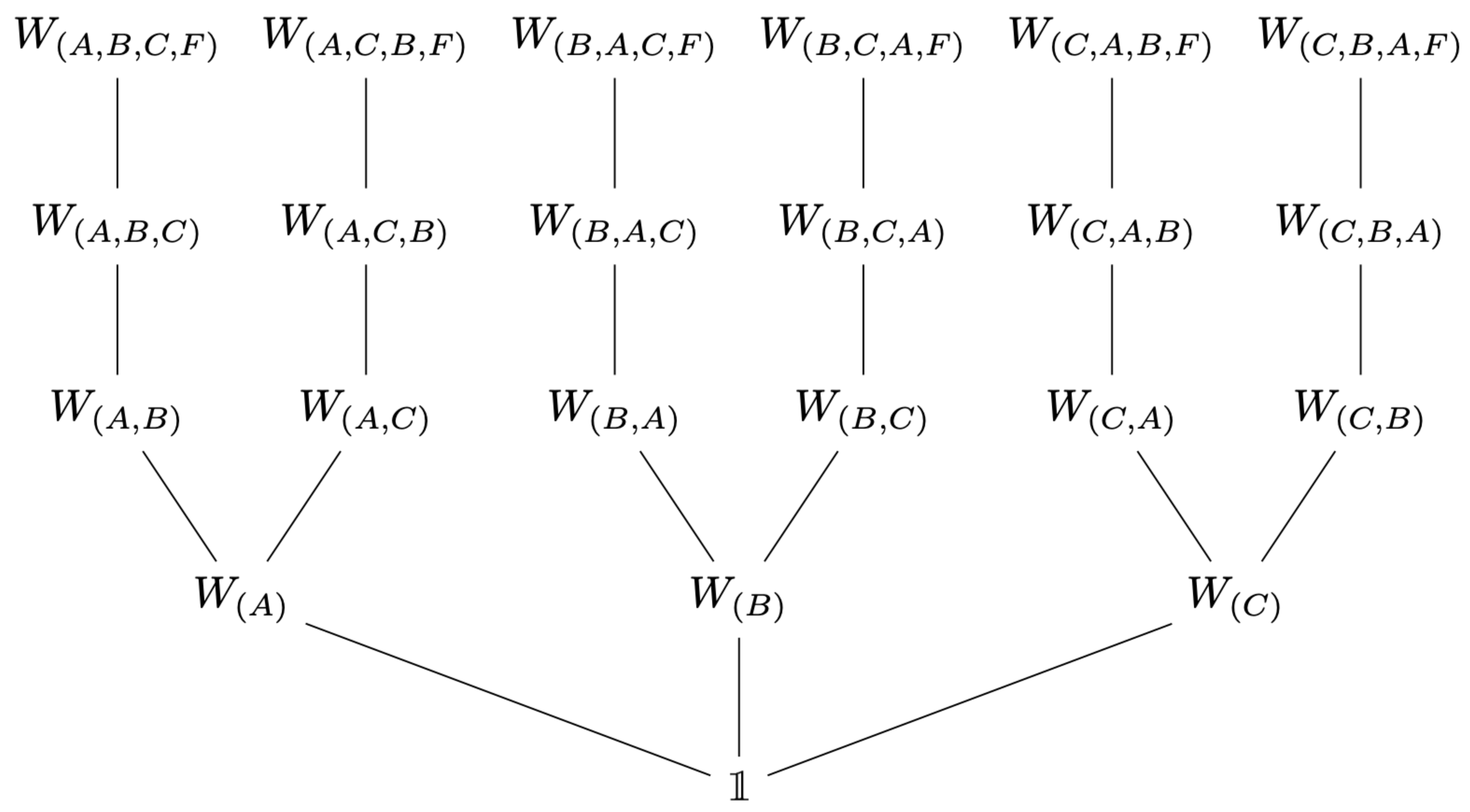
$(N \prec 2 \prec \ldots \prec 1)$
PAR
QCCC
QCQC
PUR
GEN
S_{CS}
S_{QS}
$S_{\mathrm{AF/BW}}$
S_{OCB}
S_{CS}
Chiribella et al., PRA (2013)
S_{QS}
Chiribella et al., PRA (2013)
$S_{\mathrm{AF/BW}}$
Araújo et al., Quantum (2017)
Baumeler and Wolf, NJP (2016)
S_{OCB}
Oreshkov et al., Nat. Commun. (2012)
quantum circuits
$\mathrm{PAR} \subset \mathrm{SEQ} \subset \mathrm{CONV} \subset \mathrm{QCCC} \subset \mathrm{QCQC} \subset \mathrm{PUR} \subset \mathrm{GEN}$
definite causal order
indefinite causal order

Quantum circuits with classical control

QCCC



(ref. & figure:) Wechs et al., PRX Quantum (2021)



(ref. & figure:) Mothe et al., arXiv [quant-ph] (2025)

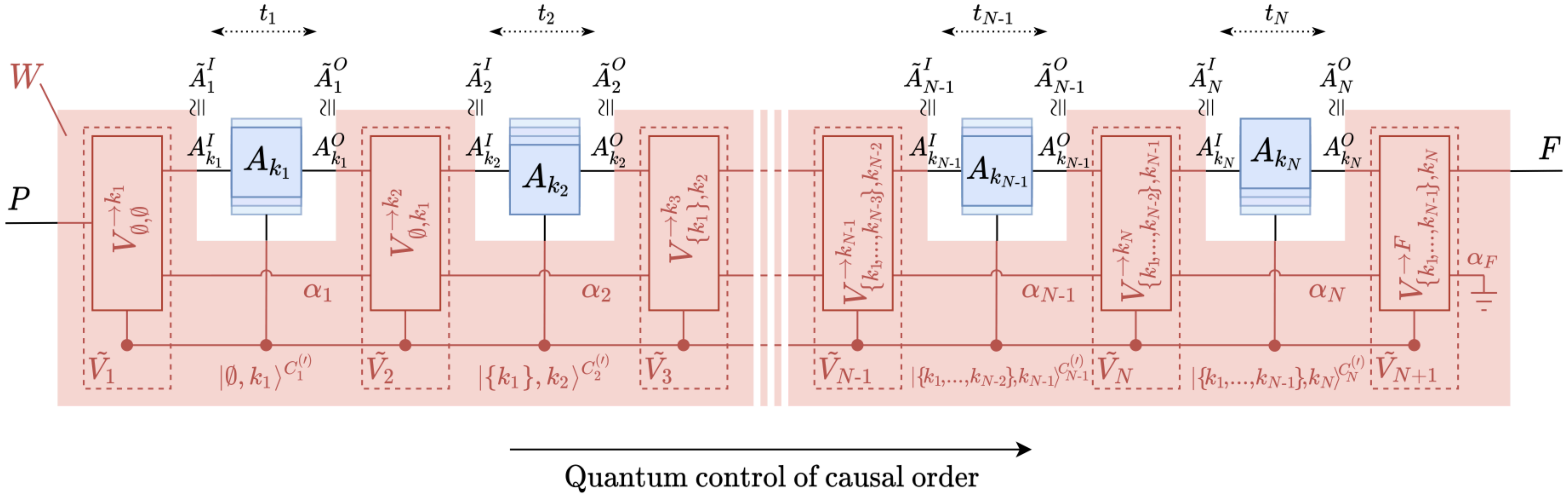
example: classical switch

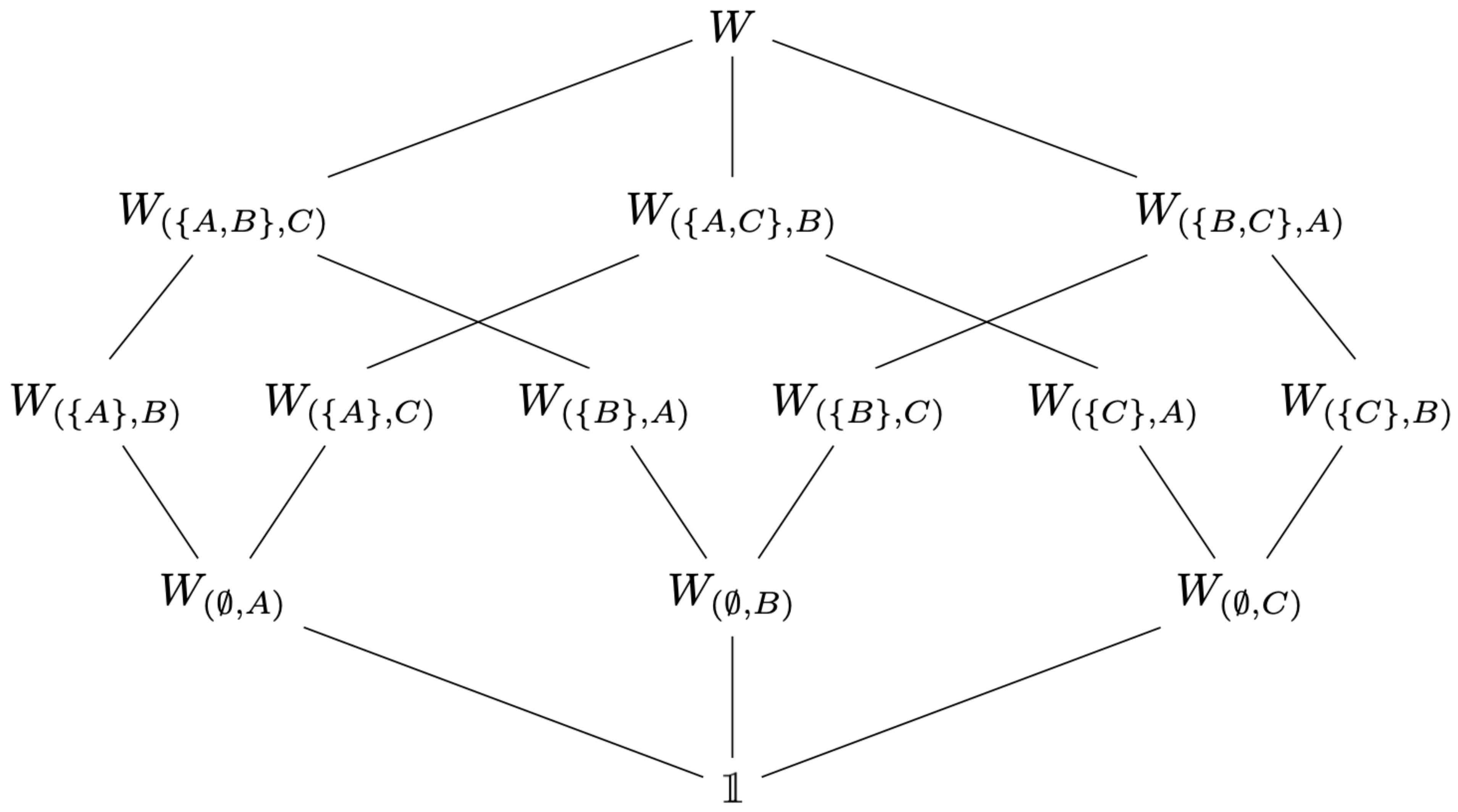
(quantum switch with fully dephased control)

Chiribella et al., PRA (2013)

Quantum circuits with quantum control

QCQC





(ref. & figure:) Wechs et al., PRX Quantum (2021)

(ref. & figure:) Mothe et al., arXiv [quant-ph] (2025)

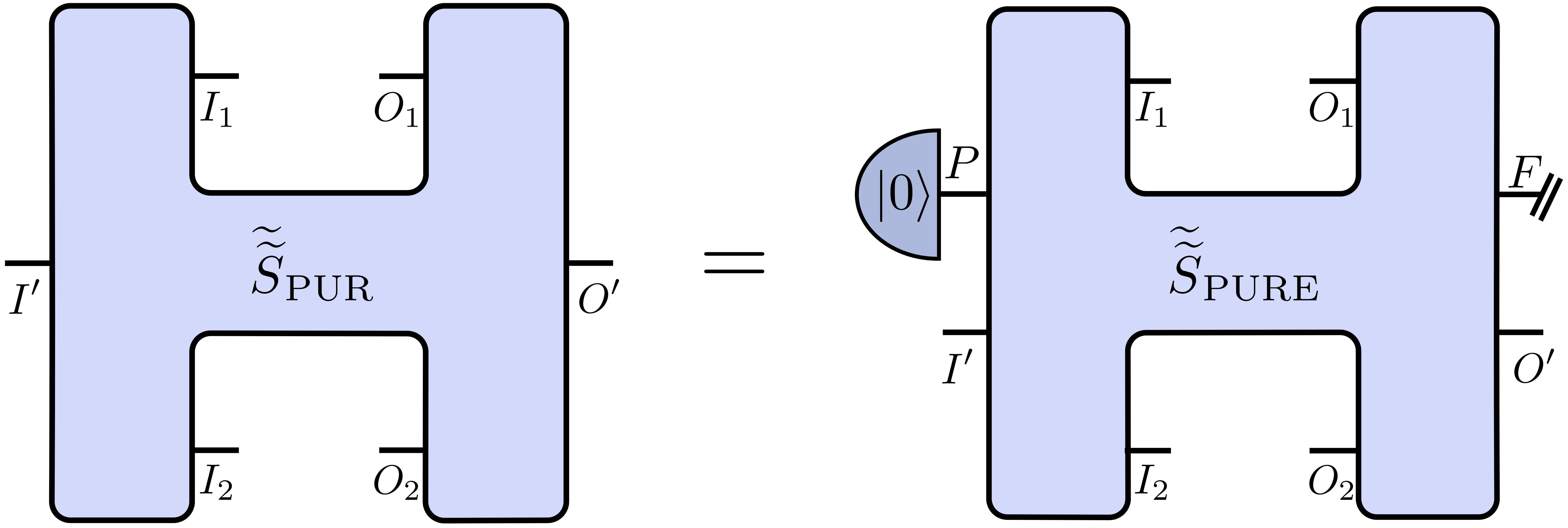
example: quantum switch

Chiribella et al., PRA (2013)

Purifiable (unitarily extensible) superchannels

PUR

Purifiable superchannels: A superchannel is purifiable if it can be expressed as the marginal of a pure superchannel.



$$S_{\mathrm{PUR}} = \mathrm{tr}_F(|0\rangle\langle 0| * S_{\mathrm{PURE}})$$

Araújo et al., Quantum (2017)

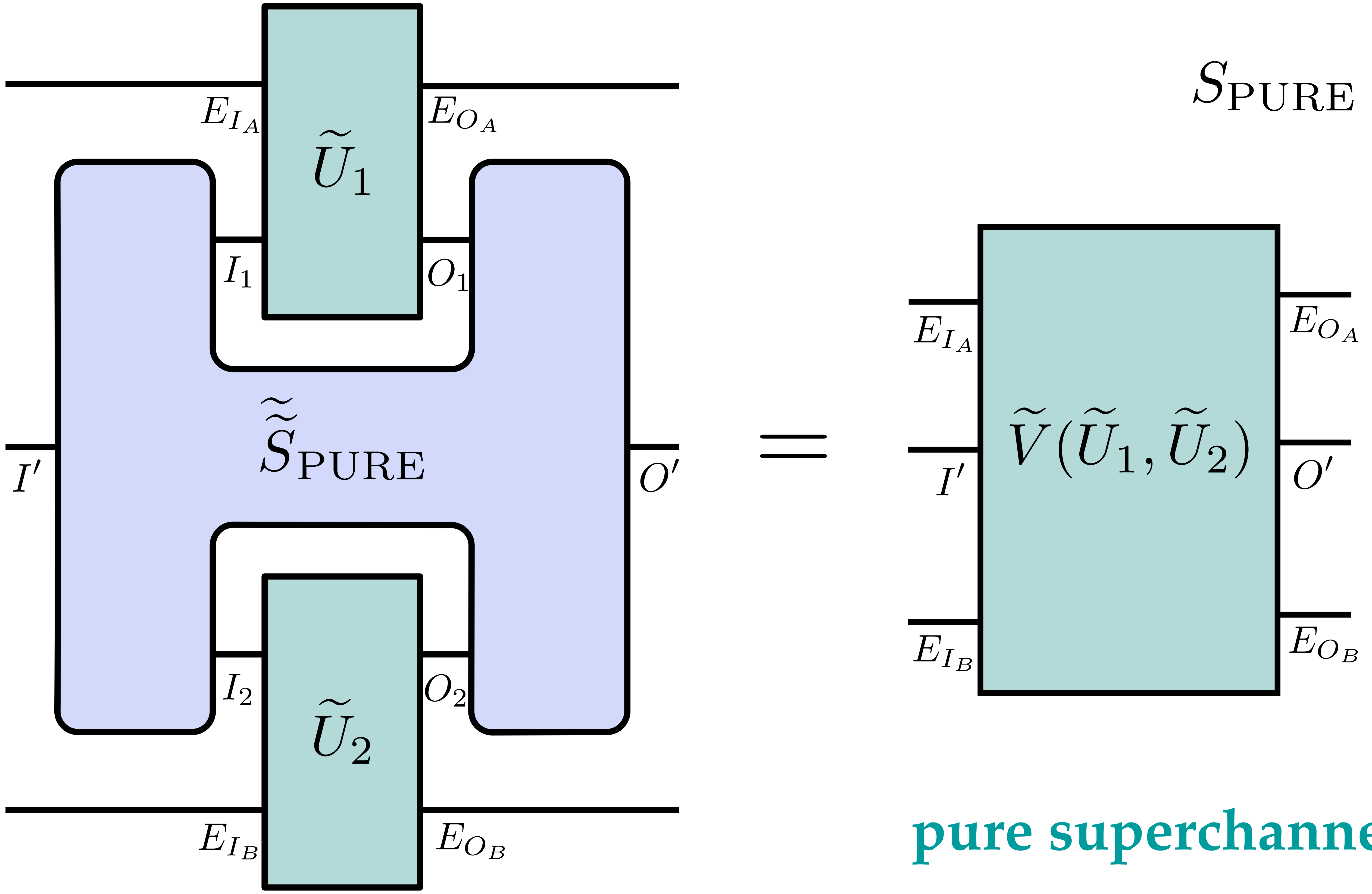
Purifiable (unitarily extensible) superchannels

PUR

Purifiable superchannels: A superchannel is purifiable if it can be expressed as the marginal of a pure superchannel.

Pure superchannels: A superchannels is pure if it maps all input unitary channels into an output channel that is also unitary.

$$S_{\text{PURE}} = |\mathsf{U}\rangle\!\rangle\langle\!\langle\mathsf{U}|$$



$\forall$ unitary $\widetilde{U}_1, \widetilde{U}_2$

pure superchannel = reversibility preserving

= unitary process

Araújo et al., Quantum (2017)

other advantages of indefinite causal order

Advantages of indefinite causal order

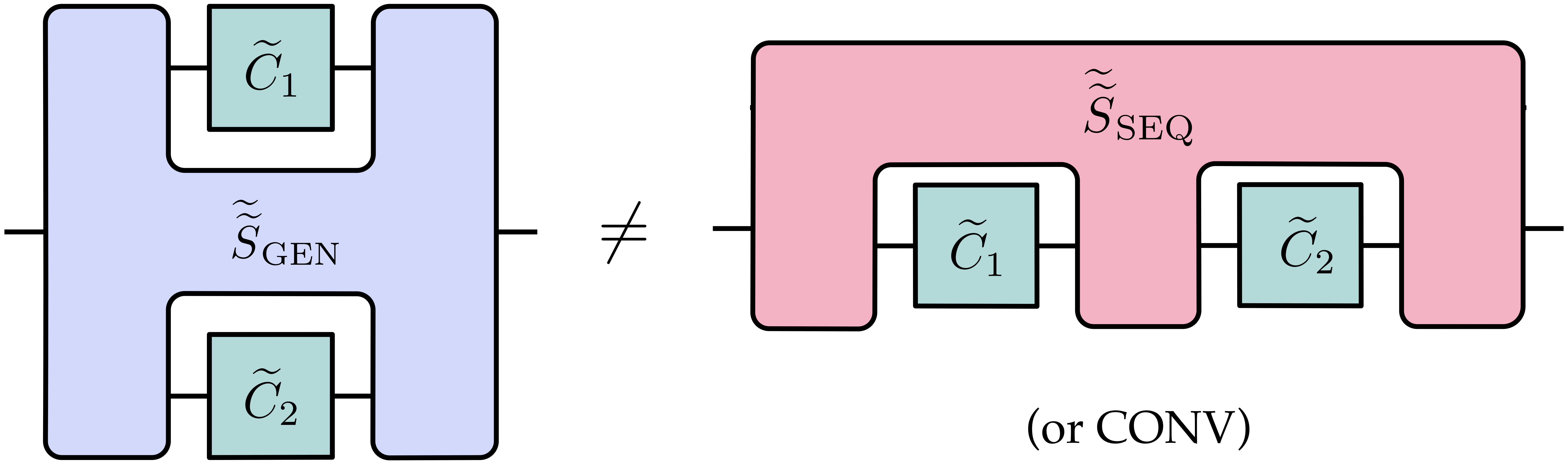
REVIEW: Costa et al., *Indefinite Quantum Causality*, arXiv:2606.19438 [quant-ph]

- **Channel discrimination**
  Chiribella, PRA (2012)
  Araújo et al., NJP (2015)
  Bavaresco et al., PRL (2021); J. Math. Phys. (2022)
  …
- **State discrimination**
  Kunjwal and Baumeler, PRL (2023)
  …
- **Quantum query complexity**
  Araújo et al., PRL (2014)
  Renner et al., PRL (2022)
  Abbott et al., PRResearch (2024); QPL (2025)
  …
- **Universal unitary transformations**
  Quintino et al., PRL (2019); PRA (2019)
  Yoshida et al., Quantum (2023)
  …
- **Metrology**
  Zhao et al., PRL (2020)
  Liu et al., PRL (2023)
  …
- **Parameter estimation**
  Meyer et al., Quantum (2025)
  André et al., QST (2026)
  …
- **… and more**

QCCC advantage over quantum circuits
Bavaresco et al., Nat. Commun. (2025)

Advantages of indefinite causal order

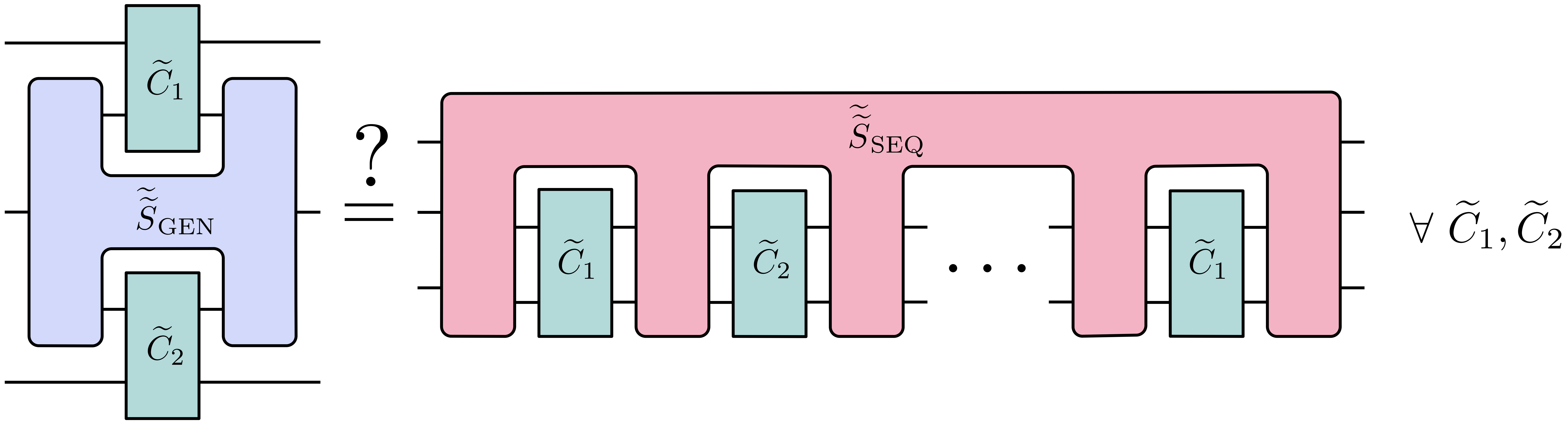
Advantages of indefinite causal order: Indefinite causal order is typically deemed to be advantageous for a certain task if there exists at least one superchannel with indefinite causal order that performs better at the given task than **any superchannel with definite causal order** that acts on the **same number of queries** of the input quantum channels.



(or CONV)
(or QCCC)

Simulation of indefinite causal order

Simulation of indefinite causal order: But how does it compare to a superchannel with definite causal order that **acts on a higher number of queries of the input quantum channels**?



(or CONV)
(or QCCC)

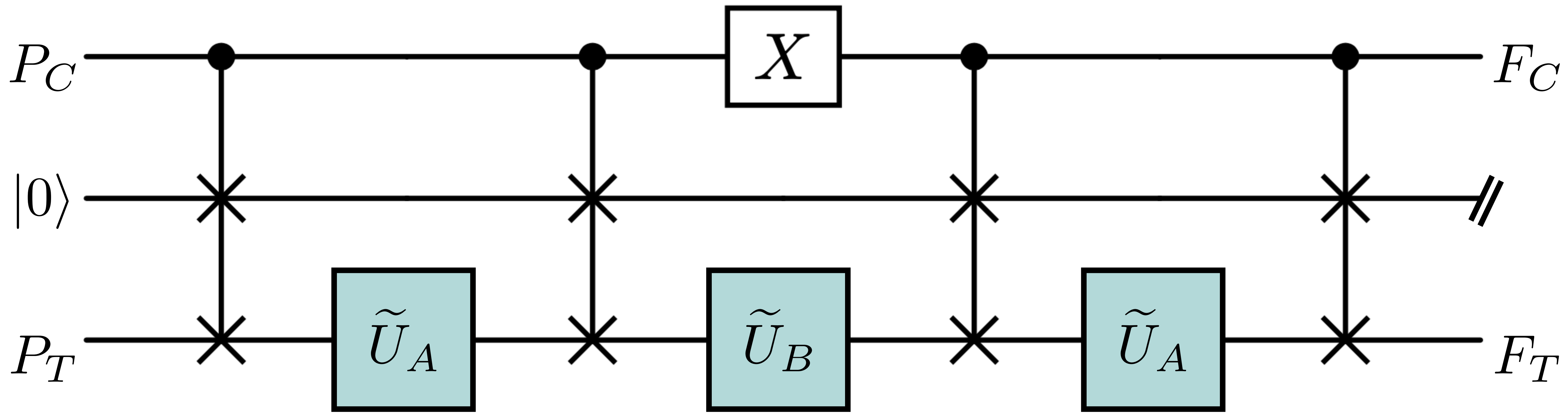
Simulation of the quantum switch

How many queries of each input channel are needed to simulate (deterministically and exactly)

- the quantum switch acting on **unitary channels** using **quantum circuits**:

$$N_A = 2, \; N_B = 1$$ (independent of the dimension)

Chiribella et al., PRA (2013)



Simulation of the quantum switch

How many queries of each input channel are needed to simulate (deterministically and exactly)

- the quantum switch acting on **unitary channels** using **quantum circuits**:

$$N_A = 2,\ N_B = 1$$ (independent of the dimension)

Chiribella et al., PRA (2013)

- the quantum switch acting on **mixtures of unitary channels** using **QCCCs**:

impossible with $N_A < 2^n,\ N_B = 1$ (where n is the number of qubits)

impossible with $N_A = 2,\ N_B = 2$ (independent of the dimension)

Bavaresco et al., Nat. Commun. (2025)

Implementation of indefinite causal order

How?

- Current experimental implementations suffer from loopholes.
- Not known how to close some loopholes even from a theoretical point of view.

Why not?

- No known logical contradictions.
- No known implausible consequences (i.e. computational superpower).

REVIEW: Rozema et al., *Experimental Aspects of Indefinite Causal Order in Quantum Mechanics*, Nat. Rev. Phys. (2024)

Take-home message

Higher-order quantum theory shows that **basic requirements** of transforming every valid set of independent quantum operation into another valid quantum operation (complete admissibility) is **not enough to achieve at a definite causal order**—more assumptions are required to arrive at the quantum circuit model.

Indefinite causal order in higher-order quantum theory has several concrete applications and **advantages to quantum information processing tasks**. However, their conclusive **implementation remains an open question**.

Thank you!

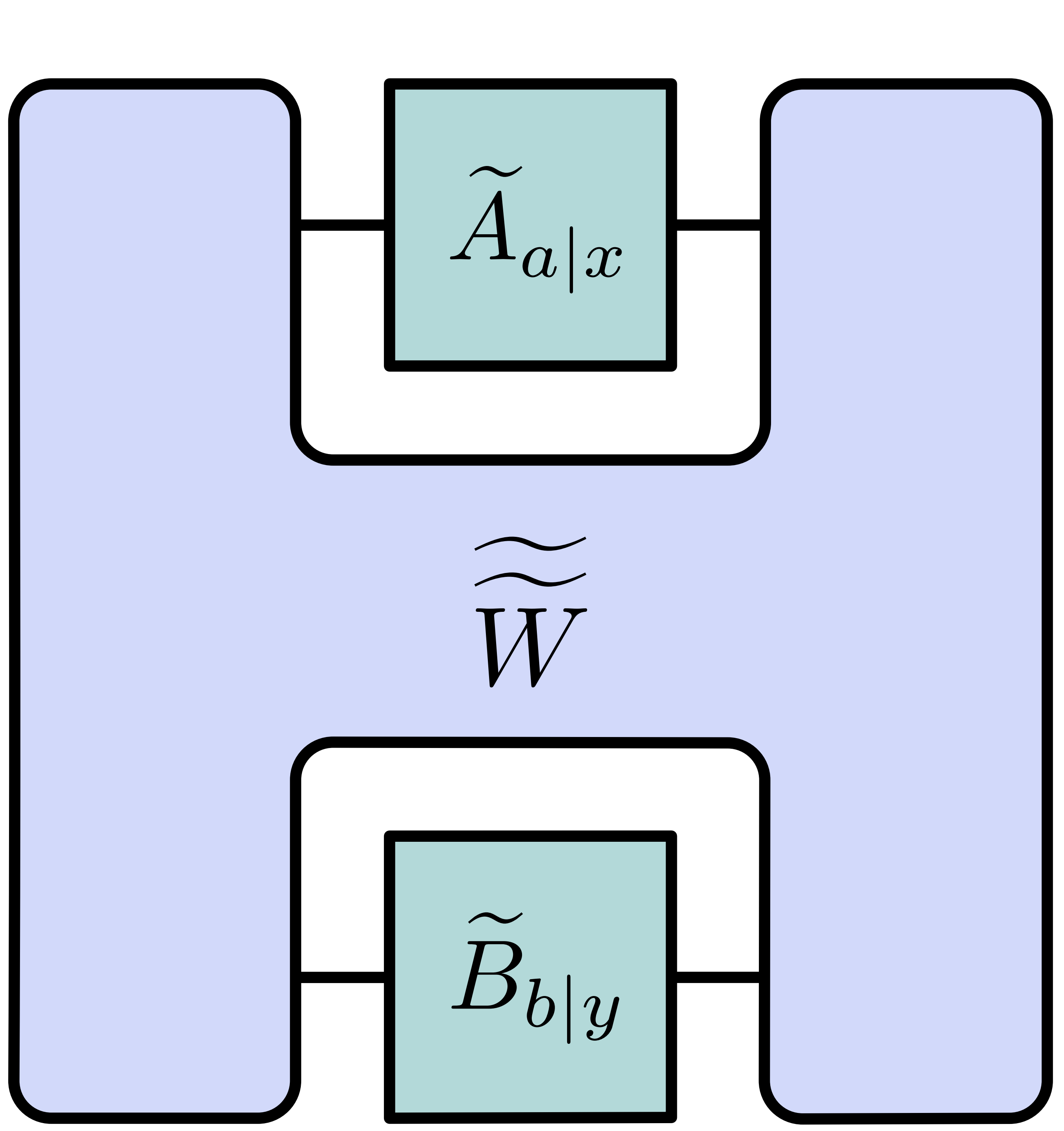
Extra

an operational approach to causality:
causal correlations

Operational approach to indefinite causal order

Superchannel with trivial $\mathcal{H}_{I'}, \mathcal{H}_{O'}$

"process matrix"



$$= p(ab|xy)$$

$$\widetilde{\widetilde{W}} : [(\mathcal{L}(\mathcal{H}_{I_A}) \to \mathcal{L}(\mathcal{H}_{O_A})) \otimes (\mathcal{L}(\mathcal{H}_{I_B}) \to \mathcal{L}(\mathcal{H}_{O_B}))] \to \mathbb{C}$$

$$W \in \mathcal{L}(\mathcal{H}_{I_A} \otimes \mathcal{H}_{O_A} \otimes \mathcal{H}_{I_B} \otimes \mathcal{H}_{O_B})$$

quantum instruments:

$$\{A_{a|x}\};\ A_{a|x} \geq 0,\ \sum_a A_{a|x} = C_x^A \text{ is CPTP}$$

$$\{B_{b|y}\};\ B_{b|y} \geq 0,\ \sum_b B_{b|y} = C_y^B \text{ is CPTP}$$

$$p(ab|xy) = \mathrm{tr}\left[W(A_{a|x} \otimes B_{b|y})\right]$$

process matrix correlations

Oreshkov et al., Nat. Commun. (2012)

Operational approach to indefinite causal order

Causal correlations

$$p_{\text{causal}}(ab|xy) = q\, p_{A \prec B}(ab|xy) + (1-q)\, p_{B \prec A}(ab|xy)$$

$$\sum_b p_{A \prec B}(ab|xy) = \sum_b p_{A \prec B}(ab|xy') \longrightarrow$$ nonsignalling from Bob to Alice

$$\sum_a p_{B \prec A}(ab|xy) = \sum_a p_{B \prec A}(ab|x'y) \longrightarrow$$ nonsignalling from Alice to Bob

$$q \in [0,1]$$

Oreshkov et al., Nat. Commun. (2012)

Operational approach to indefinite causal order

Causal inequalities

$$\boxed{\sum_{a,b,x,y} \alpha_{xy}^{ab}\, p(ab|xy) \leq B_{\text{causal}}}$$

$$\alpha_{xy}^{ab} \in \mathbb{R}$$

$$B_{\text{causal}} := \max_{p_{\text{causal}}} \sum_{a,b,x,y} \alpha_{xy}^{ab}\, p_{\text{causal}}(ab|xy)$$

Oreshkov et al., Nat. Commun. (2012)
Branciard et al., NJP (2016)

Operational approach to indefinite causal order

Causal inequalities

$$P_{\mathrm{GNYI}} = \frac{1}{4} \sum_{a,b,x,y} \delta_{a,y}\, \delta_{b,x}\, p(ab|xy) \leq \frac{1}{2}$$

$$P_{\mathrm{LGNYI}} = \frac{1}{4} \sum_{a,b,x,y} \delta_{x(a\oplus y),0}\, \delta_{y(b\oplus x),0}\, p(ab|xy) \leq \frac{3}{4}$$

$$P_{\mathrm{OCB}} = \frac{1}{8} \sum_{a,b,x,y,y'} \delta_{(y'\oplus 1)(a\oplus y),0}\, \delta_{y'(b\oplus x),0}\, p(ab|xyy') \leq \frac{3}{4}$$

process matrix correlations:

$$P_{\mathrm{GYNI}} \approx 0.6218$$

$$P_{\mathrm{LGYNI}} \approx 0.8194$$

$$P_{\mathrm{OCB}} \approx 0.8536$$

Oreshkov et al., Nat. Commun. (2012)
Branciard et al., NJP (2016)

Operational approach to indefinite causal order

(all) Superchannels in QCQC **cannot** violate causal inequalities.

(at least some) Superchannels in PUR and GEN **can** violate causal inequalities.

Araújo et al., NJP (2015)
Wechs et al., PRX Quantum (2021)
Oreshkov et al., Nat. Commun. (2012)
Branciard et al., NJP (2016)
Araújo et al., Quantum (2017)
Baumeler and Wolf, NJP (2016)

Operational approach to indefinite causal order

Process matrix correlations